Weak cosmic censorship conjecture in BTZ black holes with scalar fields*

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Abstract: The weak cosmic censorship conjecture in the near-extremal BTZ black hole has been tested using test particles and fields. It has been claimed that such a black hole can be overspun. In this paper, we review the thermodynamics and weak cosmic censorship conjecture in BTZ black holes using the scattering of a scalar field. The first law of thermodynamics in the non-extremal BTZ black hole is recovered. For the extremal and near-extremal black holes, due to the divergence of the variation of entropy, we test the weak cosmic censorship conjecture by evaluating the minimum of the function *f*, and find that both the extremal and near-extremal black holes cannot be overspun.

Keywords: weak cosmic censorship conjecture, scalar field, thermodynamics

DOI: 10.1088/1674-1137/44/1/015101

1 Introduction

It is widely believed that spacetime singularities are formed at the end of a gravitational collapse. At the singularities, all physical laws break down. To avoid the destruction caused by the singularities, Penrose proposed the weak cosmic censorship conjecture (WCCC) in 1969 [1]. This conjecture states that naked singularities cannot be produced in a physical process from regular initial conditions. For black holes, their singularities should be hidden behind event horizons without any access to distant observers. Since the time the conjecture was put forward, considerable amount of research has been done to demonstrate its validity. However, no concrete evidence has been found to prove it and no unanimous conclusion has been reached.

The Gedanken experiment proposed by Wald is the first attempt to test the validity of WCCC [2]. In this experiment, a test particle with large enough charge and angular momentum is thrown into a Kerr-Newman black hole to test whether the black hole exceeds its extremal limit. The Kerr-Newman solution describes a charged and rotating spacetime. Its charge and angular momentum per unit mass are bounded by the mass as $a^2+Q^2 \le M^2$. When $M^2 < a^2+Q^2$, the black hole exceeds its extremal limit and the event horizon disappears. Thus the singularity is naked and WCCC is violated. Following this seminal work, other researches investigated the validity of WCCC from the aspect of particles and fields. Jacobson and Sotiriou studied the absorption of an object with spin and orbital angular momentum in a near-extremal Kerr black hole. They found that without consideration of the radiative and self-force effects, the black hole can be overspun [3]. The overcharge of the near-extremal Reissner-Nordstrom black hole was first found in [4], and then investigated taking into consideration the tunnelling effects in [5]. However, when the radiative, backreaction and self-force effects are taken into account, particles may escape from the black holes and naked singularities can be avoided [6-12]. This result was confirmed in a recent work where the self-force and finite size effects were considered [13, 14]. A counter example to WCCC in four-dimensional anti de Sitter (AdS) spacetime was presented in [15]. In this spacetime, constant time slices have planar topology. It was shown that it is just a pure AdS in the past. In the future, the curvature grows without bound and leaves regions of spacetime with arbitrarily large curvatures. These regions are naked to boundary observers. This work is important for the weak gravity conjecture [16].

The validity of WCCC was examined from the aspect of fields in Refs. [17-20]. In this research, the field has a finite energy, which indicates the existence of a wave packet. Initially, the field does not exist and there is only a black hole. The field comes in from infinity and interacts with the black hole. Due to the interaction, the energy, charge and angular momentum are transferred between the field and the black hole. Part of the field is reflected back to infinity. Finally, the field decays leaving behind another spacetime with the new energy,

Received 18 August 2019, Published online 6 November 2019

* Supported by the National Natural Science Foundation of China (11205125, 11875095)

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charge and angular momentum [18]. Whether the black hole is overspun or overcharged can be judged from the change of the event horizon. Based on this point of view, the interaction between a dyonic Kerr-Newman black hole and a complex massive scalar field was discussed in Ref. [19]. It was found that this interaction does not destroy the WCCC. The same result was derived by Toth [18]. His derivation is based on the null energy condition, the conservation laws and the electromagnetic field of the black hole. In a recent work, Gwak calculated the variations of the energy and angular momentum of the Kerr-(Anti-)de Sitter black hole during an infinitesimal time interval using the fluxes of the energy and angular momentum of a scalar field [20]. He found that the black hole keeps the initial states and is not overspun. This result is different from that obtained for the near-extremal black hole by throwing a test particle into it. When the initial data are non-generic, naked singularities can develop [21-27]. Other tests of WCCC are given in Refs. [28-54] and the references therein.

