Thermodynamics and weak cosmic censorship conjecture of BTZ black holes in extended phase space*

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Abstract: As a charged fermion drops into a BTZ black hole, the laws of thermodynamics and the weak cosmic censorship conjecture are investigated in both the normal and extended phase space, where the cosmological parameter and renormalization length are regarded as extensive quantities. In the normal phase space, the first and second law of thermodynamics, and the weak cosmic censorship are found to be valid. In the extended phase space, although the first law and weak cosmic censorship conjecture remain valid, the second law is dependent on the variation of the renormalization energy dK. Moreover, in the extended phase space, the configurations of extremal and near-extremal black holes are not changed, as they are stable, while in the normal phase space, the extremal and near-extremal black holes evolve into non-extremal black holes.

Keywords: weak cosmic censorship conjecture, thermodynamics, BTZ black holes

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1 Introduction

The pioneering work of Hawking [1, 2] showed that a black hole can be regarded as a thermodynamic system. Similarly to typical thermodynamic systems, there are four laws of thermodynamics that govern the behavior of black holes. The event horizons of the black holes play a key role in the thermodynamic systems, as both the temperature and entropy are related to it. In addition, the event horizon will hide the singularity of the spacetime, in the contrary the weak cosmic censorship conjecture proposed by Penrose [3] is violated. The Kretschmann scalar can be used to investigate the weak cosmic censorship conjecture, since it is independent of the choice of coordinates [4–6]. The location where the Kretschmann scalar is infinite denotes the singularity.

The laws of thermodynamics and weak cosmic censorship conjecture can be investigated considering a test particle [7–18] or a test field [19–26]. In Ref. [27], the first law, second law, and the weak cosmic censorship conjecture of a BTZ black hole have been investigated. The first and second laws were valid and the weak cosmic censorship conjecture was held for the extremal black hole. Subsequently, the work in Ref. [27] was extended to a D-dimensional charged AdS black hole in the extended phase space, where the negative cosmological constant and its conjugate were regarded as the pressure and volume, respectively [28]. Interestingly, the study presented in Ref. [28] indicated that the second law is violated for the extremal and near-extremal black holes as the contributions of the pressure and volume are considered. Moreover, extremal black holes are found to be stable for the absorbed particles, as they will not change the configurations of the black holes. Recently, Refs. [29] and [30] investigated thermodynamics and weak cosmic censorship conjecture in the Born-Infeld AdS black holes and phantom Reissner-Nordström AdS black holes. Differently from the result in Ref. [28], they found that extremal black holes change into non-extremal black holes. The reason is that they did not employ any approximation employed by Ref. [28].

The works mentioned above only considered the case where the black holes absorb scalar particles. In this study, we will study the case of fermions with the Dirac equation. We intend to explore whether we can obtain the same result, taking BTZ black holes as an example.

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There are two viewpoints on the thermodynamics of BTZ black holes in the extended phase space. Identifying the cosmological constant and its conjugate as the thermodynamic pressure and thermodynamic volume, the first law was derived directly in Ref. [31]. However, in their treatment, the reverse isoperimetric inequality is violated, and the black holes are always superentropic. Moreover, the thermodynamic volume defined by the first law is related to the charge. Soon after, Ref. [32] introduced a new extensive quantity, namely the renormalization length, in the first law. In this framework, the reverse isoperimetric inequality is satisfied, and the standard definition of the thermodynamic volume is retained. In this study, we investigate whether the first law proposed by Ref. [32] can be reproduced under a charged fermion absorption. Apart from the first law, we investigate the second law and the weak cosmic censorship conjecture. Resultantly, the first law and the weak cosmic censorship conjecture are found to be valid, while the second law is found to be related to the variation of the renormalization energy dK.

This paper is outlined as follows. In Section 2, the motion of a charged fermion in the BTZ black holes is investigated. In Section 3 and Section 4, the first law, second law, and the weak cosmic censorship conjecture are investigated in the normal phase space and extended phase space. Section 5 is devoted to our conclusions. Throughout this study, we set $G = c = 1$.

