Revisiting radiative leptonic *B* decay[*]

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Abstract: In this paper, we summarize the existing methods of solving the evolution equation of the leading-twist *B*-meson LCDA. Then, in the Mellin space, we derive a factorization formula with next-to-leading-logarithmic (NLL) resummation for the form factors $F_{A,V}$ in the $B \to \gamma \ell \nu$ decay at leading power in Λ/m_b. Furthermore, we investigate the power suppressed local contributions, factorizable non-local contributions (which are suppressed by $1/E_\gamma$ and $1/m_b$), and soft contributions to the form factors. In the numerical analysis, which employs the two-loop-level hard function and the jet function, we find that both the resummation effect and the power corrections can sizably decrease the form factors. Finally, the integrated branching ratios are also calculated for comparison with future experimental data.

Keywords: *B* physics, RG equation, QCD factorization, form factors

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1   Introduction

The radiative leptonic decay of the *B* meson is of interest as it is the most important channel to extract the parameters of the *B*-meson light-cone distribution amplitudes (LCDAs) and to test the factorization theorem when the emitted photon is energetic. A precision study of this mode can also be helpful in decreasing the background to the purely leptonic decay process $B^- \to \ell^- \nu$, which is important for determining the CKM matrix element V_{ub}. The radiative leptonic $B \to \gamma \ell \nu$ decay amplitude is defined by the QCD matrix element

$$\mathcal{A}(B \to \gamma \ell \nu) = \frac{G_F V_{ub}}{\sqrt{2}} \left\langle \gamma(p) \ell(p_\ell) \nu(p_\nu) \left| \left[\bar{\ell} \gamma_\mu (1 - \gamma_5) \nu \right] \right. \right.$$
$$\left. \left. \times \left[\bar{u} \gamma^\mu (1 - \gamma_5) b \right] \right| B^-(p_B) \right\rangle . \tag{1}$$

In the rest frame of the *B* meson with momentum $p_B = m_B v$, it is convenient to introduce two light-cone vectors n_μ and $\bar{n}_\mu$ with the definitions

$$p_\mu = \frac{n \cdot p}{2} \bar{n}_\mu \equiv E_\gamma \bar{n}_\mu , \qquad v_\mu = \frac{n_\mu + \bar{n}_\mu}{2} . \tag{2}$$

At leading order in QED and considering the constraints from the Ward identity, the amplitude can be parameterized as [1, 2]

$$\mathcal{A}(B \to \gamma \ell \nu) \to \frac{G_F V_{ub}}{\sqrt{2}} (i g_{em} \epsilon_\nu^*) v \cdot p \left\{ -i \epsilon_{\mu\nu\rho\sigma} n^\rho v^\sigma F_V(E_\gamma) + g_{\mu\nu} \hat{F}_A(E_\gamma) \right\} , \tag{3}$$

where the contribution from final-state radiation is accounted for by the redefinition of the axial form factor $\hat{F}_A(E_\gamma)$.

At leading power in Λ/m_b, the QCD factorization for-

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mula has been derived for the $B \to \gamma$ form factors $F_{A,V}$ [3, 4] and was confirmed under the framework of soft-collinear effective theory (SCET) [5, 6]. The form factors $F_{A,V}(E_\gamma)$ can be factorized into a convolution of the hard function, jet function, and B-meson LCDA. The hard function arises from the matching between heavy-to-light current in the QCD and SCET_I operators, and it has been calculated up to the two-loop level [1]. The jet function can be obtained from the matching between SCET_I and SCET_II [5], and the next-to-next-to-leading-order (NNLO) correction has been recently obtained [7]. The matrix elements of the SCET_II operators are actually the definition of B-meson LCDA. All the ingredients in the factorization formula depend on the factorization scale, and the radiative corrections lead to large logarithmic terms, which need to be resummed. For the hard function, the three-loop anomalous dimension is known [8-11], and the two-loop-level anomalous dimensions both for the B-meson LCDA and the jet function have recently been calculated [7, 12]. Therefore, the sufficient condition for a complete NLL resummation is readily available. It was first derived in [13] by performing a Laplace transformation of the B-meson LCDA.

Although the factorization formula of $B \to \gamma \ell \nu$ decay is well established at leading power, the power corrections are important for finite bottom-quark mass. The power-suppressed corrections of $O(1/m_b)$ were considered at tree level [1], where a symmetry-preserving form factor $\xi(E_\gamma)$ was introduced to parameterize the non-local power corrections. The soft contribution from the endpoint region of the momentum of the light quark inside the B meson was first studied using dispersion relation and quark-hadron duality in [14]. The QCD correction to the soft contribution at one loop and the contribution from three-particle LCDAs were computed in [2]. In a comprehensive study on the local and non-local power suppressed contributions, the soft contribution and the higher-twist contribution to the $B \to \gamma \ell \nu$ decay were presented [15]. The contribution from the hadronic structure of the photon, which can be defined by the matrix elements of power-suppressed SCET operators, was studied in [16, 17]. Moreover, based on transverse-momentum-dependent factorization, the power corrections to $B \to \gamma \ell \nu$ decay were investigated in [18]. All these studies indicate that the power-suppressed contribution is sizable and should not be neglected in the determination of the first inverse moment of the B-meson LCDA.

In this paper, we make improvements from two aspects. The first is to derive the scale-independent factorization formula at the NLL level in the Mellin space, and the second is to perform the phenomenological analysis after combining the NLL leading-power result with the power corrections. This paper is organized as follows. In the next section, we review the evolution of the leading-

twist B-meson LCDA. In the third section, we derive the scale-independent factorization formula of the $B \to \gamma$ form factors and discuss the power-suppressed contributions; this is followed by the phenomenological analysis. Concluding discussions are presented in the final section.

