# Quark confinement and the fractional quantum Hall effect *

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**Abstract** Working in the physics of Wilson factor and Aharonov-Bohm effect, we find in the fluxtube-quark system the topology of a baryon consisting of three heavy flavor quarks resembles that of the fractional quantum Hall effect (FQHE) in condensed matter. This similarity yields the result that the constituent quarks of baryon have the "filling factor" 1/3, thus the previous conjecture that quark confinement is a correlation effect is confirmed. Moreover, by deriving a Hamiltonian of the system analogous to that of FQHE, we predict an energy gap for the ground state of a heavy three-quark system.

**Key words** QCD, confinement, Wilson factor, Aharonov-Bohm effect, fractional quantum Hall effect

**PACS** 12.38.Aw, 12.39.Jh

## 1 Introduction

Understanding the confinement of quarks in hadrons remains a challenge, notwithstanding a mature quantum chromodynamics (QCD). The attempts to explain why quarks are confined in hadrons started from t'Hooft's work a quarter of a century ago[1], and various mechanisms within QCD have been proposed ever since[2—4]. In no mechanisms can we escape the task to reduce the effective degrees of freedom of QCD using some gauge fixing conditions to separate the physical and unphysical parts. Unlike QED, in a non-Abelian gauge field like QCD, the physical results can depend on the gauge conditions, and in general the ghost fields cannot be decoupled from the vector fields. To understand quark confinement, QCD has been formulated in several gauges. Under the maximum Abelian gauge fixing condition it has been proved that the appearance of magnetic monopoles[1] can provide a picture to explain why $q\bar{q}$ is confined[2]. The binding of a quark and an anti-quark can be understood as the result of condensation of monopoles that squeeze the force between a quark and an anti-quark to a string, known as linear confining potential. The other most widely used gauge is the Coulomb gauge[3, 4], in which the contribution from ghosts lies in the resultant potential. There are still other gauges used to reduce the gauge symmetry of QCD in discussing the confinement of quarks, such as the Landau gauge[5, 6], the temporal gauge[7] and the axial gauge condition[8]. More directly, some authors lower the dimension of QCD to $QCD_2$[9] or $QCD_3$[10] to reduce the number of gauge degrees and then come back to compare the results with $QCD_4$. But in all of these approaches, we encounter difficulties to derive the interacting potential in the presence of ghost-vector coupling.

There have been quite a few works[11—13] from the angle of condensed matter physics conjecturing the properties of confinement of strong interaction, but one still lacks solid proofs from the viewpoint of particle physics or chromo-dynamics supporting these conjectures. If the efforts of correlating these two sides can be realized, the table-top experiments of condensed matters can be used to test the speculations on hadron physics - where we have encountered many difficulties originating from the confining quarks and hence the nonperturbative properties.

Here we propose a potential model that does not fit into the paradigms of the quark–anti-quark confinement, based on which we take topological properties to explain the general quark-confinement in

Received 20 November 2007, Revised 9 April 2008
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hadrons. Inspired by Laughlin's statement[14] that quark confinement might come from collective excitation, we have succeeded in choosing a reasonable gauge condition resembling the physical conditions for the FQHE to appear (named FQHE gauge hereafter). We find that under the FQHE gauge, the topology of the three-quark color-singlet system is equivalent to the topology of the FQHE with a filling factor of 1/3. In other words, imposing the FQHE to quark systems would induce naturally the topology of the FQHE. Together with the heavy-quark approximation, the FQHE gauge makes the complex interaction between ghosts and vector bosons irrelevant to quark confinement.

## 2 Theoretical formulation of the model

The topology of FQHE[15] depends on that of Aharonov-Bohm effect[16], and thus on closed Wilson lines (named as nonintegrable phase factor in some other references). Since QCD can be equivalently formulated as a theory of Wilson lines, many authors started with Wilson loops to derive the strong interaction between quarks[17—20]. An excellent treatment in the derivation is to study the systems consisting of heavy flavor quarks ($m \gg \Lambda_{\rm QCD}$). These systems are near the perturbative region and the non-relativistic Schrödinger equation is a fairly good approximation to leading order. In this paper, without affecting conclusions, we qualitively make the quarks in a baryon have the same masses.