2 Motion of a charged fermion in the BTZ black holes

The three dimensional theory of gravity with Maxwell tensor is [33]

$$I = \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{-g}\left(R-2\Lambda-4\pi G F_{\mu\nu}F^{\mu\nu}\right), \quad (1)$$

where G is the gravitational constant, R is the Ricci scalar, g is the determinant of the metric tensor $g_{\mu\nu}$, Λ is the cosmological constant that relates to the AdS radius with the relation $\Lambda = -1/l^2$, and $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, where A_μ is the electrical potential. The charged BTZ black hole solutions can be derived from Eq. (1), that is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\phi^2, \quad (2)$$

where

$$f(r) = -m + \frac{r^2}{l^2} - 2q^2\log\left(\frac{r}{l}\right), \quad (3)$$

in which m and q are the parameters that relate to the mass and charge of the black hole. The non-vanishing component of the vector potential of this black hole is [33]

$$A_t = q\log\left(\frac{r}{l}\right). \quad (4)$$

Applying Gauss law, the electric charge of the black hole can be obtained by calculating the flux of the electric field at infinity [34], which yields

$$Q = \frac{q}{2}. \quad (5)$$

Moreover, the total mass can be obtained using the Hamiltonian approach or the counterterm method [34], which leads to

$$M = \frac{m}{8}. \quad (6)$$

We turn to investigate the dynamics of a charged fermion as it is absorbed by the BTZ black hole. We employ the Dirac equation for electromagnetic field

$$i\gamma^\mu\left(\partial_\mu + \Omega_\mu - \frac{i}{\hbar}eA_\mu\right)\psi - \frac{\mu}{\hbar}\psi = 0, \quad (7)$$

where u is the rest mass, e is the charge of the fermions, $\Omega_\mu = \frac{i}{2}\Gamma_\mu^{\alpha\beta}\sum_{\alpha\beta}$, $\sum_{\alpha\beta} = \frac{i}{4}\left[\gamma^\alpha,\gamma^\beta\right]$, γ^μ matrices satisfy $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}I$. To obtain the solution of the Dirac equation, we first should choose γ^μ matrices. In this study, we set

$$\gamma^\mu = \left(-if^{-\frac{1}{2}}\sigma^2, f^{\frac{1}{2}}\sigma^1, \frac{1}{r}\sigma^3\right), \quad (8)$$

where σ^μ are the Pauli sigma matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

For a fermion with spin 1/2, the wave function has a spin up state and spin down state. In this study, we only investigate the spin up case for the case of a similar spin down. We use the ansatz for the two-component spinor ψ as

$$\psi = \begin{pmatrix} A(t,r,\phi) \\ B(t,r,\phi) \end{pmatrix}\exp\left(\frac{i}{\hbar}I(t,r,\phi)\right). \quad (10)$$

Inserting Eq. (10) into Eq. (7), we have the following two simplified equations

$$A\left(\mu + \frac{1}{r}\partial_\phi I\right) + B\left[\sqrt{f}\partial_r I - \left(\frac{1}{\sqrt{f}}\partial_t I - \frac{1}{\sqrt{f}}eA_t\right)\right] = 0, \quad (11)$$

$$A\left[\sqrt{f}\partial_r I + \left(\frac{1}{\sqrt{f}}\partial_t I - \frac{1}{\sqrt{f}}eA_t\right)\right] + B\left(\mu - \frac{1}{r}\partial_\phi I\right) = 0. \quad (12)$$

These two equations have a non-trivial solution for A and B if and only if the determinant of coefficient matrix vanishes, which implies

$$\frac{1}{r^2}\left(\partial_\phi I\right)^2 - \mu^2 + \left(\sqrt{f}\partial_r I\right)^2 - \left(\frac{1}{\sqrt{f}}\partial_t I - \frac{1}{\sqrt{f}}eA_t\right)^2 = 0. \quad (13)$$

There are two Killing vectors in the charged BTZ spacetime, so we can make the separation of variables for $I(t,r,\phi)$ as

$$I = -\omega t + L\phi + I(r) + K, \tag{14}$$

where ω and L are fermion's energy and angular momentum, respectively, and K is a complex constant. Inserting Eq. (14) into Eq. (13), we obtain

$$\partial_r I(r) = \pm\frac{1}{f}\sqrt{(\omega + eA_t)^2 + f\left(\mu^2 - \frac{L^2}{r^2}\right)}. \tag{15}$$

