Analytical solution of transverse oscillation in cyclotron using LP method

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Abstract: We have carried out an approximate analytical solution to precisely consider the influence of magnetic field on the transverse oscillation of particles in a cyclotron. The differential equations of transverse oscillation are solved from the Lindstedt-Poincare method. After careful deduction, accurate first-order analytic solutions are obtained. The analytical solutions are applied to the magnetic field from an isochronous cyclotron with four spiral sectors. The accuracy of these analytical solutions is verified and confirmed from comparison with a numerical method. Finally, we discussed the transverse oscillation at $v_0=\frac{N}{2}$, using the same analytical solution.

Keywords: cyclotron, perturbation method, transverse oscillation, LP method

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1 Introduction

The motion of particles in a cyclotron can be precisely investigated from numerical methods, but the effect of magnetic field on particle motion cannot be visualized. An analytical formula with certain accuracy can consider and visualize the effect of the magnetic field.

There are many analytical formulas for transverse oscillation which describe synchrotrons and classic cyclotrons at relatively simple magnetic field, as shown in Ref. [1]. However, no one has reported such a formula for a cyclotron with spiral sectors.

Formulas for calculating transverse oscillation frequency have been reported by several investigators, as shown in Refs. [2–5]. However, the accuracy of these formulas is limited.

In this paper, we solve the linear equations for transverse oscillation through the Lindstedt-Poincare (LP) method. We have obtained different formulas for calculating transverse oscillation and oscillation frequency. Finally, these formulas are compared with numerical results and shown to be accurate.

2 Linear equations for transverse oscillation

In Ref. [6],the linear equations for transverse oscillation around the equilibrium orbit are given as:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{p_{re}}{p_{\theta e}}\cdot x+\dfrac{r_e\cdot p^2}{p_{\theta e}^3}\cdot p_x \\ \dfrac{\mathrm{d}p_x}{\mathrm{d}\theta}=-\dfrac{e}{P}\left.\dfrac{\partial rB_z}{\partial r}\right|_{seo}\cdot x-\dfrac{p_{re}}{p_{\theta e}}\cdot p_x, \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\theta}=\dfrac{r_e}{p_{re}}\cdot p_z \\ \dfrac{\mathrm{d}p_z}{\mathrm{d}\theta}=\dfrac{e}{P}\left.\left(r\dfrac{\partial B_z}{\partial r}-\dfrac{p_r}{p_\theta}\dfrac{\partial B_z}{\partial \theta}\right)\right|_{seo} x. \end{cases} \tag{2}$$

Equation (1) is for radial oscillation and Eq. (2) is for axial oscillation; x and z are respectively the radial and axial offset of the equilibrium orbit, and p_x and p_z are correspondingly the radial and axial momentum offset of the equilibrium orbit.

The coefficients of the above linear differential equations are determined from the parameters of equilibrium orbit such as r_e and p_{re}. As long as the magnetic fields are given, the equilibrium orbit can be determined from numerical integration or Gordon's formulas [3] followed by the determination of the coefficients of the equation.

In order to simplify these equations, let

$$\begin{cases} a(\theta)=\dfrac{p_{re}}{p_{\theta e}}, \quad b(\theta)=\dfrac{r_e}{p_{\theta e}^3}, \\ c(\theta)=-\dfrac{e}{P}\left.\dfrac{\partial rB_z(r,\theta)}{\partial r}\right|_{seo}, \quad d(\theta)=-\dfrac{p_{re}}{p_{\theta e}}, \\ e(\theta)=\dfrac{r_e}{p_{re}}, \quad f(\theta)=\dfrac{e}{P}\left.\left(r\dfrac{\partial B_z}{\partial r}-\dfrac{p_r}{p_\theta}\dfrac{\partial B_z}{\partial \theta}\right)\right|_{seo}. \end{cases} \tag{3}$$

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Then the radial oscillation equations can be written as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=a(\theta)\cdot x+b(\theta)\cdot p_x \\ \dfrac{\mathrm{d}p_x}{\mathrm{d}\theta}=c(\theta)\cdot x+d(\theta)\cdot p_x. \end{cases} \tag{4}$$