The Bañados, Teitelboim and Zanelli (BTZ) black hole is a solution of the Einstein field equation in the three-dimensional spacetime and describes a rotating AdS geometry [55, 56]. The properties of BTZ black holes were discussed in Refs. [49, 57-59]. In [57], Rocha and Cardoso tested WCCC in the extremal BTZ black hole by using a test particle. They found that this black hole cannot be overspun, and the initial black hole was designated as extremal. However, Düztas claimed that a near-extremal black hole has the possibility to exceed its extremal limit whether or not superradiation occurs [49]. If superradiation does not occur, overspinning becomes generic and also applies to the extremal BTZ black hole. If there is superradiation, the black hole is overspun in a certain range of frequencies. He elaborated these results using respectively massive test particles and test fields. Due to the similarities of rotating black holes, the same result was obtained for the Kerr black hole [17]. Obviously, this result is different from that derived by Wald and Sorce [13, 14].

In this paper, we review the thermodynamics and WCCC in BTZ black holes using the scattering of a scalar field. The BTZ black holes are generic in the sense that they constitute an open set in the space of solutions of the Einstein equation. The change of the energy and angular momentum of the black hole during a time interval relies on the fluxes of energy and angular momentum of the ingoing wave function. To get the wave function, we introduce a tortoise coordinate and solve the second-order differential equation at the event horizon. Here, the time interval is infinitesimal. For the non-extremal BTZ black hole, the increase of the event horizon ensures that WCCC is valid. The first law of thermodynamics is recovered by the scattering. For the near-extremal and extremal BTZ black holes, Eq. (21) is divergent at the event horizons. Therefore, we need to resort to other methods to test the validity of WCCC. Its validity is tested by evaluating the minimum value of the function f. It is found that the horizons do not disappear and the singularities are always hidden behind them.

The paper is organized as follows. The BTZ black hole solution is given and its thermodynamics are discussed in the next section. In Section 3, the first law of thermodynamics for the non-extremal BTZ black hole is recovered by the scattering of a scalar field. In Section 4, the validity of WCCC in the extremal and near-extremal BTZ black holes is tested using the minimum of the function f. Section 5 is devoted to our discussion and conclusion.

2 BTZ black holes

The BTZ metric is given by [55, 56]

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + \frac{1}{f}\mathrm{d}r^2 + r^2\left(\mathrm{d}\varphi - \frac{J}{2r^2}\mathrm{d}t\right)^2, \tag{1}$$

where

$$f = f(M,J,r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}. \tag{2}$$

It describes a local three-dimensional rotating AdS spacetime. The parameter l^2 is related to the cosmological constant Λ as $l^2 = -\frac{1}{\Lambda}$. M and J are the ADM mass and angular momentum. They determine the asymptotic behavior of the spacetime. The event (inner) horizons are located at $r_+(r_-)$ and satisfy the relations

$$Ml^2 = r_+^2 + r_-^2, \qquad J^2 l^2 = 4r_+^2 r_-^2. \tag{3}$$

When $r_+ = r_-$, the two horizons are coincident and the black hole becomes extremal.

The entropy, Hawking temperature, angular velocity and ADM mass are respectively

$$\begin{aligned} S &= 4\pi r_+, \qquad T = \frac{r_+}{2\pi l^2} - \frac{J^2}{8\pi r_+^3}, \\ \Omega &= \frac{J}{2r_+^2}, \qquad M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2}. \end{aligned} \tag{4}$$

Here, the expression for the entropy used in [55] is adopted, which shows that the entropy is equal to twice the perimeter length of the horizon. When the mass of a BTZ black hole varies, other thermodynamic quantities of the black hole, such as entropy, temperature and angular velocity, also vary. These thermodynamic quantities obey the first law of thermodynamics

$$\mathrm{d}M = T\mathrm{d}S + \Omega\mathrm{d}J. \tag{5}$$

We show in the next section that the first law of thermodynamics is recovered by the scattering of a scalar field.