Before studying the topology of QCD, we first reduce the degrees of freedom of the three-quark baryon system. Our model is a hybrid scenario in the sense of combining the covariant gauge condition $\partial_\mu A^\mu = 0$ and the situation that FQHE happens (FQHE gauge). The FQHE takes place in a 2-dimensional system penetrated by a vertical magnetic field. Since our focus here is on the origin and degeneration of the states in the energy bands, the electric field perpendicular to the magnetic field in FQHE is neglected in our discussion. The introduction of such a gauge scheme forces us to lower the dimension of the interactions in QCD down to 2. However, as shown in the following, the field $A_\mu$ here is generated in a way different from $\rm QCD_2$. In a three-quark color-singlet system, one quark (hereafter referred to as non-local quark) moves in the field induced by the other two quarks (hereafter referred to as di-quark). Evidence for the existence of stable di-quarks came recently from lattice calculations by Alexandrou et al[21]. The magnetic part of the field induced by the di-quark penetrates (in contrast to the case of the FQHE not vertically) instantaneously through a plane in which the non-local quark is located due to momentum and angular momentum conservation. The dynamics of the non-local quark is determined by the normal QCD Lagrangian:

$$\mathcal{L} = \bar{\psi}({\rm i}\,\not{D})\psi - \frac{1}{4}(F^i_{\mu\nu})^2 - m\bar{\psi}\psi \ . \tag{1}$$

Here $D_\mu = \partial_\mu - {\rm i}\,g\,A^a_\mu \dfrac{\lambda^a}{2}$ ($\lambda^a$ are the Gellmann matrices) and $F^i_{\mu\nu}$ is defined as

$$F_{\mu\nu} \ = \ F^a_{\mu\nu}\frac{\lambda^a}{2} = \partial_\mu A^a_\nu \frac{\lambda^a}{2} - \partial_\nu A^a_\mu \frac{\lambda^a}{2} - {\rm i}\,g\left[A^a_\mu \frac{\lambda^a}{2}, A^b_\nu \frac{\lambda^b}{2}\right], \tag{2}$$

where $A_\mu = A^a_\mu \dfrac{\lambda^a}{2}$ is the vector field induced by the di-quark. Gluons are assumed to be massless here. There are two reasons for us to choose such an ordinary Lagrangian for the non-local quark. First, the di-quark interacts as a whole with the non-local quark. Second, under the condition that the three-quark system should be in a color-singlet state, the di-quark behaves (with respect to color) like an anti-quark of the non-local quark.

The covariant gauge fixing $\partial_\mu A^\mu = 0$ is imposed before the non-local quark is required to be located in a plane. To simulate the situation of FQHE, we further idealize the scalar component $A_0$ in $A_\mu = (A_0, \boldsymbol{A})$ to be nontrivial only at infinity or the central interaction region (Pauli principle) of the plane. Since we discuss only heavy quarks ($m \gg \Lambda_{\rm QCD}$) that need not move too far away from the center of the field induced by the di-quark to meet the infrared requirement, $A_0$ is taken as zero in what follows. With $\partial_\mu A^\mu = 0$ and $A_0 \approx 0$, the spatial degrees of freedom of $A_\mu$ are reduced to two1). Under the heavy-quark approximation, the tree level diagram plus renormalization give a good description of the scattering amplitude and the changes of wave functions. The ghosts' contribution to the wave function, which appears only in higher order corrections and hence in the renormalization, may not be relevant in our case because the renormalization will not affect the topology.

Now, let's turn to the topology of QCD. With the reduced degrees of freedom, we can write the field strength Eq. (2) of the di-quark as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} - {\rm i}g\,\boldsymbol{A} \times \boldsymbol{A} \ , \tag{3}$$

We note that $\boldsymbol{A}$ lies in the plane, and the second term

1) Here we assume the remaining degrees are just that lying in the plane, e.g., $A_x$ and $A_y$ at any sufficiently short moment.

prevents $\boldsymbol{B}$ from being perpendicular to the plane[1)]. Since the quark moves in the plane within a restricted region, its paths (wave) form loops. It therefore justifies our use of the form of closed Wilson lines in non-Abelian fields. In order to show the analogy to the FQHE, let's study the Wilson line along an infinitesimal closed path. In QCD, the Wilson line is defined as[14]

$$U_P(x,y)=P\left[\mathrm{e}^{\mathrm{i}g\int_y^x A_\mu(z)\mathrm{d}z^\mu}\right], \tag{4}$$

where the symbol $P$ is responsible for the path ordering due to the noncommutativity of different $A_\mu$'s in a non-Abelian field. Except for the noncommutativity of different parts of the path, Eq. (4) is usually path-dependent, just as in QED in the presence of a uniform magnetic field. Thus we use a superscript $s$ to denote a particular path. The differential equation for $U_P(x,y)$ that holds in QED also works in QCD along any path[22]:

$$D_\mu^s U_P(x,y)=0\ . \tag{5}$$

The denotation $s$ here indicates that the differential is taken along an arbitrary path $s$. With Eq. (5) it is straightforward to verify that the Wilson line Eq. (4) acts as an evolution factor of the wave function for massless quarks

