Legendre transformations and the thermodynamic geometry of 5D black holes [*]

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Abstract: This paper studies the thermodynamic properties of the 5D black hole in Einstein-Gauss-Bonnet gravity from the viewpoint of geometrothermodynamics. It is found that the Legendre invariant metrics of the 5D black holes in Einstein-Yang-Mills-Gauss-Bonnet theory and Einstein-Maxwell-Gauss-Bonnet theory reproduce the behavior of the thermodynamic interaction and phase transition structure of the corresponding black hole configurations correctly. It is shown that they are both curved and that the curvature scalar provides information about the phase transition point.

Key words: black hole, Legendre invariance, curvature scalar, phase transition

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1   Introduction

In 1975, Weinhold proposed a geometrical way of studying thermodynamics and gave a Riemannian metric, defined as the second derivatives of internal energy [1]. According to Weinhold's theory, an interesting inner product of the equilibrium thermodynamic state space in the energy representation was the Hessian matrix of internal energy U with respect to the extensive thermodynamic variables N^a, namely $g_{ij}^W = \partial_i \partial_j M(U, N^a)$. However, there was no physical interpretation associated with this metric structure [2]. As a modification, Ruppeiner introduced a Riemannian metric $(g_{ij}^R = -\partial_i \partial_j S(U, N^a))$ into the thermodynamic system once more, and defended it as the second derivative of entropy S (here, the entropy is a function of the internal energy U and other extensive variables N^a of a thermodynamic system) [3]. Based on Ruppeiner's theory above, and including the Weinhold metric, Ferrara et al. [4] investigated the critical points of moduli space by using the Weinhold and Ruppeiner metrics. Since then, black hole thermodynamic geometry has been one of the focuses in theoretical physics. Until now, there have been many geometrical descriptions of equilibrium thermodynamics. For example, Aman et al. [5–7] showed the relation between the thermodynamic (Riemannian) curvature of the Reissner-Nordstrom black hole and the higher dimensional black hole. Sarkar et al. [8, 9] gave a brief review on the geometrical method of thermodynamics, and applied this approach to the BTZ black hole and ex-

tremal black holes in string theory. Their studies showed that Ruppeiner geometry can overcome problems such as the covariance and self-consistency of general thermodynamics, which has the phase structure information of the thermodynamic system, and this was applied into all kinds of thermodynamic modes [10–13]. In addition, Ruppeiner gave a systematic discussion on how to make the correct metric choice, and also demonstrated several limiting results matching extreme Kerr-Newman black hole thermodynamics to the two-dimensional Fermi gas, showing that connection to a 2D model is consistent with the membrane paradigm of black holes [14, 15]. With Ruppeiner's thermodynamics geometry theory, it is shown that Ruppeiner geometry can perform the physical meanings of various thermodynamic systems [16–22], such as the ideal gas, the van der Waals gas, and so on. The results revealed the fact that the scalar curvature is zero and the Ruppeiner metric is flat, for the van der Waals gas, while the curvature is non-zero and diverges only as the phase transition take place. The focus of the above problems is on the thermodynamic potential, which is generally believed to be the internal energy rather than the simple mass. But the above studies have shown that Weinhold and Ruppeiner's thermodynamic metrics are not invariant under Legendre transformations.

Recently, Quevedo et al. [23] presented a new formalism of geometrothermodynamics (GTD) as a geometric approach that incorporates Legendre invariance in a natural way, and allows us to derive Legendre invariant me-

trics in the space of equilibrium states. Considering the Legendre invariant, they presented a unified geometry where the metric structure can give a good description of various types of black hole thermodynamics [24–26]. One of the aims of the application of different thermodynamic geometries is to describe phase transitions in terms of curvature singularities. For a thermodynamic system, it is quite interesting to investigate the corresponding relationship between the curvature of the Weinhold metric, the Ruppeiner metric and the Legendre invariant metric, and the phase transitions. In fact the above viewpoints have been applied to various black holes [27–32], even though it is widely believed that the thermodynamic geometry of a black hole is still a most fascinating and unresolved subject at present. The main purpose of the present work is to show that the Legendre invariant metric can be used to correctly reproduce the thermodynamics of the 5D black hole and the EMGB black hole. This case has been analyzed previously by using a different approach in which Legendre invariance is not taken into account [22].