By using the following formula:

$$x(\theta)=\sqrt{b(\theta)}\cdot u(\theta), \tag{5}$$

Eq. (4) can be rewritten as a Hill's equation,

$$u''+G(\theta)\cdot u=0, \tag{6}$$

where

$$G(\theta)=-\frac{3}{4}(\frac{b'}{b})^2+\frac{1}{2}\frac{b''}{b}+(ad-bc)+a\frac{b'}{b}-a'. \tag{7}$$

Here $G(\theta)$ has the same periodicity as the equilibrium orbit. Expanding $G(\theta)$ into a Fourier series,

$$G(\theta)=v^2+\sum_n P_n\cos n\theta+Q_n\sin n\theta. \tag{8}$$

In order to simplify, let

$$g(\theta)=\sum_n P_n\cos n\theta+Q_n\sin n\theta. \tag{9}$$

Then Eq. (6) becomes

$$u''+\nu^2 u=-g(\theta)u. \tag{10}$$

Equation (10) is a simplified radial oscillation equation. We can get the simplified axial equation directly by replacing Eqs. (5) and (7) with the following two formulas:

$$z(\theta)=\sqrt{e(\theta)}\cdot u(\theta), \tag{11}$$

$$G(\theta)=\frac{1}{2}\frac{e''}{e}-\frac{3}{4}(\frac{e'}{e})^2-ef. \tag{12}$$

3 Solving oscillation equations by LP method

Equation (10) is a linear differential equation with variable coefficients, and the LP method is very effective for solving such equations.

Now, we want to solve Eq. (10) by using the LP method. First, we introduce a small parameter ε with the condition of $0\leqslant\varepsilon\leqslant 1$ and rewrite Eq. (10) as follows:

$$u''+\nu^2 u=-\varepsilon\cdot g(\theta)u. \tag{13}$$

According to the LP method [7, 8], we will try to solve Eq. (13) by inserting a perturbation series:

$$u(\theta)=u_0(\theta)+\varepsilon\cdot u_1(\theta)+\varepsilon^2\cdot u_2(\theta)+\cdots, \tag{14}$$

$$\nu^2=\nu_0^2+\varepsilon\cdot\nu_1^2+\varepsilon^2\cdot v_2^2+\cdots, \tag{15}$$

and then find the expressions of $u_0(\theta)$, $u_1(\theta)$, $u_2(\theta)$ as well as the values of v_0^2, v_1^2, v_2^2. Finally, the solutions of Eq. (10) are obtained by letting $\varepsilon=1$.

Substituting Eqs. (14) and (15) into Eq. (13), we can get

$$\begin{aligned} &(u_0''+\varepsilon\cdot u_1''+\varepsilon^2\cdot u_2''+\cdots)+(\nu_0^2+\varepsilon\cdot\nu_1^2+\varepsilon^2\cdot\nu_2^2+\cdots) \\ &\times(u_0+\varepsilon\cdot u_1+\varepsilon^2\cdot u_2+\cdots) \\ =\ &\varepsilon\cdot g(\theta)(u_0+\varepsilon\cdot u_1+\varepsilon^2\cdot u_2+\cdots). \end{aligned} \tag{16}$$

The coefficient of each power of ε should be zero, hence we can get:

$$\begin{cases} u_0''+\nu_0^2\cdot u_0=0 \\ u_1''+\nu_0^2\cdot u_1=-\nu_1^2\cdot u_0-g(\theta)\cdot u_0 \\ u_2''+v_0^2\cdot u_2=-v_2^2\cdot u_0-v_1^2\cdot u_1-g(\theta)\cdot u_1. \end{cases} \tag{17}$$

From the above procedures, we transform Eq. (13) into several ordinary differential equations. The solution of the first equation of Eq. (17) is

$$u_0(\theta)=A\cos v_0\theta+B\sin v_0\theta. \tag{18}$$

Substituting Eq. (18) into the second equation of Eq. (17) gives:

$$\begin{aligned} u_1''+v_0^2\cdot u_1 =\ & -v_1^2(A\cos v_0\theta+B\sin v_0\theta) \\ & -\sum_n P_n A\cos n\theta\cos v_0\theta+P_n B\cos n\theta\sin v_0\theta \\ & +Q_n A\sin n\theta\cos v_0\theta+Q_n B\sin n\theta\sin v_0\theta, \end{aligned} \tag{19}$$

where $-v_1^2(A\cos v_0\theta+B\sin v_0\theta)$ on the right-hand side is the secular term. The LP method requires avoidance of the secular term by choosing a suitable value of v_1^2. It is obvious that v_1^2 must be zero to eliminate the secular term, which is

$$v_1^2=0. \tag{20}$$

The solution of Eq. (19) is:

$$\begin{aligned} u_1 =\ & \sum_n -\frac{P_n A}{2}\left[\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right] \\ & -\frac{P_n B}{2}\left[\frac{\sin(n+v_0)\theta}{v_0^2-(n+v_0)^2}-\frac{\sin(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right] \\ & -\frac{Q_n A}{2}\left[\frac{\sin(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\sin(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right] \\ & +\frac{Q_n B}{2}\left[\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}-\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right]. \end{aligned} \tag{21}$$