2   The evolution of the B-meson LCDA

The B-meson LCDA is one of the most important ingredients of the QCD factorization formula for exclusive B decays. The two-particle LCDAs of the B meson in the heavy-quark effective theory (HQET) can be obtained from the coordinate-space matrix elements [19]

$$\langle 0|\bar{q}^\beta(z)[z,0]h_v^\alpha(0)|\bar{B}(v)\rangle = -\frac{\mathrm{i}\tilde{f}_B m_B}{4}\left[\frac{1+\not{v}}{2}\left\{\tilde{\Phi}^+(t,z^2)\right.\right.$$
$$\left.\left.+\frac{\tilde{\Phi}^-(t,z^2)-\tilde{\Phi}^+(t,z^2)}{t}\not{z}\right\}\right]^{\alpha\beta}. \quad (4)$$

The LCDAs $\tilde{\Phi}^\pm(t,z^2)$ (in curly brackets) can be expanded around $z^2 = 0$. In the limit, $z^2 \to 0$, $t \to \tau = n \cdot z/2$, the B-meson LCDAs in the momentum space are defined through the Fourier transformation

$$\phi^\pm(\omega) = \int \frac{\mathrm{d}\tau}{2\pi} \mathrm{e}^{\mathrm{i}\omega\tau} \tilde{\Phi}^\pm(\tau). \quad (5)$$

At leading power, only $\phi^+(\omega)$ is relevant in the $B \to \gamma \ell \nu$ decay, and the evolution equation of $\phi^+(\omega)$ is the well-known Lange-Neubert equation [20]:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\phi^+(\omega,\mu) = -\int_0^\infty \mathrm{d}\omega'\Gamma_+(\omega,\omega',\mu)\phi^+(\omega',\mu),$$
$$\Gamma_+(\omega,\omega',\mu) = \left(\Gamma_\text{cusp}\ln\frac{\mu}{\omega}+\gamma\right)\delta(\omega-\omega')+\omega\Gamma_\text{cusp}\Gamma(\omega,\omega'),$$
$$(6)$$

where μ is the renormalization scale. At the one-loop level, the anomalous dimensions are

$$\Gamma_\text{cusp} = \sum_{n=1}\Gamma_\text{cusp}^{(n)}\left(\frac{\alpha_s}{4\pi}\right)^n, \qquad \gamma = \sum_{n=1}\gamma^{(n)}\left(\frac{\alpha_s}{4\pi}\right)^n,$$
$$\Gamma_\text{cusp}^{(1)} = 4C_F, \qquad \gamma^{(1)} = -2C_F,$$
$$\Gamma(\omega,\omega') = -\left[\frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega(\omega-\omega')}\right]_+, \quad (7)$$

with the "plus" function defined as

$$\int_0^\infty \mathrm{d}y\left[f(x,y)\right]_+ g(y) = \int_0^\infty \mathrm{d}y\, f(x,y)\left[g(y)-g(x)\right]. \quad (8)$$

In the position space, the evolution equation of the B-meson LCDA takes the form [21]

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\tilde{\Phi}^+(t,\mu) = -[\Gamma_\text{cusp}(\alpha_s)\ln it\tilde{\mu}+\gamma_+(\alpha_s)$$
$$-\gamma_F(\alpha_s)]\tilde{\Phi}^+(t,\mu)+\int_0^1 \mathrm{d}z\, K(z,\alpha_s)\tilde{\Phi}^+(zt,\mu), \quad (9)$$

where at the one-loop level,

$$\tilde{\mu} = \mu \, e^{\gamma_E}, \qquad \gamma_+(\alpha_s) = -\frac{\alpha_s C_F}{4\pi},$$
$$\gamma_F(\alpha_s) = -\frac{3\alpha_s C_F}{4\pi}, \qquad K(z,\alpha) = \frac{\alpha_s C_F}{\pi}\left(\frac{z}{1-z}\right)_+. \quad (10)$$

Whether in the momentum space or in the position space, the evolution equation of the B-meson LCDA is the integro-differential equation. It is difficult to solve directly: it must be simplified by an integral transformation. To date, there exist the following treatments:

● Performing the Fourier transformation with respect to $\ln(\omega/\mu)$ (or the Mellin transform $\langle\omega^{N-1}\rangle$ for $N = i\theta$):

$$\varphi^+(\theta,\mu) = \int_0^\infty \frac{d\omega}{\omega} \phi^+(\omega)\left(\frac{\omega}{\mu}\right)^{-i\theta}. \quad (11)$$

Then, the evolution kernel of $\varphi^+(\theta,\mu)$ is obtained as [22]

$$\varphi_B^+(\theta,\mu) = e^{V(\mu,\mu_0)-2\gamma_E g}\left(\frac{\mu}{\mu_0}\right)^{i\theta}\frac{\Gamma(1-i\theta)\Gamma(1+i\theta-g)}{\Gamma(1+i\theta)\Gamma(1-i\theta+g)}$$
$$\times \varphi_B^+(\theta+ig,\mu_0). \quad (12)$$

Through the inverse Fourier transformation, we arrive at the solution to the evolution equation in the momentum space:

$$\phi^+(\omega,\mu) = e^{V(\mu,\mu_0)-2\gamma_E g}\frac{\Gamma(2-g)}{\Gamma(g)}\int_0^\infty \frac{d\omega'}{\omega'}\phi^+(\omega',\mu_0)$$
$$\times \left(\frac{\omega_>}{\mu_0}\right)^g \frac{\omega_<}{\omega_>} {}_2F_1\left(1-g,2-g;2;\frac{\omega_<}{\omega_>}\right), \quad (13)$$

where $\omega_< = \min(\omega,\omega'), \omega_> = \max(\omega,\omega')$, and the functions V and g take the form

$$V(\mu,\mu_0) = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\left[\Gamma_{\text{cusp}}(\alpha)\int_{\alpha_s(\mu_0)}^\alpha \frac{d\alpha'}{\beta(\alpha')} + \gamma(\alpha)\right],$$
$$g \equiv g(\mu,\mu_0) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}. \quad (14)$$

● Performing the Mellin transformation to the evolution equation in the position space [21, 23]:

$$\tilde{\varphi}^+(j,\mu) = \frac{1}{2\pi i}\int_{-i0}^{-i\infty}\frac{dt}{t}(it\tilde{\mu})^{-j}\tilde{\Phi}^+(t,\mu). \quad (15)$$

In the Mellin space, the evolution equation takes a simple form:

$$\left[\frac{d}{d\ln\mu} + \hat{V}(j,\alpha_s)\right]\tilde{\varphi}^+(j,\mu) = 0, \quad (16)$$

with

$$\hat{V}(j,\alpha_s) = j + \gamma_+ - \gamma_F + \Gamma_{\text{cusp}}\left[\psi(j+2)-\psi(2)+\vartheta(j)\right], \quad (17)$$

where $\vartheta(j) = 0$ at the one-loop level. The solution in the Mellin space can be obtained directly:

$$\tilde{\varphi}^+(j(\mu),\alpha_s(\mu),\mu) = \tilde{\varphi}^+(j(\mu_0),\alpha_s(\mu_0),\mu_0)$$
$$\times \exp\left\{-\int_{\mu_0}^\mu \frac{ds}{s}\hat{V}[j(s),\alpha_s(s)]\right\}. \quad (18)$$