The variations are caused by the interaction between the scalar field and the black hole. Due to this interaction, the energy and angular momentum are transferred, and are evaluated by the energy flux and angular momentum flux. Therefore, we first write the action and calculate the energy-momentum tensor.

3 Thermodynamics of non-extremal BTZ black holes

The action of the minimally coupled complex scalar field in the BTZ spacetime is

$$I = \int \mathrm{d}t\mathrm{d}r\mathrm{d}\varphi \sqrt{-g}\mathcal{L} = -\frac{1}{2}\int \mathrm{d}t\mathrm{d}r\mathrm{d}\varphi \sqrt{-g}[\partial_\mu\Phi\partial^\mu\Phi^\star + \mu_0^2\Phi\Phi^\star], \quad (6)$$

where $\mathcal{L}$ is the Lagrangian density and μ_0 is the mass [60]. The energy-momentum tensor is obtained from the action, namely, $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I}{\delta g^{\mu\nu}}$, and we get

$$T^\mu_\nu = \frac{1}{2}\partial^\mu\Phi\partial_\nu\Phi^\star + \frac{1}{2}\partial^\mu\Phi^\star\partial_\nu\Phi + \delta^\mu_\nu\mathcal{L}. \quad (7)$$

To evaluate the energy-momentum tensor, we need to know the wave equation Φ which obeys the equation of motion for the scalar field. This equation is obtained from the action (6) and takes the form

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right) - \mu_0^2\Phi = 0. \quad (8)$$

Due to the existence of the Killing vectors $(\frac{\partial}{\partial t})^a$ and $(\frac{\partial}{\partial\varphi})^a$ in the BTZ background spacetime, we carry out the separation of variables

$$\Phi = \mathrm{e}^{-\mathrm{i}(\omega t - j\varphi)}R(r), \quad (9)$$

where ω and j denote the energy and angular momentum. Inserting the contravariant components of the BTZ metric and the separation of variables (9) into the Klein-Gordon equation (8), yields a second-order differential equation. To solve this equation, we introduce the tortoise coordinate [61]

$$r_\star = r + \frac{1}{2\kappa}\ln\frac{r - r_+}{r_+}, \quad (10)$$

where $\kappa = 2\pi T$ is the surface gravity at the event horizon and T is the Hawking temperature. The second-order differential equation then becomes

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r_\star^2} + \frac{f^2 + f(2\kappa r + 1)}{(f+1)^2 r}\frac{\mathrm{d}R(r)}{\mathrm{d}r_\star} + \frac{4\omega^2 r^4 - 4\omega J j r^2 + 4j^2 r^2 f - J^2 j^2 - 4f\mu_0^2 r^4}{4(f+1)^2 r^4}R(r) = 0. \quad (11)$$

Since we are interested in the scattering near the event horizon, the equation needs to be solved at the horizon. Let $r \to r_+$. Eq. (11) then reduces to

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r_\star^2} + (\omega - j\Omega)R(r) = 0. \quad (12)$$

In the above equation, $\Omega = \frac{J}{2r_+^2}$ is the angular velocity at the event horizon. From Eq. (12), we get the radial solutions

$$R(r) \sim \mathrm{e}^{\pm\mathrm{i}(\omega - j\Omega)r_\star}, \quad (13)$$

where the solutions with +(−) denote the outgoing (ingoing) radial waves. Therefore, the standard wave equations are

$$\Phi = \mathrm{e}^{-\mathrm{i}(\omega t - j\varphi)\pm\mathrm{i}(\omega - j\Omega)r_\star}. \quad (14)$$

The interaction between the field and the black hole transfers the energy and angular momentum. Since we are discussing the changes in the horizon after the black hole absorbs the energy and angular momentum, we focus our attention on the ingoing wave equation.