We are interested in the radial momentum of the particle $p^r \equiv g^{rr} p_r = g^{rr}\partial_r I(r)$. In addition, we want to investigate the thermodynamics, so we will focus on the near horizon region. In this case, we obtain

$$\omega = |p^r_+| - eA_t(r_+), \tag{16}$$

which is obviously the same as that of the scalar particles [27]. A positive sign should be endowed in front of the $|p^r_+|$ term. This choice is to assure that the signs in front of ω and p^r_+ are the same and positive in the positive flow of time.

3 Thermodynamics and weak cosmic censorship conjecture in the normal phase space

The electrostatic potential difference between the black hole horizon and the infinity is

$$\Phi = -2Q\log\left(\frac{r_+}{l}\right), \tag{17}$$

in which r_+ is the event horizon of the black hole, which is determined by $f(r_+) = 0$. Based on the definition of surface gravity, the Hawking temperature can be expressed as

$$T = \frac{r_+}{2\pi l^2} - \frac{2Q^2}{\pi r_+}. \tag{18}$$

For the three dimensional BTZ black hole, the black hole entropy can be expressed as

$$S = \frac{1}{2}\pi r_+. \tag{19}$$

In addition, with Eqs. (3) and (6), the mass of the BTZ black hole can be expressed as

$$M = \frac{-8l^2 Q^2 \log\left(\frac{r_+}{l}\right) + r_+^2}{8l^2}. \tag{20}$$

As a charged fermion is absorbed by the black hole, the variation of the internal energy and charge of the black hole satisfy

$$\omega = \mathrm{d}M, \quad e = \mathrm{d}Q, \tag{21}$$

where the energy conservation and charge conservation have been imposed. In this case, Eq. (16) can be rewritten as

$$\mathrm{d}M = \Phi \mathrm{d}Q + p^r_+. \tag{22}$$

The absorbed fermions will change the configurations of the black holes. There is a shift for the horizon of the black hole, labeled as $\mathrm{d}r_+$. In the new horizon, there is also a relation, $f(r_+ + \mathrm{d}r_+) = 0$. Hence, the change of the horizon should satisfy

$$\mathrm{d}f_+ = f(r_+ + \mathrm{d}r_+) - f(r_+) = \frac{\partial f_+}{\partial M}\mathrm{d}M + \frac{\partial f_+}{\partial Q}\mathrm{d}Q + \frac{\partial f_+}{\partial r_+}\mathrm{d}r_+ = 0. \tag{23}$$

Here, q and m in Eq. (3) have been substituted by Q and M in Eqs. (5) and (6). Inserting Eq. (22) into Eq. (23), we can delete dM. Interestingly, dQ is meanwhile eliminated. Solving this equation, we can obtain $\mathrm{d}r_+$ directly, which is

$$\mathrm{d}r_+ = -\frac{4l^2 p^r_+ r_+}{4l^2 Q^2 - r_+^2}. \tag{24}$$

Based on Eq. (24), we can obtain the variation of entropy by making use of Eq. (19), that is

$$\mathrm{d}S = -\frac{\pi}{2}\frac{4l^2 p^r_+ r_+}{4l^2 Q^2 - r_+^2}. \tag{25}$$

With Eqs. (18) and (25), we find there is a relation

$$T\mathrm{d}S = p^r_+. \tag{26}$$

In this case, the internal energy in Eq. (22) can be rewritten as

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q, \tag{27}$$

which is the first law of black hole thermodynamics. Thus, as a fermion drops into the black hole, the first law is valid in the normal phase space.

Next, we consider the second law of thermodynamics, which states that the entropy of the black holes never decreases in the clockwise direction. As a fermion is absorbed by the black hole, the entropy of the black hole increases according to the second law of the thermodynamics. We employ Eq. (25) to verify whether this is true.