$$\psi_P^s(x,x_0)=U_P^s(x,x_0)\psi(x_0)\ , \tag{6}$$

where $\psi(x,x_0)$ is the solution of the Dirac equation

$$(\mathrm{i}\,\partial\!\!\!/ - g\,A\!\!\!/)\psi(x,x_0)=0\ , \tag{7}$$

and $\psi(x_0)$ satisfies $(\mathrm{i}\,\partial\!\!\!/ - m)\psi(x_0)=0$ with $m=0$. If $m\neq 0$ in Eq. (7), the mass term would contribute nothing to the phase factor when the path $x_0\rightarrow x(x_0)$ forms a loop. Therefore, when dealing with closed paths, we take in the following also for the massive quarks $U_P(x,y)$ as the evolution factor of the wave function.

A loop (closed path)

$$U_P(x,x)=P\left[\mathrm{e}^{\mathrm{i}g\oint A_\mu(z)\mathrm{d}z^\mu}\right] \tag{8}$$

defined on the basis of Eq. (4) in a non-Abelian field is not gauge invariant. So, in order to construct a gauge invariant Lagrangian of the gauge field, one calculates the trace of Eq. (8) which is a gauge invariant quantity. Such a quantity is defined as Wilson loop, from which the effective interaction potential between heavy quarks can be obtained[18—20]. Here, instead of calculating the trace, we will focus on Eq. (8) to investigate the topological property of the wave function of Eq. (6). Applying Stokes's theorem to a loop, we can rewrite Eq. (8) as

$$U_P(x,x)=\mathrm{e}^{\mathrm{i}\frac{g}{2}\int_\Sigma F_{\mu\nu}(z)\mathrm{d}\sigma^{\mu\nu}}\ , \tag{9}$$

where $\Sigma$ is the infinitesimal surface inside the closed loop $P$, $\mathrm{d}\sigma^{\mu\nu}$ is a differential surface element, and $F_{\mu\nu}$ is the field strength as defined in Eq. (2). Note that here in our selected plane, $F_{\mu\nu}$ reduces to the form shown in Eq. (3). Thus the phase factor in Eq. (9) has now the form $\int_\Sigma \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$, where $\mathrm{d}\boldsymbol{S}$ replaced the $\mathrm{d}\sigma^{\mu\nu}$. Similar to the Aharonov-Bohm phase in QED, Eq. (9) can be written as

$$U_P(x,x)=\mathrm{e}^{\mathrm{i}\frac{g}{2}\phi_0^a\lambda^a}\ , \tag{10}$$

where $\phi_0^a=F_{xy}^a\varepsilon^2$ is the color flux and $\varepsilon^2$ denotes the infinitesimal area enclosed by the loop.

A quantitative discussion of the Aharonov-Bohm phase of a non-Abelian field needs the definition of a unit of $\phi_0^a$. Let $\Omega(x)=\exp\left[\mathrm{i}\,\alpha^a(x)\frac{\lambda^a}{2}\right]$ be an infinitesimal transformation near the identity. The gauge transformation of the vector field $A_\mu$ is $A_\mu\rightarrow \Omega(x)\left(A_\mu+\frac{1}{g}\partial_\mu\right)\Omega^\dagger(x)$, which can be derived from the corresponding transformation of the Wilson line $U(x+\varepsilon,x)\rightarrow\Omega(x+\varepsilon)U(x+\varepsilon,x)\Omega^\dagger(x)$ by collecting all coefficients of the terms linear in $\varepsilon$. The transformation of the Wilson line $U(x+\varepsilon,x)$ originates from the gauge transformation of Eq. (6). Now let's consider the integral in Eq. (8) after performing a gauge transformation on $A_\mu$. The original gauge field $A_\mu(x)$ is regular, and so is $\Omega(x)A_\mu(x)\Omega^\dagger(x)$. The term $\Omega(x)\frac{1}{g}\partial_\mu\,\Omega^\dagger(x)$ is singular which can be seen when it is rewritten as $-\frac{1}{g}\partial_\mu\ln\Omega(x)$ by using $\partial_\mu(\Omega\Omega^\dagger)=0$. The analysis suggests that in the integral of Eq. (8) only the singular term $\Omega(x)\frac{1}{g}\partial_\mu\,\Omega^\dagger(x)$ remains. Expanding the transformation of $A_\mu$ and collecting the coefficients of the terms linear in generators $\lambda^a$, one has

$$A_\mu^a\rightarrow A_\mu^a+\frac{1}{g}\partial_\mu\,\alpha^a+f^{abc}A_\mu^b\alpha^c\ . \tag{11}$$

From this it seems to be natural to take $\frac{1}{g}$ as the unit of $\phi_0^a$. This is so because the phase factor of Eq. (8) can be rewritten as an integral with integrand $\frac{1}{g}\frac{\partial_\mu\,\alpha^a\lambda^a}{2}$,

$$\mathrm{e}^{\mathrm{i}g\oint\frac{1}{g}\partial_\mu\,\alpha^a\frac{\lambda^a}{2}\mathrm{d}x^\mu}=\mathrm{e}^{\mathrm{i}\oint\frac{\lambda^a}{2}\mathrm{d}\alpha^a}\ , \tag{12}$$

in which $\alpha^a$ becomes formally identical with a winding angle.