● It was found that, if the B-meson LCDA is transformed into the so-called "dual" space, the evolution kernel is diagonalized [24]. The LCDA in the dual space can be obtained by

$$\rho^+(\omega',\mu) = \int_0^\infty \frac{d\omega}{\omega}\sqrt{\frac{\omega}{\omega'}}J_1\left(2\sqrt{\frac{\omega}{\omega'}}\right)\phi^+(\omega,\mu), \quad (19)$$

which satisfies an ordinary differential equation:

$$\mu\frac{d}{d\mu}\rho^+(\omega',\mu) = -\left[\Gamma_{\text{cusp}}\ln\frac{\mu}{\hat{\omega}'} + \gamma\right]\rho^+(\omega',\mu). \quad (20)$$

It is then simple to write the solution:

$$\rho^+(\omega',\mu) = e^V\left(\frac{\mu_0}{\hat{\omega}'}\right)^{-g}\rho^+(\omega',\mu_0) = e^{\bar{V}}\left(\frac{\mu\mu_0}{(\hat{\omega}')^2}\right)^{-g/2}\rho^+(\omega',\mu_0), \quad (21)$$

with

$$\hat{\omega}' = e^{-2\gamma_E}\omega', \qquad \bar{V}(\mu,\mu_0) = \frac{1}{2}(V(\mu,\mu_0)-V(\mu_0,\mu)).$$

The method mentioned above is equivalent, and the LCDAs $\phi^+(\omega)$, $\varphi^+(\theta)$, $\tilde{\Phi}^+(t)$, $\tilde{\varphi}^+(j)$, $\rho^+(\omega')$ are different expressions of an identical objective. Because the momentum space and the position space are related through a standard Fourier transformation, we are able to derive

$$\tilde{\varphi}_+(j) = \frac{\Gamma(-j)}{2\pi i}\int_0^\infty d\omega\left(\frac{\omega}{\tilde{\mu}}\right)^j \phi_+(\omega) \quad (22)$$

and

$$\tilde{\varphi}^+(-i\theta) = \frac{\Gamma(i\theta)}{2\pi i}\frac{\mu^{1-i\theta}}{\tilde{\mu}^{-i\theta}}\varphi^+(\theta+i). \quad (23)$$

$\rho^+(\omega')$ is related to $\varphi^+(\theta)$ by definition:

$$\varphi^+(\theta,\mu) = \frac{\Gamma(1-i\theta)}{\Gamma(1+i\theta)}\int_0^\infty \frac{d\omega'}{\omega'}\rho^+(\omega',\mu)\left(\frac{\mu}{\omega'}\right)^{i\theta}. \quad (24)$$

Then, we have

$$\tilde{\varphi}^+(j,\mu) = \frac{\tilde{\mu}}{2\pi i}\Gamma(2+j)\int_0^\infty \frac{d\omega'}{\omega'}\rho^+(\omega',\mu)\left(\frac{\tilde{\mu}}{\omega'}\right)^{-1-j}. \quad (25)$$

At the one-loop level, the most convenient method is to work in the dual space since the Bessel function is the eigenfunction of the Lange-Neubert kernel, which is confirmed in [25, 26]. The Lange-Neubert kernel can be expressed as a logarithm of the generator of special conformal transformations along the light cone. When the eigenfunction of the generator is transformed to the momentum space, it is simply the Bessel function in Eq. (19).

The two-loop-level anomalous dimension of the B-meson LCDA was first calculated in the coordinate space in [12]; it is more simply expressed in the dual space:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(a)\frac{\partial}{\partial a}+\Gamma_{\text{cusp}}(a)\ln(\tilde{\mu}e^{\gamma_E}s)+\gamma(a)\right]\eta_+(s,\mu)$$

$$=4C_F a^2\int_0^1\frac{du}{u}\,\bar{u}h(u)\eta_+(\bar{u}s,\mu),\qquad(26)$$

where $s\eta_+(s)=\rho^+(1/s)$ and $a=\alpha_s/(4\pi)$. This equation is also transformed into the momentum space in [7], resulting in the two-loop-level Lange-Neubert equation. The advantage of solving the evolution equation at the two-loop level in the dual space does not hold as the two-loop evolution kernel is not diagonal in this space. On the contrary, the elegant form of the evolution equation (Eq. (16)) in the Mellin space is maintained. Thus, Eq. (18) is still the solution to the evolution equation up to the two-loop level with [23]

$$\vartheta(j)=a\vartheta^{(1)}(j)=a\Big\{(\beta_0-3C_F)\big(\psi'(j+2)-\psi'(2)\big)$$

$$+2C_F\Big(\frac{1}{(j+1)^3}+\psi'(j+2)(\psi(j+2)-\psi(1))$$

$$+\psi'(j+1)(\psi(j+1)-\psi(1))-\frac{\pi^2}{6}\Big)\Big\},$$

$$\gamma_+(a)=-aC_F+a^2C_F\Big\{4C_F\Big[\frac{21}{8}+\frac{\pi^2}{3}-6\zeta_3\Big]$$

$$+C_A\Big[\frac{83}{9}-\frac{2\pi^2}{3}-6\zeta_3\Big]+\beta_0\Big[\frac{35}{18}-\frac{\pi^2}{6}\Big]\Big\},$$

$$\gamma_F(a)=-3aC_F+a^2C_F\Big\{C_F\Big[\frac{5}{2}-\frac{8\pi^2}{3}\Big]$$

$$+C_A\Big[1+\frac{2\pi^2}{3}\Big]-\frac{5}{2}\beta_0\Big\}.\qquad(27)$$

In a recent paper [13], an alternative approach to solving the evolution equation at the two-loop level was proposed. The essential idea of this approach is to perform a Laplace transformation on the B-meson LCDA,

$$\tilde{\phi}^+(\eta,\mu)=\int_0^\infty\frac{d\omega}{\omega}\Big(\frac{\omega}{\bar{\omega}}\Big)^{-\eta}\phi^+(\omega,\mu),\qquad(28)$$