For extremal black holes, the temperature vanishes at the horizon for the inner horizons, while the outer horizons are coincident. With Eq. (18), we can obtain the mass of the extremal black hole and substitute it into Eq. (25), we find

$$\mathrm{d}S_{\text{extreme}} = \infty. \tag{28}$$

The divergence of dS implies the second law for the extremal black hole is meaningless, since the thermodynamic system is a zero temperature system.

The temperatures of non-extremal black holes are larger then zero, which implies

$$r_+^2 > 4l^2 Q^2, \tag{29}$$

where we have used Eq. (18). In this case, dS in Eq. (25) is positive. The second law of thermodynamics is therefore valid.

In the normal phase space, we can also verify the validity of the weak cosmic censorship conjecture, which states that the singularity of spacetime cannot be observed for an observer located at future null infinity. Hence, singularities need to be hidden by the event horizon for a black hole. Thus, an event horizon should exist to assure the validity of the weak cosmic censorship conjecture. As a fermion is absorbed by a black hole, we intend to investigate whether there is an event horizon, i.e., whether the equation $f(r)=0$ has solutions.

For BTZ black holes, there is a minimum value for $f(r)$ with the radial coordinate r_m. When $f(r_m)>0$, no horizon exists, while when $f(r_m)\leqslant 0$, there are always horizons. At r_m, the following relations [35–37]

$$f(r)\big|_{r=r_m} \equiv f_m = \epsilon \leqslant 0, \quad \partial_r f(r)\big|_{r=r_m} \equiv f'_m = 0, \tag{30}$$

should be satisfied. For extremal black holes, $\epsilon=0$, r_+ and r_m are coincident. For the near extreme black holes, ϵ is a small quantity, and r_m is distributed between the inner and outer horizon. As a fermion drops into the black hole, the mass and charge of the black hole change into $M+\mathrm{d}M, Q+\mathrm{d}Q$, respectively. Correspondingly, the locations of the minimum value and event horizon change into $r_m+\mathrm{d}r_m$, $r_++\mathrm{d}r_+$. There is also a shift for $f(r)$, which can be written as

$$\mathrm{d}f_m = f(r_m+\mathrm{d}r_m) - f_m = \left(\frac{\partial f_m}{\partial M}\mathrm{d}M + \frac{\partial f_m}{\partial Q}\mathrm{d}Q\right), \tag{31}$$

where we have used $f'_m=0$ in Eq. (30). We first discuss the extremal black holes, for which the horizons are located at r_m. In this case, Eq. (22) can be used. Inserting Eq. (22) into Eq. (31), we find that $\mathrm{d}Q$ disappears. In this case, Eq. (31) can be simplified to

$$\mathrm{d}f_m = -8p^r_+. \tag{32}$$

This result shows that $f(r_m+\mathrm{d}r_m)$ is smaller than $f(r_m)$ as a charged fermion is absorbed by the black hole.

For near-extremal black holes, Eq. (22) is not valid at r_m, as it holds true only at the horizon. With the condition $r_+ = r_m+\delta$, we expand Eq. (22) at r_m, which leads to

$$\mathrm{d}M = p^r_+ - 2\mathrm{d}QQ\log\left(\frac{r}{R}\right) - \frac{2Q\mathrm{d}Q}{r}\delta + O(\delta)^2. \tag{33}$$

Substituting Eq. (33) into Eq. (23), we obtain

$$\mathrm{d}f_m = -8p^r_+ + \frac{32\mathrm{d}Q}{l}\delta + O(\delta)^2. \tag{34}$$

Because δ is a small quantity, while l is relatively large, the last two terms can be neglected as an approximation. In this case, Eq. (34) takes the same form as Eq. (32), indicating that the weak cosmic censorship conjecture is also valid for near-extremal black holes.

The second term in Eq. (34) is small compared with $8p^r_+$. In fact, the higher order corrections are important to discuss the weak cosmic censorship conjecture. However, in our method, we find they can be neglected after calculation strictly if the dominant term is too large.