The organization of this paper is outlined as follows. We show the GTD of the 5D black hole in Einstein-Yang-Mills-Gauss-Bonnet (EYMGB) theory in Section 2. Then in Section 3, the GTD of the 5D black hole in Einstein-Maxwell-Gauss-Bonnet (EMGB) theory is described. Section 4 provides some discussions and conclusions. Throughout this paper, the units $c=\hbar=G=1$ are used, and the computer algebra system Mathematica 7.0 was used for most of the calculations.

2   Geometrothermodynamics of the 5D black hole in EYMGB theory

In this section we first describe the black hole solution in Einstein-Yang-Mills-Gauss-Bonnet (EYMGB) theory and then study the thermodynamic properties in the next subsection. In EYMGB gravity theory, the 5D spherically symmetric solution obtained recently by Mazharimousavi and Halisoy [33] has the metric ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \qquad (1)$$

in which the S^3 line element can be expressed in the alternative form

$$d\Omega_3^2 = \frac{1}{4}(d\theta^2 + d\phi^2 + d\psi^2 - 2\cos\theta d\phi d\psi), \qquad (2)$$

where

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi, \quad \psi \leqslant 2\pi.$$

The energy-momentum tensor

$$T_{\mu\nu} = 2F_\mu^{i\alpha}F_{\nu\alpha}^i - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}^i F^{i\alpha\beta}, \qquad (3)$$

where $F_{\alpha\beta}^i$ is the Yang-Mills field 2-forms such that $F_{\alpha\beta}^i F^{i\alpha\beta} = 6Q^2/r^4$, Q is the only non-zero gauge charge. The modified Einstein equations in EYMGB theory are [33]

$$G_{\mu\nu} - \alpha\left[\frac{1}{2}g_{\mu\nu}(R_{\kappa\lambda\rho\sigma}R^{\kappa\lambda\rho\sigma} - 4R_{\rho\sigma}R^{\rho\sigma} + R^2) - 2RR_{\mu\nu}\right.$$
$$\left. + 4R_\mu^\lambda R_\nu^\lambda + 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - R_\mu^{\rho\sigma\lambda}R_{\nu\rho\sigma\lambda}\right] = T_{\mu\nu}.$$

The solution of field equations is given by

$$f(r) = 1 + \frac{r^2}{4\alpha} \pm \sqrt{\left(\frac{r^2}{4\alpha}\right)^2 + \left(1 + \frac{M}{2\alpha}\right) + \frac{Q^2\ln r}{\alpha}}, \quad (4)$$

in which M is the usual integration constant to be identified as mass. In the limit $\alpha \to 0$, the Einstein-Yang-Mills solution was obtained with M as a mass of the system, provided the negative branch of the above solution is chosen as the form [22, 33]

$$f(r) \to 1 - \frac{M}{r^2} - \frac{2Q^2\ln r}{r^2}. \qquad (5)$$

The metric coefficient $f(r)$ in equation (3) is identical to that of the Einstein-Yang-Mills solution, and hence "M" is interpreted as the mass of the system. In Eq. (4) the expression within the square root is positive definite for $\alpha > 0$, while the geometry has a curvature singularity at the surface $r = r_{\min}$ for $\alpha < 0$. Here, $r_{\min}$ is the minimum value of the radial coordinate such that the function under the square root is positive. Moreover, according to the values of the parameters (M, Q, α), the singular surface can be surrounded by an event horizon with radius r_h. However, if no event horizon exists, there will be a naked singularity.

Now the metric described by equation (4) has a singularity at the greatest real and positive solution (r_s) of the equation

$$\frac{r^4}{16\alpha^2} + \left(1 + \frac{M}{2\alpha}\right) + \frac{Q^2\ln r}{\alpha} = 0. \qquad (6)$$

Note that if Eq. (4) has no real positive solution, then the metric diverges at $r = 0$. However, the singularity is surrounded by the event horizon r_h, which is the positive root of (the larger one if there are two positive real roots)

$$r^2 - M - 2Q^2\ln r = 0. \qquad (7)$$

If $r_s < r_h$, then the singularity will be covered by the event horizon, while the singularity will be naked for $r_s \geqslant r_h$. In this connection one may note that the event horizon is independent of the coupling parameter α.