In order to simplify, let

$$\begin{cases} C_1=v_0^2-(n+v_0)^2 \\ C_2=v_0^2-(n-v_0)^2, \end{cases} \tag{22}$$

and substitute Eq. (21) into the third formula of Eq. (17), so the secular term can be written as

$$-v_2^2(A\cos v_0\theta+B\sin v_0\theta)+\sum_n \frac{P_n^2A}{4}\left(\frac{\cos v_0\theta}{C_1}+\frac{\cos v_0\theta}{C_2}\right)+\frac{P_n^2A}{4}\left(\frac{\sin v_0\theta}{C_1}+\frac{\sin v_0\theta}{C_2}\right)$$
$$+\frac{Q_nP_nA}{4}\left(\frac{\sin v_0\theta}{C_1}-\frac{\sin v_0\theta}{C_2}\right)-\frac{Q_nP_nB}{4}\left(\frac{\cos v_0\theta}{C_1}-\frac{\cos v_0\theta}{C_2}\right)+\frac{Q_nP_nA}{4}\left(-\frac{\sin v_0\theta}{C_1}+\frac{\sin v_0\theta}{C_2}\right)$$
$$+\frac{Q_nP_nB}{4}\left(\frac{\cos v_0\theta}{C_1}-\frac{\cos v_0\theta}{C_2}\right)+\frac{Q_n^2A}{2}\left(\frac{\cos v_0\theta}{C_1}+\frac{\cos v_0\theta}{C_2}\right)+\frac{Q_n^2B}{2}\left(\frac{\sin v_0\theta}{C_1}+\frac{\sin v_0\theta}{C_2}\right). \quad (23)$$

The secular term can be eliminated by choosing the value of v_2^2. Letting Eq. (23) be equal to zero, we can get

$$v_2^2=\sum_n\frac{P_n^2+Q_n^2}{4}\left[\frac{1}{v_0^2-(n+v_0)^2}+\frac{1}{v_0^2-(n-v_0)^2}\right]. \quad (24)$$

Substituting Eqs. (20) and (24) into Eq. (15), and letting $\varepsilon=1$, Eq. (15) becomes

$$v^2=v_0^2+\sum_n\frac{P_n^2+Q_n^2}{4}\left[\frac{1}{v_0^2-(n+v_0)^2}+\frac{1}{v_0^2-(n-v_0)^2}\right]. \quad (25)$$

Since the values of v^2, P_n and Q_n have been determined from Eq. (8), we can get the value of v_0 by solving the above equation with the help of the bisection method.

So far, we have found the expressions of u_0, u_1 and the values of v_0,v_1,v_2. By letting $\varepsilon=1$, eventually we can get the solution of Eq. (10):

$$\begin{aligned}u\ =\ & u_0+u_1=A\cos v_0\theta+B\sin v_0\theta\\ &+\sum_n-\frac{P_nA}{2}\left[\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right]\\ &-\frac{P_nB}{2}\left[\frac{\sin(n+v_0)\theta}{v_0^2-(n+v_0)^2}-\frac{\sin(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right]\\ &-\frac{Q_nA}{2}\left[\frac{\sin(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\sin(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right]\\ &+\frac{Q_nB}{2}\left[\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}-\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right],\end{aligned} \quad (26)$$

where A and B can be determined from the initial conditions of $u(0)$ and $u'(0)$. Finally, according to the Eq. (5), we can get the expression of $x(\theta)$

$$x(\theta)=\sqrt{b(\theta)}\cdot u(\theta). \quad (27)$$

The solution for axial oscillation $z(\theta)$ can be found from the same method, but the details of the calculation are not shown here.