The two Killing vectors $(\frac{\partial}{\partial t})^a$ and $(\frac{\partial}{\partial\varphi})^a$ correspond to the two local conservation laws in the BTZ spacetime. The corresponding conservative quantities are the energy E and the angular momentum L. When the fluxes of energy and angular momentum flow into the event horizon and are absorbed by the black hole, the energy and angular momentum of the black hole change. The energy flux and the angular momentum flux are respectively

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int T^r_t\sqrt{-g}\mathrm{d}\varphi, \quad \frac{\mathrm{d}L}{\mathrm{d}t} = -\int T^r_\varphi\sqrt{-g}\mathrm{d}\varphi. \quad (15)$$

Combining the fluxes with the energy-momentum tensor and the ingoing wave equation yields

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2\pi r_+\omega(\omega - j\Omega), \quad \frac{\mathrm{d}L}{\mathrm{d}t} = 2\pi r_+ j(\omega - j\Omega). \quad (16)$$

In this derivation, $\frac{\mathrm{d}r_\star}{\mathrm{d}r} = \frac{f+1}{f}$ obtained from Eq. (10), was used. The energy of the black hole is its ADM mass, and the angular momentum is expressed as J. Therefore, the increase of the energy and angular momentum during a time interval $\mathrm{d}t$ is

$$\mathrm{d}M = 2\pi r_+\omega(\omega - j\Omega)\mathrm{d}t, \quad \mathrm{d}J = 2\pi r_+ j(\omega - j\Omega)\mathrm{d}t, \quad (17)$$

which may be negative or positive, depending on the sign of $\omega - j\Omega$. When $\omega > j\Omega$, the energy and angular momentum of the black hole increase and the fluxes flow into the event horizon. There is no change of the energy and angular momentum for $\omega = j\Omega$. $\omega < j\Omega$ implies a decrease of the energy and angular momentum. The energy and angular momentum are extracted by the scattering, and superradiation occurs. In fact, the appearance of superradiation should satisfy the boundary condition that the scalar field is in the asymptotic region. Here, we follow the work of Gwak and focus on the infinitesimal change in the BTZ spacetime [20]. Therefore, our discussion does not rely on the asymptotic boundary conditions.

In the following discussion, the time interval is assumed to be infinitesimal, and the variations of the energy and angular momentum are also infinitesimal. The scattering changes the function f and the horizon radius r_+. The variations are labeled as δf and $\mathrm{d}r_+$, and satisfy

$$\begin{aligned}\delta f =& f(M+\mathrm{d}M, J+\mathrm{d}J, r_+ +\mathrm{d}r_+) - f(M,J,r_+)\\ =& \left.\frac{\partial f(M,J,r)}{\partial M}\right|_{r=r_+}\mathrm{d}M + \left.\frac{\partial f(M,J,r)}{\partial J}\right|_{r=r_+}\mathrm{d}J\\ &+\left.\frac{\partial f(M,J,r)}{\partial r}\right|_{r=r_+}\mathrm{d}r_+,\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\left.\frac{\partial f(M,J,r)}{\partial M}\right|_{r=r_+} = -1, \quad \left.\frac{\partial f(M,J,r)}{\partial J}\right|_{r=r_+} = \frac{J}{2r_+^2},\\ &\left.\frac{\partial f(M,J,r)}{\partial r}\right|_{r=r_+} = 4\pi T.\end{aligned} \tag{19}$$

To derive $\mathrm{d}r_+$, one can assume that the final state is still a black hole after the absorption of the infinitesimal energy and angular momentum [20, 35]. This implies $f(M+\mathrm{d}M, J+\mathrm{d}J, r_+ +\mathrm{d}r_+) = f(M,J,r_+) = 0$. Thus, the variation of the horizon radius is

$$\mathrm{d}r_+ = \frac{r_+(\omega - j\Omega)^2\mathrm{d}t}{2T}. \tag{20}$$

When $\omega \neq j\Omega$, we get $\mathrm{d}r_+ > 0$, and for $\omega = j\Omega$, we have $\mathrm{d}r_+ = 0$. Therefore, the horizon radius does not decrease when the black hole absorbs the ingoing wave. This implies that the singularity is hidden behind the event horizon and cannot be observed by external observers of the black hole. Using the relation between the entropy and the horizon radius, we get