where $\bar{\omega}$ is a fixed reference scale, which can be used to eliminate the logarithmic moment σ_1 in the factorization formula of $B\to\gamma\nu\ell$. Then, one could derive

$$\Big(\frac{d}{d\ln\mu}+\Gamma_{\text{cusp}}(\alpha_s)\frac{\partial}{\partial\eta}\Big)\tilde{\phi}_+(\eta,\mu)=\Big[\Gamma_{\text{cusp}}(\alpha_s)\Big(\ln\frac{\bar{\omega}}{\mu}+\mathcal{F}(\eta)\Big)$$

$$-\gamma(\alpha_s)+\mathcal{G}(\eta,\alpha_s)\Big]\tilde{\phi}_+(\eta,\mu),\qquad(29)$$

with the definition

$$\mathcal{F}(\eta)=\int_0^\infty dx\,\Gamma(1,x)x^\eta=-[H(\eta)+H(-\eta)],$$

$$\mathcal{G}(\eta;\alpha_s)=\int_0^\infty dx\,\hat{\gamma}_+(1,x;\alpha_s)x^\eta,\qquad(30)$$

where $\hat{\gamma}_+(1,x;\alpha_s)$ starts from the two-loop level, and the

specific expression can be seen in [13]. After the Laplace transformation, the solution to the evolution equation reads

$$\tilde{\phi}_+(\eta,\mu)=N(\mu_s,\mu)\frac{\Gamma(1+\eta+a_\Gamma(\mu_s,\mu))\Gamma(1-\eta)}{\Gamma(1-\eta-a_\Gamma(\mu_s,\mu))\Gamma(1+\eta)}$$

$$\times\exp\Big[\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta(\alpha)}\mathcal{G}(\eta+a_\Gamma(\mu_\alpha,\mu),\alpha)\Big]$$

$$\times e^{2\gamma_E a_\Gamma(\mu_s,\mu)}\tilde{\phi}_+(\eta+a_\Gamma(\mu_s,\mu),\mu_s).\qquad(31)$$

The normalization $N(\mu_s,\mu)$ depends on the factorization scale through

$$N(\mu_s,\mu)=\Big(\frac{\bar{\omega}}{\mu_s}\Big)^{-a_\Gamma(\mu_s,\mu)}e^{S(\mu_s,\mu)+a_\gamma(\mu_s,\mu)},\qquad(32)$$

where the quantities a_γ, a_Γ, and $S(\mu_s,\mu)$ are given as

$$S(\mu_s,\mu)=-\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu)}d\alpha\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\int_{\alpha_s(\mu_s)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')},$$

$$a_\Gamma(\mu_0,\mu)=-\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}d\alpha\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)},$$

$$a_\gamma(\mu_0,\mu)=-\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}d\alpha\frac{\gamma(\alpha)}{\beta(\alpha)}.\qquad(33)$$

We note that the LCDA $\varphi^+(j)$ is related to $\tilde{\phi}^+(\eta)$ through

$$\tilde{\varphi}^+(j,\mu)=\frac{\Gamma(-j)}{2\pi i}\tilde{\mu}\Big(\frac{\bar{\omega}}{\tilde{\mu}}\Big)^{j+1}\tilde{\phi}^+(-j-1,\mu).\qquad(34)$$

3 $B\to\gamma$ form factors

At leading power in Λ/m_b, the QCD factorization formula for the $B\to\gamma$ form factors can be written as

$$F_{V,\text{LP}}(E_\gamma)=F_{A,\text{LP}}(E_\gamma)=\frac{Q_u m_B}{2E_\gamma}\tilde{f}_B(\mu)C_\perp(E_\gamma,\mu)$$

$$\times\int_0^\infty d\omega\frac{\phi_B^+(\omega,\mu)}{\omega}J_\perp(E_\gamma,\omega,\mu).\qquad(35)$$

At the one-loop level, the hard function and jet function are given as [5, 6, 27]

$$C_\perp(E_\gamma,\mu)=1-\frac{\alpha_s C_F}{4\pi}\Big[2\ln^2\frac{\mu}{2E_\gamma}+5\ln\frac{\mu}{m_b}-2\text{Li}_2\Big(1-\frac{1}{r}\Big)$$

$$-\ln^2 r+\frac{3r-2}{1-r}\ln r+\frac{\pi^2}{12}+6\Big]+O(\alpha_s^2)$$

$$J_\perp(E_\gamma,\omega,\mu)=1+\frac{\alpha_s C_F}{4\pi}\Big[\ln^2\frac{\mu^2}{2E_\gamma\omega}-\frac{\pi^2}{6}-1\Big]+O(\alpha_s^2),$$

$$(36)$$

with $r = 2E_\gamma/m_b$. The results of the two-loop level hard function and jet function can be found in [1,7]. As the hard function and jet function contain large logarithmic terms, it is important to perform resummation to improve the convergence of the perturbative series. The first complete NLL resummation is given in [13]:

$$
\begin{aligned}
F_{A/V,\mathrm{LP}}(E_\gamma) &= \frac{Q_u m_B}{2E_\gamma} \exp\Big[S(\mu_h,\mu_j) + S(\mu_s,\mu_j) - a_{\gamma_H}(\mu_h,\mu_j) + a_\gamma(\mu_s,\mu_j) + 2\gamma_E a_\Gamma(\mu_s,\mu_j) \Big] \\
&\quad \times \tilde{f}_B(\mu_h) C_\perp(E_\gamma,\mu_h) \left(\frac{2E_\gamma}{\mu_h}\right)^{-a_\Gamma(\mu_h,\mu_j)} \mathcal{J}(\partial_\eta,\mu_j) \left(\frac{2E_\gamma\bar\omega}{\mu_j^2}\right)^\eta \frac{\Gamma(1-\eta+a_\Gamma(\mu_s,\mu_j))\Gamma(1+\eta)}{\Gamma(1+\eta-a_\Gamma(\mu_s,\mu_j))\Gamma(1-\eta)} \\
&\quad \times \exp\Bigg[\int_{\alpha_s(\mu_s)}^{\alpha_s(\mu_j)} \frac{\mathrm{d}\alpha}{\beta(\alpha)} \mathcal{G}(-\eta+a_\Gamma(\mu_\alpha,\mu_j),\alpha) \Bigg] \left(\frac{\bar\omega}{\mu_s}\right)^{-a_\Gamma(\mu_s,\mu_j)} \tilde\phi_+(-\eta+a_\Gamma(\mu_s,\mu_j),\mu_s) \Bigg|_{\eta=0},
\end{aligned}
\tag{37}
$$