As the event horizon r_h satisfies Eq. (7), we have

$$M = r_h^2 - 2Q^2\ln r_h. \qquad (8)$$

The surface area of the event horizon is given by $A =$

$2\pi^2 r_{\mathrm{h}}^3$. The entropy of the black hole takes the form as

$$S=\frac{c^3 K_{\mathrm{B}} A}{4G\hbar}=\frac{c^3 K_{\mathrm{B}}\pi^2}{2G\hbar}r_{\mathrm{h}}^3. \qquad (9)$$

Considering $c=\hbar=G=1$, and the Boltzmann constant appropriately [22], Eq. (9) can be expressed as

$$S=r_{\mathrm{h}}^3. \qquad (10)$$

From Eq. (8), M can be obtained as a function of S and Q in the form

$$M=S^{2/3}-\frac{2}{3}Q^2\ln S. \qquad (11)$$

From the energy conservation law of the black hole $\mathrm{d}M=T\mathrm{d}S+\phi\mathrm{d}Q$, the thermodynamic temperature and electric potential of the black hole can be given by

$$T=\left(\frac{\partial M}{\partial S}\right)_Q=\frac{2}{3}S^{-\frac{1}{3}}-\frac{2Q^2}{3S}, \qquad (12)$$

and

$$\phi=\left(\frac{\partial M}{\partial Q}\right)=-4Q\ln S. \qquad (13)$$

Now we turn to using the recent geometric formulation of the extended thermodynamic behavior of the 5D black hole.

The formulation of the GTD of the black hole is based on the use of contact geometry as a framework for thermodynamics [23]. Consider the $(2n+1)$-dimensional thermodynamic phase space $\Im$ coordinates $Z^A=\{\phi,E^a,I^a\}$ with $A=0,\cdots,2n$ and $a=1,\cdots,n$. In ordinary thermodynamics, ϕ corresponds to the thermodynamic potential, and E^a and I^a are the extensive and intensive variables, respectively. The fundamental differential form Θ can then be written in a canonical manner as $\Theta=\mathrm{d}\phi-\delta_{ab}I^a\mathrm{d}E^b$, where δ_{ab} is the Euclidean metric. Consider $\Im$ as a non-degenerate metric $G=G(Z^A)$, and the Gibbs 1-form with $\delta_{ab}=\mathrm{diag}(1,1,\cdots,1)$. The set $(\Im,\Theta,G)$ defines a contact Riemannian manifold if the condition $\Theta\wedge(\mathrm{d}\Theta)^n\neq 0$ is satisfied. This arbitrariness is restricted by the condition that G must be invariant with respect to Legendre transformations. This is a necessary condition for our description of thermodynamic systems being independent of the thermodynamic potential, and implies that $\Im$ must be a curved manifold [23] because the special case of a metric with vanishing curvature turns out to be non-Legendre invariant. The Gibbs 1-form Θ is also invariant with respect to Legendre transformations. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

The thermodynamic phase space is $\Im$, which in the case of the $(2+1)$-dimensional black hole with a coulomb-like field, can be defined as a four-dimensional space with coordinates $Z^A=\{M,S,T,Q\}$, $A=0,\cdots,3$. Eq. (11) represents the fundamental relationship $M=(S,Q)$ from which all thermodynamic information can be obtained,

therefore we would like to consider a four-dimensional phase space $\Im$ with coordinates (M,S,T,Q), a contact 1-form

$$\Theta=\mathrm{d}M-T\mathrm{d}S-\phi\mathrm{d}Q, \qquad (14)$$

and an invariant metric

$$G=(\mathrm{d}M-T\mathrm{d}S-\phi\mathrm{d}Q)^2+(TS+\phi Q)(-\mathrm{d}T\mathrm{d}S+\mathrm{d}\phi\mathrm{d}Q). \qquad (15)$$

The triplet $(\Im,\Theta,G)$ defines a contact Riemannian manifold that plays an auxiliary role in GTD. It is used to properly handle the invariance with respect to Legendre transformations. In fact, for the charged black hole, a Legendre transformation involves in general all the thermodynamic variables M, S, Q, T and ϕ, so that they must be independent of each other as they are in the phase space. We also introduce the geometric structure of the space of equilibrium states ε in the following manner: ε is a two-dimensional submanifold of $\Im$ that is defined by the smooth embedding map φ: $\varepsilon\mapsto\Im$, satisfying the condition that the "projection" of the contact form Θ on ε vanishes, namely $\varphi^*(\Theta)=0$, where φ^* is the pullback of φ, and that G induces a Legendre invariant metric g on ε by means of ε. In principle, any two-dimensional subset of the set of coordinates of $\Im$ can be used to co-ordinative ε. For the sake of simplicity, we will use the set of extensive variables S and Q, which corresponds to the energy representation in ordinary thermodynamics. Then, the embedding map for this specific choice is φ: $\{S,Q\}\mapsto\{M(S,Q),S,Q,T(S,Q),\phi(S,Q)\}$. The condition $\varphi^*(\Theta)=0$ is equivalent to the first law of thermodynamics and the conditions of thermodynamic equilibrium