$$\mathrm{d}S = \frac{2\pi r_+(\omega - j\Omega)^2\mathrm{d}t}{T}, \tag{21}$$

which shows that the entropy does not decrease with the scattering of the field. This result supports the second law of thermodynamics, and is a simple consequence of the fact that the system satisfies the null energy condition. From Eqs. (17) and (21), we get

$$\mathrm{d}M = T\mathrm{d}S + \Omega\mathrm{d}J. \tag{22}$$

Therefore, the first law of thermodynamics in the non-extremal BTZ black hole is recovered by the scattering of the scalar field.

In the thought experiments, it is usually preferred to study systems which are inferred to be close to the critical condition. In the next section, we investigate this case, namely the near-extremal and extremal BTZ black holes. For these black holes, Eq. (21) is divergent at the event horizons. Thus, the above method cannot be applied to the extremal and near-extremal BTZ black holes, and we need to resort to other methods to test WCCC.

4 WCCC in near-extremal and extremal BTZ black holes

WCCC in the near-extremal and extremal BTZ black holes has been tested, and it was found that the near-extremal BTZ black hole has the possibility to be overspun [49]. However, the extremal BTZ black hole cannot be overspun [57]. In this section, we review the validity of WCCC in the near-extremal and extremal BTZ black holes using the minimum of the function f in the final state. Due to the interaction between the black hole and the field, the energy and angular momentum of the black hole change, and the value of the function f changes. In the metric, there are two roots (corresponding to the inner and event horizons) for $f<0$, and one root (corresponding to the event horizon) for $f=0$. For $f>0$, the event horizon disappears and the singularity is naked.

The time interval is assumed to be infinitesimal, and the transferred energy and angular momentum in the scattering are also infinitesimal. The minimum value of f is expressed as $f_0 = f(M,J,r_0) = -M+\frac{r_0^2}{l^2}+\frac{J^2}{4r_0^2}$, where r_0 is the location corresponding to f_0. r_0 is not an independent variable, as it depends on M and J. Thus we get

$$\begin{aligned}&f(M+\mathrm{d}M, J+\mathrm{d}J, r_0+\mathrm{d}r_0)\\ =& f_0 + \left.\frac{\partial f(M,J,r)}{\partial M}\right|_{r=r_0}\mathrm{d}M + \left.\frac{\partial f(M,J,r)}{\partial J}\right|_{r=r_0}\mathrm{d}J\\ &+\left.\frac{\partial f(M,J,r)}{\partial r}\right|_{r=r_0}\mathrm{d}r_0\\ =& -\left(\frac{\omega}{j}\right)^2 2\pi j^2 r_+\mathrm{d}t + \left(\frac{\omega}{j}\right)2\pi j^2 r_+(\Omega+\Omega_0)\mathrm{d}t\\ &+ f_0 - 2\pi j^2 r_+\Omega\Omega_0\mathrm{d}t,\end{aligned} \tag{23}$$

where

$$\begin{aligned}&\left.\frac{\partial f(M,J,r)}{\partial M}\right|_{r=r_0} = -1, \quad \left.\frac{\partial f(M,J,r)}{\partial J}\right|_{r=r_0} = \frac{J}{2r_0^2},\\ &\left.\frac{\partial f(M,J,r)}{\partial r}\right|_{r=r_0} = 0,\end{aligned} \tag{24}$$

and $\Omega_0 = \frac{J}{2r_0^2}$ is the angular velocity at the location r_0. The formulae in Eq. (17) were used to derive Eq. (23). The above equation is a quadratic equation in $\frac{\omega}{j}$, and its maximum can be adjusted using $\frac{\omega}{j}$. If its maximal value is greater than zero, there is no horizon. Otherwise, the event horizons exist.