4 Comparison with numerical solution

The analytical solutions are applied to the magnetic field from a SC200 cyclotron. The SC200 cyclotron is a compact superconducting proton cyclotron with four spiral sectors, used for proton therapy. The highest energy of the SC200 is 200 MeV, the extracting radius is 60 cm, and the central magnetic field is 2.95 T. The transverse oscillation under the initial conditions of E_k=100 MeV, x_0=2 mm, x'_0=0, z_0=2 mm, and z'_0=0 was calculated from Eqs. (25), (26) and (27).

We also calculated the equilibrium orbit and the general orbit by solving the equation of motion using the 4th order Runge-Kutta method, with the above initial conditions. The equation of motion and the numerical method of calculating the equilibrium orbit come from Ref. [9]. The transverse oscillation is then obtained by comparing the general orbit with the equilibrium orbit.

A comparison of the result from the formulas with those of the above numerical method is shown in Figs. 1–4.

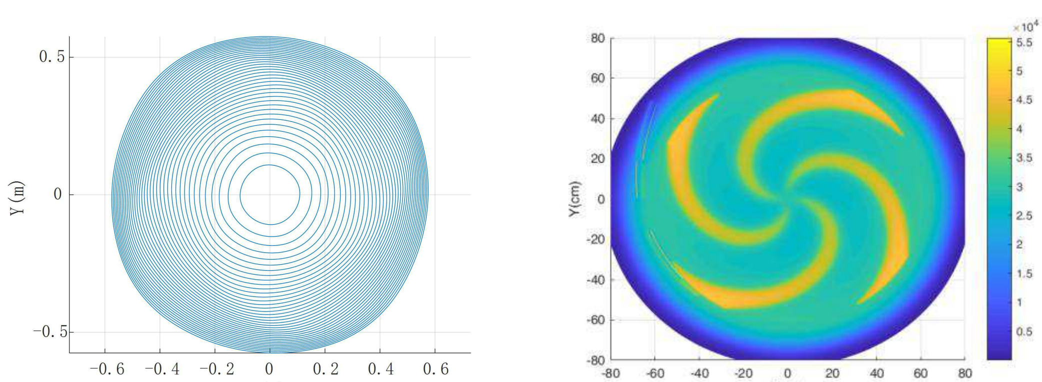

Fig. 1. (color online) (left) 5-180 MeV equilibrium orbits of SC200 cyclotron, with energy step of 5 MeV. (right) Magnetic field in mid-plane.

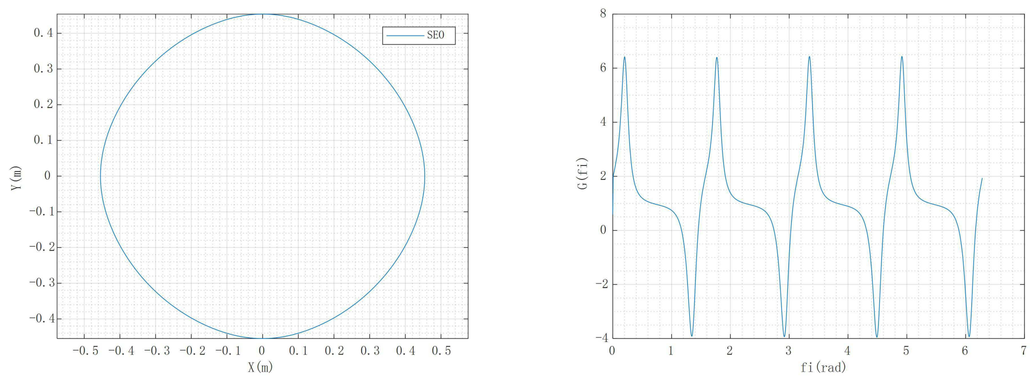

Fig. 2. (color online) (left) Equilibrium orbit for a 100 MeV particle; (right) $G(\theta)$, obtained from Eq. (7).