$$\mathrm{d}M=T\mathrm{d}S+\phi\mathrm{d}Q, \quad T=\frac{\partial M}{\partial S}, \quad \phi=\frac{\partial M}{\partial Q}, \qquad (16)$$

whereas the induced metric becomes

$$g=\left(S\frac{\partial M}{\partial S}+Q\frac{\partial M}{\partial Q}\right)\left(-\frac{\partial^2 M}{\partial S^2}\mathrm{d}S^2+\frac{\partial^2 M}{\partial Q^2}\mathrm{d}Q^2\right). \qquad (17)$$

This metric determines all the geometric properties of the equilibrium space ε. In order to obtain the explicit form of the metric, it is necessary to specify the thermodynamic potential M as a function of S and Q. In ordinary thermodynamics this function is usually referred to as the fundamental equation from which all the equations of state can be derived.

Substituting Eq. (11) into Eq. (17), we can obtain the Legendre metric components of the 5D black hole as

$$g=\frac{4\left(3Q^2-S^{2/3}\right)\left(Q^2-S^{2/3}+2Q^2\ln S\right)}{27S^2}\mathrm{d}S^2$$

$$+\frac{8}{9}\left(Q^2-S^{2/3}+2Q^2\ln S\right)\ln S\mathrm{d}Q^2. \qquad (18)$$

After some calculations, the Legendre metric scalar cur-

vature is obtained

$$\Re=\frac{N}{8(\ln S)^2(S^{2/3}-3Q^2)^2(Q^2\ln S-S^{2/3}+Q^2)^3}. \quad (19)$$

$$N = 27(Q^2-S^{2/3})(3Q^2-S^{2/3})+18(9Q^6+8Q^4S^{2/3}$$

$$-13Q^2S^{4/3}+4S^2)\ln S+24(27Q^6+3Q^4S^{2/3}$$

$$-11Q^2S^{4/3}+S^2)(\ln S)^2+48Q^2S^{4/3}(\ln S)^3. \quad (20)$$

Obviously, the non-flatness of the Legendre metric indicates that the black hole thermodynamics have statistical mechanical interactions. According to Davies' approach [34], the phase transition structure of the 5D black hole can be derived from the heat capacity

$$C_Q=T\left(\frac{\partial S}{\partial T}\right)_Q=\frac{3S(S^{2/3}-Q^2)}{3Q^2-S^{2/3}}. \quad (21)$$

Hence, $C_Q=0$ at $S^{2/3}=Q^2$. Another interesting point is

$$3Q^2=S^{2/3}. \quad (22)$$

At this point, C_Q changes sign and the scalar curvature diverges. Therefore, there is a phase transition which corresponds to this critical point. This behavior is shown in Fig. 1.
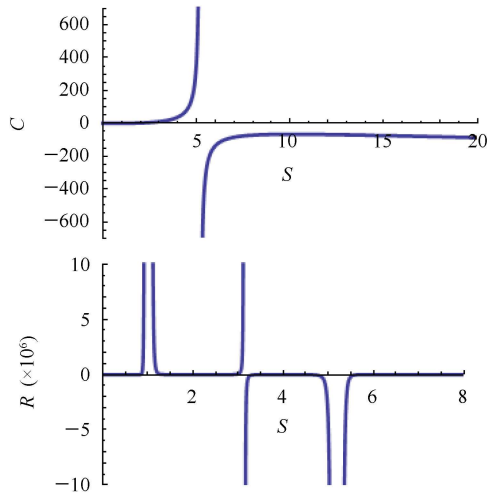

Fig. 1. The behavior of the heat capacity and scalar curvature as functions of the entropy of the black hole with $Q=1$. Their singularities are located at $S\approx5$.