where the jet function $\mathcal{J}(L_p,\mu_j) \equiv J_\perp(-p^2,\mu_j) = J_\perp(n\cdot p\omega,\mu_j)$ with $L_p = \ln(p^2/\mu_j^2)$. We now derive the scale independent factorization formula at the NLL level in the Mellin space. The evolution of the B-meson LCDA is known in the Mellin space. We thus need to perform the Mellin transformation to the jet function, although it is not well-defined. Alternatively, we follow the method in [13] to replace the first argument of $\mathcal{J}(L_p,\mu_j)$ by a derivative operator, i.e.,

$$
\int_0^\infty \frac{\mathrm{d}\omega}{\omega} \phi^+(\omega,\mu) J_\perp(E_\gamma,\omega,\mu) = \mathcal{J}_\perp(\partial_j,\mu) \left(\frac{2E_\gamma}{\mu}\right)^j \int_0^\infty \frac{\mathrm{d}\omega}{\omega} \phi^+(\omega,\mu) \left(\frac{\omega}{\mu}\right)^j \Bigg|_{j=0}.
\tag{38}
$$

Taking advantage of Eq. (22), we have

$$
\int_0^\infty \frac{\mathrm{d}\omega}{\omega} \phi^+(\omega,\mu) J_\perp(E_\gamma,\omega,\mu) = 2\pi i \mathcal{J}_\perp(\partial_j,\mu) \frac{1}{\tilde\mu} \left(\frac{2E_\gamma \mathrm{e}^{2\gamma_E}}{\mu}\right)^j \frac{1}{\Gamma(1-j)} \tilde\varphi^+(j-1,\mu) \Bigg|_{j=0}.
\tag{39}
$$

Employing the evolution function of the hard function and B-meson LCDA, we obtain

$$
\begin{aligned}
F_{A/V,\mathrm{LP}} &= \frac{Q_u m_B}{2E_\gamma} \Big[U_2(E_\gamma,\mu_h',\mu) \tilde f_B(\mu_h') \Big] \Big[U_1(E_\gamma,\mu_h,\mu) C_\perp(E_\gamma,\mu_h) \Big] \times 2\pi i \mathcal{J}_\perp(\partial_j,\mu_j) \frac{1}{\tilde\mu} \\
&\quad \times \left(\frac{2E_\gamma \mathrm{e}^{2\gamma_E}}{\mu}\right)^j \frac{1}{\Gamma(1-j)} \tilde\varphi^+(j(\mu_s)-1,\alpha_s(\mu_s),\mu_s) \exp\left\{ -\int_{\mu_s}^{\mu} \frac{\mathrm{d}s}{s} V[j(s),\alpha_s(s)] \right\} \Bigg|_{j=0},
\end{aligned}
\tag{40}
$$

where $U_1(E_\gamma,\mu_h,\mu)$ and $U_2(E_\gamma,\mu_h',\mu)$ are the evolution factor of the hard function and B-meson decay constant in the HQET, respectively; the specific expression can be found in the appendices of [1,2]. The parameter j depends on the factorization scale through

$$
\mu\frac{\mathrm{d}}{\mathrm{d}\mu} j(\mu) = -\Gamma_{\mathrm{cusp}}(\alpha_s).
\tag{41}
$$

The resummed factorization formula in the Mellin space seems more compact than Eq. (37).

Now, we turn to the power suppressed contributions. At leading power, F_V and F_A are equal due to the left-handedness of the weak interaction current and the helicity-conservation of the quark-gluon interaction in the high-energy limit, although this relation might be broken by the power corrections. In [15], the power suppressed contribution is separated into the symmetry-preserving part ξ and symmetry-breaking part $\Delta\xi$, i.e.,

$$
F_V^{\mathrm{NLP}}(E_\gamma) = \xi(E_\gamma) + \Delta\xi(E_\gamma), \quad F_A^{\mathrm{NLP}}(E_\gamma) = \xi(E_\gamma) - \Delta\xi(E_\gamma).
\tag{42}
$$

When the power suppressed contribution is from the region where $x^2 \sim 1/\Lambda^2$, in which x denotes the separation between the quark-photon vertex and the weak current, it is called the soft contribution. Soft contributions with higher twist B-meson LCDA are considered in [15]. Because they are highly suppressed and numerically small, we neglect them in this study. Then, the symmetry breaking part only contains the local contribution and can be written as

$$
\Delta\xi(E_\gamma) = \frac{e_u f_B m_B}{4E_\gamma^2} + \frac{e_b f_B m_B}{2E_\gamma m_b}.
\tag{43}
$$

The symmetry preserving part can be divided into three parts, i.e., $\xi(E_\gamma) = \xi_{\frac{\perp}{E_\gamma}}(E_\gamma) + \xi_{\frac{\perp}{m_b}}(E_\gamma) + \xi_{\mathrm{soft}}(E_\gamma)$, and the explicit expressions for the first two parts are

$$\xi_{\frac{\perp}{E_\gamma}}(E_\gamma) = \frac{e_u f_B m_B}{4E_\gamma^2} \left[-1 + 2 \int_0^\infty d\omega \ln \omega \phi_{t3}^-(\omega) \right.$$
$$\left. -2 \int_0^\infty \frac{d\omega_2}{\omega_2} \phi_4(0,\omega_2) \right],$$

$$\xi_{\frac{\perp}{m_b}}(E_\gamma) = \frac{e_u f_B m_B}{4E_\gamma m_b} \left[\frac{\bar\Lambda}{\lambda_B} - 2 \right.$$
$$\left. +2 \int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_1+\omega_2} \phi_3(\omega_1,\omega_2) \right], \quad (44)$$

where $\phi_3(\omega_1,\omega_2)$ and $\phi_4(\omega_1,\omega_2)$ are the three particle twist-3 and twist-4 B-meson LCDAs, respectively. $\phi_{t3}^-(\omega)$ is the "genuine" twist-three contribution to the LCDA $\phi^-(\omega)$ [15]. The soft contribution with QCD corrections is obtained as