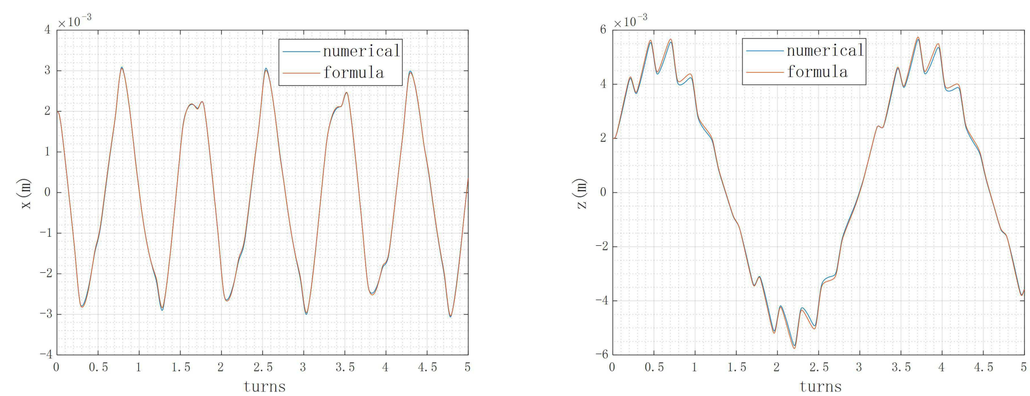

Fig. 3. (color online) Oscillation around the equilibrium orbit, obtained from Eqs. (26) and (27) (red curve), and that of the numerical method (blue curve). (left) Radial oscillation around the 100 MeV equilibrium orbit. (right) Axial oscillation around the 100 MeV equilibrium orbit.The results of the two methods are almost indistinguishable.

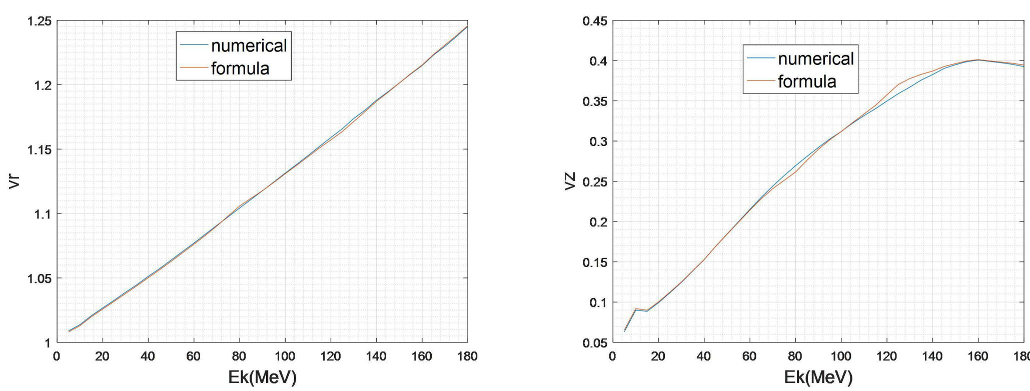

Fig. 4. (color online) The oscillation frequency v_0, obtained from Eq. (25) (red curve) and numerical method (blue curve). (left) Radial frequency ν_r and (right) axial frequency ν_z. The difference in radial frequency is negligible. The difference in axial frequency is also very small except at 80 MeV and 130 MeV. The deviations at 80 MeV and 130 MeV may be due to unavoidable error.

5 Improved formulas when $v_0=\frac{N}{2}$

The value of the denominator $v_0^2-(n-v_0)^2$ in (26) will be zero at $v_0=\frac{N}{2}$, where $n=1,2,3\cdots$ and N is an integer. So, Eq. (26) will be invalid at $v_0=\frac{N}{2}$.

Therefore, further processing of Eq. (26) is required. Actually, Eq. (26) is deduced based on initial conditions such as

$$\begin{cases} u(0)=u_0(0)+u_1(0)+u_2(0)+\cdots, \\ u'(0)=u_0'(0)+u_1'(0)+u_2'(0)+\cdots. \end{cases} \tag{28}$$

Now, we solve Eq. (10) again based on a new initial condition:

$$\begin{cases} u(0)=u_0(0), \quad u_1(0)=0, \quad u_2(0)=0,\cdots \\ u'(0)=u_0'(0), \quad u_1'(0)=0, \quad u_2'(0)=0,\cdots. \end{cases} \tag{29}$$

then $u(\theta)$ is obtained as:

$$\begin{aligned} u(\theta) \;=\; & (A+A')\cos v_0\theta+(B+B')\sin v_0\theta \\ & +\sum_n -\frac{P_nA}{2}\left[\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right] \\ & -\frac{P_nB}{2}\left[\frac{\sin(n+v_0)\theta}{v_0^2-(n+v_0)^2}-\frac{\sin(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right] \\ & -\frac{Q_nA}{2}\left[\frac{\sin(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\sin(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right] \\ & +\frac{Q_nB}{2}\left[\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}-\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}\right], \end{aligned} \tag{30}$$