3　Geometrothermodynamics of the 5D black hole in EMGB theory

The action of the five-dimensional space time $(M, g_{\mu\nu})$ that represents Einstein-Maxwell-Gauss-Bonnet theory with a cosmological constant is given by [22, 35, 36]

$$S=\frac{1}{2}\int_M \mathrm{d}^5x\sqrt{-g}\left[R-2\Lambda-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\alpha R_{GB}\right], \quad (23)$$

where

$$R_{\mathrm{GB}}=R^2-4R_{\alpha\beta}R^{\alpha\beta}+R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \quad (24)$$

which is the Gauss-Bonnet term. α is the GB coupling parameter with a dimension (length)^2, Λ is the cosmological constant and $F_{\mu\nu}=(\partial_\mu A_\nu-\partial_\nu A_\mu)$ is the usual electromagnetic field tensor with A_μ, the vector potential.

The five-dimensional spherically symmetric solution obtained has the line element

$$\mathrm{d}s^2=-B(r)\mathrm{d}t^2+B^{-1}(r)\mathrm{d}r^2+r^2(\mathrm{d}\theta_1^2+\sin^2\theta_1(\mathrm{d}\theta_2^2+\sin^2\theta_2\mathrm{d}\theta_3^2)), \quad (25)$$

with

$$0\leqslant\theta_1,\ \theta_2\leqslant\pi,\ 0\leqslant\theta_3\leqslant2\pi.$$

Solving the above field equations, one obtains [22, 34]

$$B(r)=1+\frac{r^2}{4\alpha}-\frac{r^2}{4\alpha}\sqrt{1+\frac{16M\alpha}{\pi r^4}-\frac{8Q^2\alpha}{3r^6}+\frac{4\Lambda\alpha}{3}}. \quad (26)$$

In an orthonormal frame, the non-null components of the electromagnetic tensors $F_{\hat{t}\hat{r}}=-F_{\hat{r}\hat{t}}=Q/4\pi r^3$. Ref. [22] gives an equation about Eq. (26), is well defined for $r>r_{\min}$ and $r_{\min}$ satisfies

$$1+\frac{16M\alpha}{\pi r_{\min}^4}-\frac{8Q^2\alpha}{3r_{\min}^6}+\frac{4\Lambda\alpha}{3}=0. \quad (27)$$

The surface $r=r_{\min}$ corresponds to a curvature singularity. However, depending on the values of the parameters, this singular surface may be surrounded by the event horizon ($B(r_{\mathrm{h}})=0$), and the solution (27) describes a black hole solution known as the EMGB black hole.

The metric depends on two parameters, Q and M, which are identified with the electric charge and the mass, respectively.

$$M=\pi\alpha+\frac{\pi Q^2}{6}r_{\mathrm{h}}^{-2}+\frac{\pi r_{\mathrm{h}}^2}{2}-\frac{\pi\Lambda}{12}r_{\mathrm{h}}^4. \quad (28)$$

From Eq. (10), Eq. (31) can be expressed as

$$M=\pi\alpha+\frac{\pi Q^2}{6}S^{-2/3}+\frac{\pi S^{2/3}}{2}-\frac{\pi\Lambda}{12}S^{4/3}. \quad (29)$$

This equation relates all the thermodynamic variables entering the EMGB black hole so that if we impose the first law of thermodynamics $\mathrm{d}M=T\mathrm{d}S+\phi\mathrm{d}Q$, the expressions for the temperature and the electric potential can be easily computed as $T=\partial M/\partial S$, $\phi=\partial M/\partial Q$. It is convenient to write the final results in terms of the horizon radii by using the relations

$$T=\left(\frac{\partial M}{\partial S}\right)_Q=\frac{\pi}{3}S^{-1/3}-\frac{\pi}{9}Q^2S^{5/3}-\frac{\pi\Lambda}{9}S^{1/3}, \quad (30)$$

and

$$\phi=\left(\frac{\partial M}{\partial Q}\right)=\frac{\pi Q}{3}S^{-2/3}, \quad (31)$$

Substituting Eq. (29) into Eq. (17), we can obtain the Legendre metric components of the EMGB black hole as

$$g = -\frac{\left(-2\pi Q^2-3\pi S^{4/3}+\pi\Lambda S^2\right)\left(-5\pi Q^2+3\pi S^{4/3}+\pi\Lambda S^2\right)}{243S^{10/3}}\mathrm{d}S^2+\frac{\pi\left(2\pi Q^2+3\pi S^{4/3}-\pi\Lambda S^2\right)}{27S^{4/3}}\mathrm{d}Q^2. \tag{32}$$