4 Thermodynamics and weak cosmic censorship conjecture in extended phase space

To make the charged BTZ black holes satisfy the reverse isoperimetric inequality, a new thermodynamic parameter R was introduced in the first law [32], that is

$$\mathrm{d}M = T\mathrm{d}S + V\mathrm{d}P + \Phi\mathrm{d}Q + K\mathrm{d}R, \tag{35}$$

where

$$M = \frac{r_+^2 - 8l^2Q^2\log\left(\frac{r_+}{R}\right)}{8l^2}, \tag{36}$$

$$P = -\frac{\Lambda}{8\pi} = \frac{1}{8\pi l^2}, \tag{37}$$

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,R} = \pi r_+^2, \tag{38}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_{S,Q,R} = -2Q\log\left(\frac{r}{R}\right), \tag{39}$$

$$K = \left(\frac{\partial M}{\partial R}\right)_{S,Q,P} = Q^2/R, \tag{40}$$

where R is the renormalization length scale, and K, which is the conjugate of R, is the renormalized energy. Here, the value of K is different from that in Ref. [32]. The reason stems from the definition of the electric charge Q. In fact, with the Gauss law, we noted that the charge parameter q is not the electric charge of the black hole, which has been shown in Eq. (5).

From Eq. (38), we know that in this framework, the volume recovers to the standard definition of the thermodynamic volume [32], which is more reasonable. We set out to explore whether the first law in Eq. (35) can be obtained by considering a charged fermion absorption.

In the extended, the pressure P and the renormalization length scale R are likewise state parameters of the thermodynamic system, as a fermion is absorbed by the black hole, the pressure and the renormalization length scale will also change besides the mass, charge, and entropy. In this thermodynamic system, the mass M is not the internal energy but the enthalpy, which relates to the internal energy as [32]

$$M = U + PV + KR. \tag{41}$$

As a charged fermion drops into the black hole, the energy and charge could be conserved. Namely, the energy and charge of the fermion equal to the varied energy and charge of the black hole, which implies

$$\omega = \mathrm{d}U = \mathrm{d}(M - PV - KR), \quad e = \mathrm{d}Q, \tag{42}$$

The energy in Eq. (16) changes correspondingly into

$$\mathrm{d}U = \Phi\mathrm{d}Q + p^r_+. \tag{43}$$

Considering the backreaction, the absorbed fermions will change the location of the event horizon of the black hole. However, the horizon is consistently determined by the equation $f(r)=0$, as stressed in Section 3. In the extended phase space, for the AdS radius l and renormalization length R are variables, the shift of function $f(r)$ can be expressed as

$$\mathrm{d}f_+ = \frac{\partial f_+}{\partial M}\mathrm{d}M + \frac{\partial f_+}{\partial Q}\mathrm{d}Q + \frac{\partial f_+}{\partial l}\mathrm{d}l + \frac{\partial f_+}{\partial r_+}\mathrm{d}r_+ + \frac{\partial f_+}{\partial R}\mathrm{d}R = 0. \tag{44}$$

In addition, with Eq. (41), Eq. (43) can be expressed as

$$\mathrm{d}M - \mathrm{d}(PV + KR) = \Phi\mathrm{d}Q + p^r_+. \tag{45}$$

From Eq. (44), we can obtain dl. Substituting dl into Eq. (45), we can delete it directly. Interestingly, dQ, dR, and dM are also eliminated at the same time. In this case, there is only a relation between p^r_+ and dr_+, which is

$$\mathrm{d}r_+ = -\frac{r_+(p^r_+ + \mathrm{d}KR)}{Q^2}. \tag{46}$$

Based on Eq. (46), the variations of entropy and volume of the black hole can be expressed as

$$\mathrm{d}S = -\frac{\pi r_+(p^r_+ + \mathrm{d}KR)}{2Q^2}, \tag{47}$$

$$\mathrm{d}V = -\frac{2\pi r_+^2(p^r_+ + \mathrm{d}KR)}{Q^2}. \tag{48}$$

With Eq. (47) and Eq. (48), we find

$$T\mathrm{d}S - P\mathrm{d}V - R\mathrm{d}K = p^r_+. \tag{49}$$

The internal energy in Eq. (43) thus would change into

$$\mathrm{d}U = \Phi\mathrm{d}Q + T\mathrm{d}S - P\mathrm{d}V - R\mathrm{d}K. \tag{50}$$

Moreover, from Eq. (41), we can get

$$\mathrm{d}M = \mathrm{d}U + P\mathrm{d}V + V\mathrm{d}P + K\mathrm{d}R + R\mathrm{d}K. \tag{51}$$

Substituting Eq. (51) into Eq. (50), we find

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q + V\mathrm{d}P + K\mathrm{d}R, \tag{52}$$

which is consistent with that in Eq. (35). That is, as a charged fermion is absorbed by the black hole, the first law of thermodynamics holds in the extended phase space.