$$\xi_{\text{soft}}(E_\gamma) = \frac{e_u f_B m_B}{2E_\gamma} C_\perp(E_\gamma,\mu_h) K^{-1}(\mu_h') U(E_\gamma,\mu_h,\mu_h',\mu)$$
$$\times \int_0^{\frac{s_0}{2E_\gamma}} d\omega' \left[\frac{2E_\gamma}{m_\rho^2} e^{-\frac{2E_\gamma\omega'-m_\rho^2}{M^2}} - \frac{1}{\omega'} \right] \rho_{\text{eff}}^+(\omega',\mu), \quad (45)$$

where $K(\mu)$ is the factor relating the QCD decay constant of the B-meson to the HQET decay constant, and M^2 and s_0 are the Borel mass and threshold parameter, respectively. The evolution kernel $U(E_\gamma,\mu_h,\mu_h',\mu) = U_1(E_\gamma,\mu_h,\mu)/U_2(E_\gamma,\mu_h',\mu)$. The effective LCDA $\rho_{\text{eff}}^+(\omega',\mu)$ takes the form

$$\rho_{\text{eff}}^+(\omega',\mu) = \phi^+(\omega',\mu) + \frac{\alpha_s C_F}{4\pi} \left\{ \left(\ln^2 \frac{\mu^2}{2E_\gamma\omega'} + \frac{\pi^2}{6} - 1 \right) \phi^+(\omega',\mu) \right.$$
$$+ \left(2\ln \frac{\mu^2}{2E_\gamma\omega'} + 3 \right) \omega' \int_{\omega'}^\infty d\omega \ln \frac{\omega-\omega'}{\omega'} \frac{d}{d\omega} \frac{\phi^+(\omega,\mu)}{\omega}$$
$$- 2\ln \frac{\mu^2}{2E_\gamma\omega'} \int_0^{\omega'} d\omega \ln \frac{\omega'-\omega}{\omega'} \frac{d}{d\omega} \phi^+(\omega,\mu)$$
$$\left. + \int_0^{\omega'} d\omega \ln^2 \frac{\omega'-\omega}{\omega'} \frac{d}{d\omega} \left[\frac{\omega'}{\omega} \phi^+(\omega,\mu) + \phi^+(\omega,\mu) \right] \right\}.$$
$$(46)$$

To obtain this function, one must generalize the photon momentum from $p^2 = 0$ to $-p^2 \neq 0$, calculate the generalized hard-collinear function in this Euclidean region, perform the dispersion treatment to the convolution of the hard-collinear function with the B-meson LCDA, and finally take the limit $p^2 \to 0$. The soft contribution actually includes the hadronic effect of the photon; it must overlap with the contribution of the photon LCDA, which will be investigated in future work.

4   Phenomenological analysis

The fundamental nonperturbative inputs entering the factorization formula of $B \to \gamma \nu\ell$ decay include the two-particle and three-particle B-meson distribution amplitudes up to the twist-four accuracy. The decay constant

of the B-meson and the parameters appear in the dispersion approach. In the numerical analysis, we employ the following three-parameter model for the leading twist B-meson LCDA [15]

$$\phi^+(\omega) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\omega}{\omega_0^2} e^{-\frac{\omega}{\omega_0}} U\left(\beta-\alpha, 3-\alpha, \frac{\omega}{\omega_0} \right), \quad (47)$$

where $U(\alpha,\gamma,x)$ is the confluent hypergeometric function of the second kind. In dual space, this model has a simpler expression:

$$\rho^+(\omega') = \frac{1}{\omega'} {}_1F_1\left(\alpha, \beta, -\frac{\omega_0}{\omega'} \right), \quad (48)$$

where ${}_1F_1(\alpha,\beta,z)$ is the confluent hypergeometric function of the first kind. In the leading power factorization formula, only the first inverse moment and the logarithmic moments enter the factorization formula; they are defined by

$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega} \phi^+(\omega),$$
$$\sigma_n(\mu) = \lambda_B(\mu) \int_0^\infty \frac{d\omega}{\omega} \ln^n \frac{\mu_0}{\omega} \phi^+(\omega). \quad (49)$$

For the three-parameter model, the first inverse moment and the first two logarithmic moments are obtained as

$$\lambda_B = \frac{\alpha-1}{\beta-1} \omega_0,$$
$$\sigma_1 = \psi(\beta-1) - \psi(\alpha-1) + \ln \frac{\mu_0}{\omega_0} + \gamma_E,$$
$$\sigma_2 = \sigma_1^2 + \psi'(\beta-1) - \psi'(\alpha-1) + \frac{\pi^2}{6}. \quad (50)$$

If the parameter $\alpha = \beta$, the three-parameter model is simply the familiar exponential model [19]

$$\phi^+(\omega) = \frac{\omega}{\omega_0^2} e^{-\frac{\omega}{\omega_0}}, \quad (51)$$

which is set as our default model. To estimate the error from the models, we let $\alpha - \beta$ vary in the region $-0.5 < \alpha - \beta < 0.5$. We then employ two models with $\alpha = 2.0, \beta = 1.5$ and $\alpha = 1.5, \beta = 2.0$. For the default model, $\omega_0 = \lambda_B$, whose determination has been discussed extensively in the context of exclusive B-meson decays (see [28-31] for further discussion). Here, we employ $\lambda_B(1\,\text{GeV}) = 0.35 \pm 0.05\,\text{GeV}$, which is consistent with the calculations of the semileptonic $B \to \pi$ form factors with B-meson LCDAs in the framework of light-cone sum rules [28]. The leading twist B-meson LCDA with the three-parameter model is plotted in Fig. 1. In the factorization formula (31), a new parameter $\bar\omega$ is introduced to eliminate σ_1: it can be determined once the parameters α, β, and λ_B are given. In addition, by utilizing the three-parameter B-meson LCDA model, the logarithmic moments $\sigma_{2,3,4}$ are also determined. All the parameters are listed in Table 1.

The higher-twist LCDAs must incorporate the correct

Table 1.   Numerical values of the nonperturbative parameters entering the leading twist LCDA of the B meson. Here, the energy scale is $\mu_s = 1$ GeV , and λ_B is fixed at 0.35 GeV.