After some calculations, the Legendre invariant scalar curvature can be expressed as

$$\Re = \frac{54S^{8/3}N}{(2\pi Q^2+3\pi S^{4/3}-\pi\Lambda S^2)(-5\pi Q^2+3\pi S^{4/3}+\pi\Lambda S^2)}, \tag{33}$$

$$N = 27\pi^3(4Q^4-10Q^2S^{4/3}-3S^{8/3})+\pi^3\Lambda S^{2/3}(104Q^4$$

$$-30Q^2S^{4/3}+63S^{8/3})-10\pi\Lambda^3 S^{14/3}$$

$$+\pi\Lambda^3(4Q^2S^{8/3}+6S^2). \tag{34}$$

Obviously, the non-flatness of the Legendre metric indicates that the black hole thermodynamics have statistical mechanical interactions. The expression for the heat capacity with a fixed charge is given by

$$C_Q=\frac{3S\left(\pi Q^2-3\pi S^{4/3}+\pi\Lambda S^2\right)}{-5\pi Q^2+3\pi S^{4/3}+\pi\Lambda S^2}. \tag{35}$$

The heat capacity has a zero-point at $Q^2=3S^{4/3}-\Lambda S^2$.
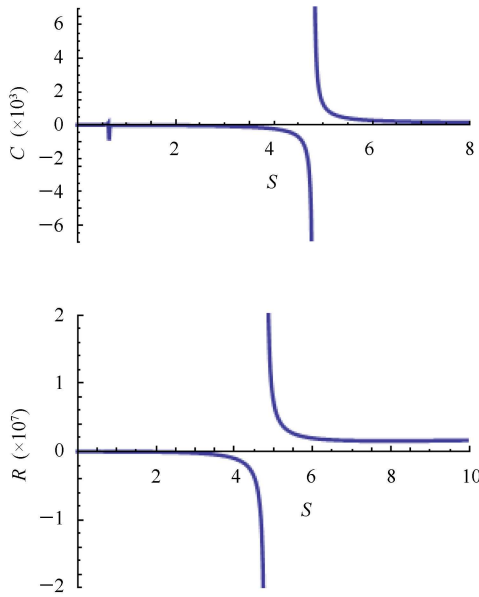

Fig. 2. The behavior of the heat capacity and scalar curvature as functions of the entropy of the black hole with $Q=1/2$. Their singularities are located at $S\approx4.9$.

Moreover, the sign change of the heat capacity and the divergence of the scalar curvature occur at $5Q^2=3S^{4/3}+\Lambda S^2$. Therefore, there will be a phase transition at this point. This behavior is shown in Fig. 2.

4 Discussions and conclusions

In this work, we reproduced thermodynamic properties such as the temperature and entropy of the 5D black hole in Gauss-Bonnet gravity theory. We studied the Legendre invariant metric of the 5D black hole. The results show that GTD delivers a particular thermodynamic metric for the 5D black hole and the EMGB black hole. Then we corroborated that the thermodynamic curvature is non-zero and its singularities reproduce the phase transition structure, which follows from the divergences of the heat capacity.

In addition, the thermodynamic metric proposed in this work was applied to the case of black hole configurations in three dimensions. It was shown that this thermodynamic metric correctly describes the thermodynamic behavior of the corresponding black hole configurations. One additional advantage of this thermodynamic metric is its invariance with respect to total Legendre transformations. This means that the results are independent of the thermodynamic potential used to generate the thermodynamic metric. In all the remaining cases, the singularities of the thermodynamic curvature correspond to points where the heat capacity diverges and phase transitions takes place. We interpret this result as an additional indication that the thermodynamic curvature, as defined in GTD, can be used to measure the thermodynamic interaction. In fact, it has been shown that in the case of more realistic thermodynamic systems [27], the ideal gas is also characterized by a vanishing thermodynamic curvature, whereas the van der Waals gas generates a nonvanishing curvature whose singularities reproduce the corresponding phase transition structure.

Furthermore, we expect that this unified geometry description may give more information about a thermodynamic system. We also conclude that GTD is, in general, duality invariant. Therefore, our results support Quevedo's viewpoint.

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