where:

$$\begin{cases} A=u(0) \\ A'=\sum_n\frac{P_nA}{2}\left[\frac{1}{C_1}+\frac{1}{C_2}\right]-\frac{Q_nB}{2}\left[\frac{1}{C_1}-\frac{1}{C_2}\right], \\ B=\frac{u'(0)}{v_0} \\ B'=\frac{\sum_n\frac{P_nB}{2}\left[\frac{n+v_0}{C_1}-\frac{n-v_0}{C_2}\right]+\frac{Q_nA}{2}\left[\frac{n+v_0}{C_1}+\frac{n-v_0}{C_2}\right]}{v_0}. \end{cases} \tag{31}$$

The initial condition Eq. (28) is equivalent to Eq. (29), while the difference between the resulting formulas Eqs. (26) and (30) is a small quantity of high order. In contrast, Eq. (26) has high accuracy, but Eq. (30) is preferred due to the constant coefficients of A and B during $v_0\to\frac{N}{2}$, which encourages further simplification.

So, we make the following transformation for the formulas in square brackets of Eq. (30), such as the first square bracket

$$\begin{aligned} & \frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2}=\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2}+\frac{\cos(n-v_0)\theta}{v_0^2-(n-v_0)^2} \\ & -\frac{\cos v_0\theta}{v_0^2-(n-v_0)^2}+\frac{\cos v_0\theta}{v_0^2-(n-v_0)^2}=\frac{\cos(n+v_0)\theta}{v_0^2-(n+v_0)^2} \\ & +\frac{\cos v_0\theta}{v_0^2-(n-v_0)^2}+\frac{-2\cdot\sin\frac{n}{2}\theta\cdot\sin\frac{n-2v_0}{2}\theta}{n(2v_0-n)}. \end{aligned} \tag{32}$$

After the above transformation, $u(\theta)$ becomes

$$\begin{aligned} u(\theta) \;=\; & u_0(\theta)+u_1(\theta)=A^*\cos v_0\theta+B^*\sin v_0\theta \\ & +\sum_n C_n^*\cdot\cos(n+v_0)\theta+D_n^*\cdot\sin(n+v_0)\theta \\ & +\sum_n\frac{P_nA}{C_2}\sin\frac{n}{2}\theta\sin\frac{n-2\nu_0}{2}\theta \\ & -\frac{P_nB}{C_2}\sin\frac{2\nu_0-n}{2}\theta\cos\frac{n}{2}\theta \\ & -\frac{Q_nA}{C_2}\cos\frac{n}{2}\theta\sin\frac{n-2v_0}{2}\theta \\ & +\frac{Q_nB}{C_2}\sin\frac{n}{2}\theta\sin\frac{n-2v_0}{2}\theta, \end{aligned} \tag{33}$$

where:

$$\begin{cases} A^*=A+A'+\sum_n-\frac{P_nA}{2}\frac{1}{C_2}-\frac{Q_nB}{2}\frac{1}{C_2} \\ B^*=B+B'+\sum_n\frac{P_nB}{2}\frac{1}{C_2}-\frac{Q_nA}{2}\frac{1}{C_2} \\ C_n^*=\left(\frac{-P_nA}{2}+\frac{Q_nB}{2}\right)\frac{1}{C_1} \\ D_n^*=\left(\frac{-P_nB}{2}-\frac{Q_nA}{2}\right)\frac{1}{C_1}. \end{cases} \tag{34}$$

It is easy to find that C_n^* and D_n^* remain constant during $v_0\to\frac{N}{2}$. According to the initial condition, A^* and B^* can also be written as:

$$\begin{cases} A^*=u(0)-\sum_n\left(\frac{-P_nA}{2}+\frac{Q_nB}{2}\right)\frac{1}{v_0^2-(n+v_0)^2} \\ B^*=\frac{u'(0)+\sum_n\frac{P_nB+Q_nA}{2}\frac{n+v_0}{v_0^2-(n+v_0)^2}+\frac{P_nB-Q_nA}{2n}}{v_0} \end{cases}, \tag{35}$$

which indicate that A^* and B^* also remain constant during $v_0\to\frac{N}{2}$.