Model	α	β	ω_0/GeV	σ_2	σ_3	σ_4	λ_B/GeV
Default	arbitrary	$\beta = \alpha$	0.350	1.64	2.4	14.6	0.35
Model 1	2.0	1.5	0.175	−1.64	−12.2	−76.3	0.35
Model 2	1.5	2.0	0.700	4.93	16.8	170.0	0.35
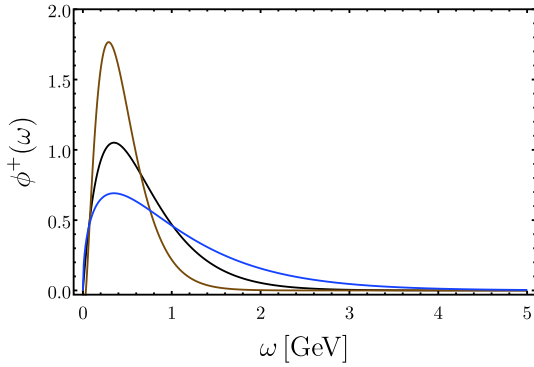
Fig. 1.   (color online) The leading-twist B-meson LCDA with the three-parameter model. Setting $\lambda_B = 0.35$ GeV, the black curve represents the exponential model, i.e., $\beta = \alpha$. The brown and blue curves represent $\alpha = 2.0, \beta = 1.5$ and $\alpha = 1.5, \beta = 2.0$, respectively.

low-momentum behaviour and satisfy the equations of motion. All the suggested models can be obtained as particular cases of a more general ansatz:

$$\phi^+(\omega) = \omega f(\omega),$$
$$\phi_3(\omega_1, \omega_2) = -\frac{1}{2} \mathcal{N} (\lambda_E^2 - \lambda_H^2) \omega_1 \omega_2^2 f'(\omega_1 + \omega_2),$$
$$\phi_4(\omega_1, \omega_2) = \frac{1}{2} \mathcal{N} (\lambda_E^2 + \lambda_H^2) \omega_2^2 f(\omega_1 + \omega_2). \quad (52)$$

Here, the function $f(\omega)$ obeys the following normalization condition:

$$\int_0^\infty \omega f(\omega) \, d\omega = 1,$$
$$\frac{1}{\mathcal{N}} = \frac{1}{2} \int_0^\infty \omega^3 f(\omega) \, d\omega = \bar{\Lambda}^2 + \frac{1}{6}(2\lambda_E^2 - \lambda_H^2). \quad (53)$$

The following results can then be derived:

$$\int_0^\infty \frac{d\omega}{\omega} \ln \omega \, \phi_-^{t3}(\omega) = \frac{1}{6} \mathcal{N} (\lambda_E^2 - \lambda_H^2),$$
$$\int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_1 + \omega_2} \phi_3(\omega_1, \omega_2) = \frac{1}{3} \mathcal{N} (\lambda_E^2 - \lambda_H^2),$$
$$\int_0^\infty \frac{d\omega_2}{\omega_2} \phi_4(0, \omega_2) = \frac{1}{2} \mathcal{N} (\lambda_E^2 + \lambda_H^2). \quad (54)$$

Taking advantage of the above results, the NLP contribution with $1/E_\gamma$ and $1/m_b$ corrections can be obtained as

$$\xi_{\frac{\perp}{E_\gamma}}(E_\gamma) = -\frac{e_u f_B m_B}{2E_\gamma^2} \left[\frac{1}{2} + \frac{2(\lambda_E^2 + 2\lambda_H^2)}{6\bar{\Lambda}^2 + 2\lambda_E^2 + \lambda_H^2} \right],$$
$$\xi_{\frac{\perp}{m_b}}(E_\gamma) = \frac{e_u f_B m_B}{4E_\gamma m_b} \left[\frac{\bar{\Lambda}}{\lambda_B} - 2 + \frac{4(\lambda_E^2 - \lambda_H^2)}{6\bar{\Lambda}^2 + 2\lambda_E^2 + \lambda_H^2} \right]. \quad (55)$$

To highlight the influence of the power suppressed contributions, we present the numerical results of the form factors with different contributions:

$$F_A = \frac{1}{\lambda_B} \left(0.102 + 0.0067\sigma_2 + 5.4 \times 10^{-5}\sigma_3 \right.$$
$$\left. + 7.6 \times 10^{-5}\sigma_4 \right)\Big|_{\text{LP}} + \left(-0.135 + \frac{0.00768}{\lambda_B} \right)\Big|_{\text{NLP}},$$

$$F_V = \frac{1}{\lambda_B} \left(0.102 + 0.0067\sigma_2 + 5.4 \times 10^{-5}\sigma_3 \right.$$
$$\left. + 7.6 \times 10^{-5}\sigma_4 \right)\Big|_{\text{LP}} + \left(-0.101 + \frac{0.00768}{\lambda_B} \right)\Big|_{\text{NLP}}, \quad (56)$$

where the photon energy E_γ is fixed at 2.2 GeV, and λ_B and σ_n are set to be free parameters. The leading power result in the first parentheses is borrowed from [13], and the power suppressed contributions include the symmetry breaking local term, the symmetry preserving $1/E_\gamma, 1/m_b$ term, and the soft contribution. It is obvious that power corrections are sizeable and more important than the σ_n terms. Therefore, the power suppressed contribution must play an important role in the determination of λ_B. We leave a more detailed study of the subleading power corrections for a future work.

To test the effect of the large logarithm resummation, we plot the E_γ dependence of the leading power form factor $F_{A/V,\text{LP}}$ in Fig. 2. For the leading logarithmic resummation, we employ the tree-level hard function and jet function as well as the one-loop level anomalous dimension and two-loop cusp anomalous dimension. For the NLL resummation case, we follow the convention in [13]. In [13], the contributions from the NNLO hard function and jet function are also considered. Strictly speaking, to resum the logarithmic terms in the NNLO jet function, we need a three-loop anomalous dimension of the B-meson LCDA, which has not yet been obtained. While it is not phenomenologically as important as the hard-collinear scale, μ_j is actually close to the soft scale μ_s. From Fig. 2, we can see that the NLL resummation effect significantly decreases the LL result, and that the NNLO result is approximately 5% smaller than the NLL result.
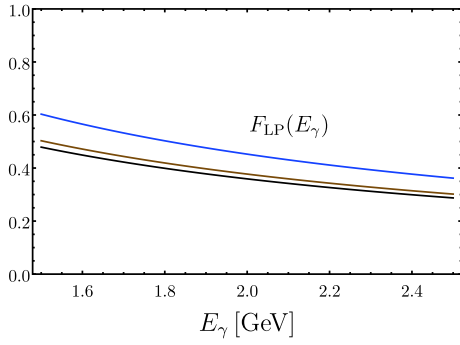
Fig. 2.　(color online) The leading power contribution to the form factors: the blue, brown, and black curves stand for the LL, NLL, and NNLO results, respectively.