For the extremal BTZ black hole, the event and inner horizons are coincident and the temperature is zero. Thus, the term $T\mathrm{d}S$ in Eq. (5) disappears and $\mathrm{d}M = \Omega\mathrm{d}J$. Using Eq. (17), we easily get $\omega = j\Omega$. The location of the event horizon coincides with the minimum of the function f, namely $r_0 = r_+$. Thus, Eq. (23) is written as

$$f(M+\mathrm{d}M, J+\mathrm{d}J, r_0+\mathrm{d}r_0) = -2\pi r_+ (\omega - j\Omega)^2 \mathrm{d}t = 0. \tag{25}$$

This result shows that the extremal black hole is also extremal after scattering with a new mass and angular momentum. Therefore, the extremal BTZ black hole cannot be overspun. This is in full accordance with the result obtained by Rocha and Cardoso in [57], where WCCC is tested by throwing a point particle into a black hole.

For the near-extremal BTZ black hole, we have $f_0 < 0$ and $|f_0| \ll 1$. To get the maximum of the function, we use $r_+ = r_0 + \epsilon$, where $0 < \epsilon \ll 1$. Also, we let $\mathrm{d}t$ be an infinitesimal scale, and $\epsilon \sim \mathrm{d}t$. Thus, the function f_0 is simplified to

$$f_0 = -\frac{2r_+}{l^2}\epsilon + \frac{J^2}{2r_+^3}\epsilon < 0. \tag{26}$$

For convenience of discussion, we rewrite Eq. (23) as a function of $\frac{\omega}{j}$ and get

$$f\left(\frac{\omega}{j}\right) = -2\pi j^2 r_+ \epsilon \left(\frac{\omega}{j}\right)^2 + 2\pi j^2 r_+ (\Omega + \Omega_0)\epsilon \left(\frac{\omega}{j}\right) - \frac{2r_+}{l^2}\epsilon + \frac{J^2}{2r_+^3}\epsilon - 2\pi j^2 r_+ \Omega\Omega_0 \epsilon. \tag{27}$$

The maximum, located at $\frac{\omega}{j} = \frac{\Omega+\Omega_0}{2}$, is

$$f\left(\frac{\omega}{j}\right)_{\max} = -\frac{2r_+}{l^2}\epsilon + \frac{J^2}{2r_+^3}\epsilon + O(\epsilon), \tag{28}$$

where $O(\epsilon) = \frac{2\pi J^2 j^2}{r_+^5}\epsilon^3$ can be neglected. Using Eq. (26), we find $f\left(\frac{\omega}{j}\right)_{\max} < 0$, which implies that there are two roots of the function. Therefore, the event and inner horizons do not disappear in the scattering of the scalar field, and the singularity is hidden behind the event horizon. In Ref. [49], Düztas found that the near-extremal BTZ black hole can be overspun when particle absorption and field effects are considered. Clearly, our result is different from that obtained in Ref. [49].

5 Discussion and conclusion

In this paper, we investigated the thermodynamics and WCCC in the BTZ black holes using the scattering of a scalar field. The variations of the energy and angular momentum of the black holes in an infinitesimal time interval were calculated. The first law of thermodynamics in the non-extremal BTZ black hole is recovered by the scattering. The increase of the horizon radius ensures that the singularity is hidden behind the event horizon of the non-extremal black hole. For the near-extremal and extremal BTZ black holes, since Eq. (21) is divergent, we tested WCCC by evaluating the minimum of the function f in the final state. We found that these black holes maintain respectively their near-extremity and extremity. This result is in full accordance with that obtained by Wald and Jorse [13, 14].

In a recent work, Düztas assumed that the horizons were destroyed and obtained a relationship between the frequency and azimuthal wave number of the incident field [49]. When superradiation occurs, the frequency is in the range $\frac{Jn}{Ml^2(1+2\epsilon)} < \omega < \frac{n}{l(1+\epsilon)}$, where n is the azimuthal wave number. When there is no superradiation, the frequency is in the range $0 < \omega < \frac{n}{l(1+\epsilon)}$. For the extremal BTZ black hole, he found that it cannot overspin if superradiation occurs. However, if there is no superradiation from the field, overspinning can appear. In our investigation, we did not assume that the black hole horizons are destroyed, and found that the extremal and near-extremal BTZ black holes cannot be overspun in any frequency range.

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