So, the expression of $u(\theta)$ under $v_0=\frac{N}{2}$ can be obtained as

$$\begin{aligned} \lim_{v_0\to\frac{N}{2}}u(\theta) \;=\; & A^*\cos\frac{N}{2}\theta+B^*\sin\frac{N}{2}\theta+\sum_n C_n^*\cos(n+\frac{N}{2})\theta+D_n^*\sin(n+\frac{N}{2})\theta+\sum_{n\neq N}\frac{P_nA}{C_2}\sin\frac{n}{2}\theta\sin\frac{n-N}{2}\theta \\ & -\frac{P_nB}{C_2}\sin\frac{N-n}{2}\theta\cos\frac{n}{2}\theta-\frac{Q_nA}{C_2}\cos\frac{n}{2}\theta\sin\frac{n-N}{2}\theta+\frac{Q_nB}{C_2}\sin\frac{n}{2}\theta\sin\frac{n-N}{2}\theta-\frac{P_NA}{2N}\cdot\theta\cdot\sin\frac{N}{2}\theta \\ & -\frac{P_NB}{2N}\cdot\theta\cdot\cos\frac{N}{2}\theta-\frac{Q_NA}{2N}\cdot\theta\cdot\cos\frac{N}{2}\theta-\frac{Q_NB}{2N}\cdot\theta\cdot\sin\frac{N}{2}\theta. \end{aligned} \tag{36}$$

Equation (36) is an improved formula for calculating oscillation at $v_0=\frac{N}{2}$. As we can see, Eq. (36) contains the terms $\theta\cdot sin\frac{N}{2}\theta$ and $\theta\cdot\cos\frac{N}{2}\theta$, which indicates that the amplitude of $u(\theta)$ increases with respect to the azimuth.

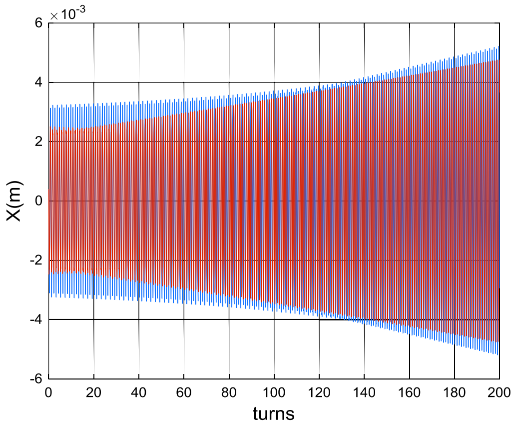

Fig. 5. (color online) Calculated results of radial oscillation from Eq. (36). The amplitude of oscillation increases with respect to azimuth at $v_0=1$.

Here we have calculated the transverse oscillation at $v_0=1$ for two different particles with the initial condition of $A=0.02$,$B=0.04$ and $A=0.006$,$B=-0.04$ using Eq. (36). The values of P_n,Q_n come from the parameters of 2.7 MeV equilibrium orbit, where the corresponding v_0 is very close to 1. The result is shown in Fig. 5.

6 Summary

In this work, we have proposed approximate formulas for the calculation of transverse oscillation, around the equilibrium orbit, using the LP method. These formulas have been further confirmed from the nice correlation of numerical and LP-based results. Moreover, these formulas are highly accurate for the calculation of transverse oscillation and oscillation frequency. Finally, we discussed the transverse oscillation at $v_0=\frac{N}{2}$, where the amplitude increases with azimuth.

The numerical method can be used to study the transverse oscillation precisely, but there are some limitations. Firstly, it is unable to directly demonstrate the influence of magnetic field structures on motions of particles. Secondly, in a given magnetic field with fixed working diagrams, particles may not cross the resonance lines that are proposed to be studied, or they do passed them but the influence imposed on motions of particles is not obvious and thus insufficient information can be obtained. However, the analytical solution seems more flexible. The transverse oscillation of particles under any resonance line can be studied by only changing the parameters of the magnetic field in the formulas concerned, with no necessity to change the whole magnetic field. The influence of the N^{th} harmonic on the transverse oscillation can be studied by changing the amplitude of the N^{th} harmonic in the given magnetic field.

The method of calculating transverse oscillation given in this paper can also be applied in the transverse oscillation equation with high order terms and coupling terms, and more information about the high-order resonances and coupling resonances may be obtained.

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