The form factors including both the LP contribution and NLP corrections are plotted in Fig. 3, where for LP contribution, we adopt the same result as in [13]. Compared with the LP result in Fig. 2, the power corrections significantly decrease the form factors, and the symmetry breaking effect from the NLP local contribution is also sizable. The uncertainties are denoted by the band in Fig. 3. To obtain the uncertainties, we consider various sources, including the following: decay constant $f_B = 0.192 \pm 0.0043$ GeV; the first inverse moment $\lambda_B = 0.35 \pm 0.05$ GeV; the hard scale μ_h and hard-collinear scale μ_j (the same as in [13]); the models of the leading twist B-meson LCDA in Table 1; the parameters $\bar{\Lambda} = 0.48 \pm 0.10$ GeV, $\quad \lambda_E^2 \in [0.027 \text{ GeV}^2, 0.088 \text{ GeV}^2]$, and
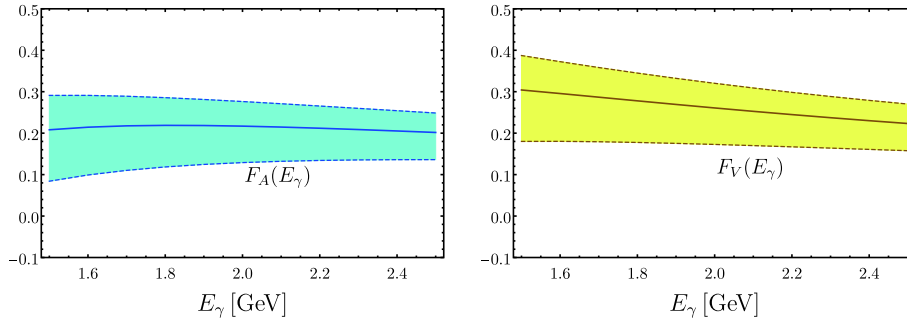
$\lambda_H^2 \in [0.045 \text{ GeV}^2, 0.222 \text{ GeV}^2]$; the Borel mass $M^2 = 1.25 \pm 0.25 \text{ GeV}^2$; and the threshold mass $s_0 = 1.5 \pm 0.1 \text{ GeV}^2$ in the soft contribution. The λ_B parameter gives rise to the most important uncertainty, as the LP result is inversely proportional to it. Having the form factors at hand, the differential decay width is expressed as

$$\frac{d\Gamma}{dE_\gamma} = \frac{\alpha_{em}^2 G_F^2 |V_{ub}|^2}{6\pi^2} m_B E_\gamma^3 \left(1 - \frac{2E_\gamma}{m_B}\right)\left(|F_V|^2 + \left|F_A + \frac{Q_\ell f_B}{E_\gamma}\right|^2\right).$$
(57)

To guarantee the reliability of our calculation, we cut the photon energy at $E_\gamma > 1.5$ GeV. We integrate over the differential decay width in the interval $[1.5 \text{ GeV}, m_B/2]$ and then multiply it by the lifetime of the B meson to obtain the branching ratio $\text{Br}(E_\gamma > 1.5 \text{ GeV})$. If we fix $\lambda_B = 0.35$ GeV, the branching ratio reads

$$\text{Br}(B \to \gamma \nu \ell) = 0.40^{+0.14}_{-0.24} \times 10^{-6},$$
(58)

where the uncertainties come from the same source as that for the form factors (except for λ_B). The dependence of the branching ratio on λ_B is presented in Fig. 4, where the parameter λ_B varies in the interval $[0.3 \text{ GeV}, 0.4 \text{ GeV}]$. We can see that the large uncertainty prevents us from precisely determining the parameter λ_B. It is thus important to reduce the uncertainty in the parameter estimates, especially the uncertainty in the B meson LCDA, and to obtain a more precise prediction of the power suppressed contribution.

Fig. 3.　(color online) The form factors $F_{V,A}$ with NLP contributions. The uncertainties arise from varying parameters, including λ_B.
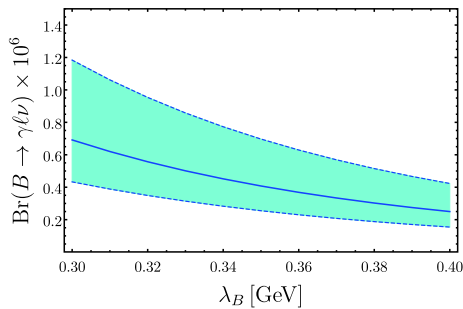

Fig. 4.　(color online) The dependence of $\text{Br}(B \to \gamma \nu \ell)$ on λ_B for the photon energy $E_\gamma > 1.5$ GeV.

5　Summary

The radiative leptonic decay mode $B \to \gamma \nu \ell$ is interesting both theoretically and experimentally. It plays an irreplaceable role in the determination of parameters of the B-meson LCDA. The factorization-scale dependence of the B-meson LCDA is governed by the LN evolution equation, which is an integro-differential equation and is not easily solved. We summarized the existing method of solving the LN evolution equation, for both the one-loop and two-loop anomalous dimensions. We then derived a

factorization formula with NLL resummation for the form factors appearing at leading power in the Mellin space, which is equivalent to the one obtained in [13] but written in a more compact form. The power corrections to the $B \to \gamma \nu \ell$ are sizeable, and much effort has been put into investigating the NLP contributions. In this paper, we included the power suppressed local contributions, factorizable non-local contributions (which are suppressed by $1/E_\gamma$ and $1/m_b$), and soft contributions.

In the numerical analysis, we found that the NLL-resummation effect significantly decreases the leading-power form factors, and that the NNLO correction brings approximately 5% additional reduction. The NLP contributions are combined with the leading-power NNLO contributions and also manifestly decrease the form factors. We also calculated the integrated branching fractions of the $B \to \gamma \nu \ell$ decay. The large uncertainty from various sources makes it difficult to determine the parameter λ_B and other logarithmic moments accurately. In future work, we will consider the NLP corrections more systematically in the framework of the SCET and in the hope of reducing the theoretical uncertainty.

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