With Eq. (47), we also can verify the second law of thermodynamics in the extended phase space. There is a term dK in Eq. (47), which depicts the variation of the renormalized energy. According to Eq. (40), dK is the function of dR, dQ. However, the existence of dQ and dR would affect the definition of Φ and K, respectively, and further violate the first law of thermodynamics. The satisfaction of the first law of thermodynamics is a necessary condition to discuss the second law of thermodynamics under particle absorption. Thus, dK can not be expressed as a linear relation of dR and dQ though we do not know the mechanism for we know little about the renormalized energy in the extended phase space. In this study, we treat the variation of the renormalized energy as an independent quantity and do not consider its form.

From Eq. (47), we know the variation of the entropy depends on the variation of the renormalized energy. For the case $\mathrm{d}K > -p^r_+/R$, dS is negative, and for the case $\mathrm{d}K < -p^r_+/R$, dS is positive. Hence, the second law is violated for the case $\mathrm{d}K > -p^r_+/R$, and valid for the case $\mathrm{d}K < -p^r_+/R$. Moreover, for $\mathrm{d}K = -p^r_+/R$, $\mathrm{d}S = 0$, indicating that the horizons of the black holes will not change as a charged fermion is absorbed.

We discuss the weak cosmic censorship conjecture in the extended phase space with the condition in Eq. (30). Because of the backreaction, the mass M, charge Q, renormalization length R, and AdS radius l of the black hole will change into $(M+\mathrm{d}M, Q+\mathrm{d}Q, R+\mathrm{d}R, l+\mathrm{d}l)$ as a charged fermion drops into the black hole. Correspondingly, the locations of the minimum value, event horizon, AdS radius, and renormalization length will change into $r_m+\mathrm{d}r_m$, $r_+ + \mathrm{d}r_+$, $l+\mathrm{d}l$, and $R+\mathrm{d}R$. In this case, the shift of $f(r)$ can be written as

$$\mathrm{d}f(r_m) = \left(\frac{\partial f_m}{\partial M}\mathrm{d}M + \frac{\partial f_m}{\partial Q}\mathrm{d}Q + \frac{\partial f_m}{\partial l}\mathrm{d}l + \frac{\partial f_m}{\partial R}\mathrm{d}R\right), \tag{53}$$

where we have used $f'_m = 0$ in Eq. (30). Next, we focus on finding the last result of Eq. (53). For extremal black holes, the horizons are located at r_m. The energy relation in Eq. (45) is valid. Substituting Eq. (45) into Eq. (53), we find

$$\mathrm{d}f(r_m) = -8p^r_+ - 8\mathrm{d}KR - \frac{2r_m\mathrm{d}r_m}{l^2}, \tag{54}$$

Substituting Eq. (49) into Eq. (54), we find

$$\mathrm{d}f(r_m) = 0. \tag{55}$$

That is, as fermions drop into extremal BTZ black holes, the black holes remain at their initial states so their configurations remain unchanged. This result is quite different from that in the normal phase space, where the extremal black holes will evolve into the non-extremal black holes with the absorption.

For the near-extremal black hole, Eq. (45) is not valid. However, we can expand the expression near the lowest point with $r_+ = r_m + \delta$. p^r_+ should also be expanded, as it is also a function of the horizon r_+. To the first order, we obtain

$$\begin{aligned}\mathrm{d}M = &-\frac{r_m^2}{4l^3}\mathrm{d}l - 2Q\log\left(\frac{r_m}{R}\right)\mathrm{d}Q + \frac{r_m\mathrm{d}r}{4l^2} - \frac{Q^2\mathrm{d}r}{r_m} + \frac{Q^2\mathrm{d}R}{R}\\ &+\left(-\frac{r_m\mathrm{d}l}{2l^3} - \frac{2\mathrm{d}QQ}{r_m} + \frac{\mathrm{d}r}{4l^2} + \frac{Q^2\mathrm{d}r}{r_m^2}\right)\delta + O(\delta)^2,\end{aligned} \tag{56}$$

Substituting Eq. (56) into Eq. (59), we can obtain

$$\begin{aligned}\mathrm{d}f(r_m) = &\left(\frac{8Q^2}{r_m} - \frac{2r_m}{l^2}\right)\mathrm{d}r_m\\ &+\left(\frac{4r_m\mathrm{d}l}{l^3} + \frac{16Q\mathrm{d}Q}{r_m} - \frac{2\mathrm{d}r}{l^2} - \frac{8Q^2\mathrm{d}r}{r_m^2}\right)\delta + O(\delta)^2,\end{aligned} \tag{57}$$

In addition, at $r_m + \mathrm{d}r_m$, there is also a relation

$$\partial_r f(r)\big|_{r=r_m+\mathrm{d}r_m} = f'_m + \mathrm{d}f'_m = 0, \tag{58}$$

which implies

$$\mathrm{d}f'_m = \frac{\partial f'_m}{\partial Q}\mathrm{d}Q + \frac{\partial f'_m}{\partial l}\mathrm{d}l + \frac{\partial f'_m}{\partial r_m}\mathrm{d}r_m = 0. \tag{59}$$

Solving this equation, we obtain

$$\mathrm{d}l = \frac{l\left(-8l^2 Q r_m \mathrm{d}Q + 4l^2 Q^2 \mathrm{d}r_m + r^2 \mathrm{d}r_m\right)}{2r_m^3}. \tag{60}$$

Based on the condition $f'_m = 0$ in Eq. (30), we can get

$$l = \frac{r_m}{2Q}. \tag{61}$$

Substituting Eq. (61) and Eq. (60) into Eq. (57), we find

$$\mathrm{d}f(r_m) = O(\delta)^2, \tag{62}$$

which shows that near-extremal black holes are also stable. This result is consistent with the extremal black holes in Eq. (55). Thus, we can conclude that the weak cosmic censorship conjecture holds for both the extremal and near-extremal black holes in the extended phase space, as configurations of the black holes are not changed as fermions are adsorbed.

5 Conclusions

In the normal and extended phase spaces, the laws of thermodynamics and weak cosmic censorship conjecture in BTZ black holes were investigated by a charged fermion absorption. We first investigated the motion of a fermion via the Dirac equation and obtained a relation between the energy and momentum near the horizon. With this relation, the first law was reproduced in the normal phase space. By studying the variation of the entropy, we also investigated the second law of thermodynamics and found that for both the extremal black holes and near-extremal black holes, the second law was valid in the normal phase space, as the variation of entropy was positive. The weak cosmic censorship for extremal black holes and near-extremal black holes was likewise studied. We found that the metric function, which determines the locations of the horizons, moved with the same scale, $-8p_+^r$, implying that there are always horizons to hide the singularity, such that the weak cosmic censorship is valid for both cases.

Employing a similar strategy, the thermodynamic laws and weak cosmic censorship conjecture were further investigated in the extended phase space. We found that the first law of thermodynamics was still valid, however the validity of the second law depended on the variation of the renormalization energy $\mathrm{d}K$. For the case $\mathrm{d}K > -p_+^r/R$, the second law is violated, whereas for the case $\mathrm{d}K \leqslant -p_+^r/R$, the second law is valid. Though the weak cosmic censorship conjecture is valid in both the normal and extended phase space for extremal and near-extremal black holes, their final states are different after absorption. Extremal and near-extremal black holes will evolve into non-extremal black holes in the normal phase space, while they are stable in the extended phase space.

The study of Ref. [27] investigated the laws of thermodynamics and the weak cosmic censorship conjecture of the BTZ black holes. In contrast, in the present work, the absorbed particles considered are fermions. Moreover, laws of thermodynamics and the weak cosmic censorship conjecture were discussed not only in the normal phase space, but also in the extended phase space in this study, while Ref. [27] only investigated the case of the normal phase space.

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