1) Since we concentrate on the stable states of confinement, quantities varying with time are omitted, i.e., $\boldsymbol{E}=-\frac{\partial\boldsymbol{A}}{\partial t}=0$

In our case, the integral path in the evolution factor of Eq. (6) should form a closed path, i.e., $\psi_P^s(x,x_0) = U_P^s(x,x)\psi(x,x_0)$. To make the definitions of the evolution factor $U_P^s(x+\varepsilon,x)$ and the infinitesimal gauge transformation $\Omega(x)$ for a wave function $\psi(x,x_0)$ consistent along a definite closed path, $U_P^s(x,y)$ is required to be single-valued along the closed path. We will show that the Eq. (6) can be consistent if the topology of the evolution factor mimics that of the Aharonov-Bohm effect. Let's put a particular color to the non-local quark in the plane, then the symmetry of the vector field $A_\mu$ produced by the di-quark would be $SU(2)$, and the group generators in the phase factor reduce from those of $SU(3)$ to the Pauli matrices $\tau^i$. For a closed path, the evolution factor in Eq. (12) would be

$$\mathrm{e}^{\mathrm{i}\oint\frac{\lambda^a}{2}\mathrm{d}\alpha^a}\rightarrow\mathrm{e}^{\mathrm{i}\oint\frac{\tau^k}{2}\mathrm{d}\alpha^k}\rightarrow\mathrm{e}^{i\pi\boldsymbol{n}\cdot\boldsymbol{\tau}}, \tag{13}$$

where $\boldsymbol{n}=(n_1,\ n_2,\ n_3)$, the $n_i$ being real for any given $i(=1,2,3)$. By applying $\mathrm{e}^{\mathrm{i}\,\boldsymbol{A}\cdot\boldsymbol{\tau}}=\cos A+\mathrm{i}\dfrac{\boldsymbol{A}\cdot\boldsymbol{\tau}}{A}\sin A$ (where $A=|\boldsymbol{A}|$), we see that $\mathrm{e}^{\mathrm{i}\pi\boldsymbol{n}\cdot\boldsymbol{\tau}}=1$ only if $|\boldsymbol{n}|=2k$($k$-integer). An equivalent, but easier way to understand the result is making only one of the $n_i$'s nonvanishing, e.g. $n_1=n_2=0$ and $n_3=2k$, while the physical meaning of $\mathrm{e}^{\mathrm{i}\pi\boldsymbol{n}\cdot\boldsymbol{\tau}}$ will not change with this particular choice. To put it in another way, only if the non-local quark has loops with even winding number does the nonintegrable phase in Eq. (13) have the same meaning as in the Aharonov-Bohm effect, and at the same time can the evolution factor of Eq. (6) for closed paths be defined uniquely and consistently. In a geometrical picture, the non-local quark moves around a flux tube of two units, $\phi_{xy}^a=2\phi_0^a$, but in fact it winds along the closed edge of a Möbius-strip[23] around a flux tube with only one unit. So in the language of fractional charge[24], the non-local quark is bound to a flux tube $2\phi_0^a$ to form a "composite particle". In the topological scenario[15] for the FQHE, we find that the filling factor $1/(2m+1/p)$ fits our model with $m=1$ and $p=1$, hence we obtain a filling factor 1/3 for the non-local quark.

In FQHE with filling factor 1/3, an energy band is already full when the number of accommodated electrons reaches one third of the number of total states and any further filling to the same band is forbidden. If we view the three states in a baryon as a degenerate energy band, a quark with a filling factor 1/3 can in fact fill all these three energy levels. This means that one quark has effectively occupied three states and any further filling by quarks with the same color is not allowed. Nevertheless, since all the three binding quarks share the $SU(3)$ symmetry, the filling factor for one defined non-local quark is simultaneously applicable to the other two quarks. As a result, every single one of the three quarks occupies three states (See Fig. 1). We cannot diagonalize the quarks' states by conventional transformation between representations. The quarks (states) appear only as an entirety containing all three colors simultaneously, but cannot be separated (See Fig. 2). Note that we initially assigned a particular color to the non-local quark, but finally the non-local quark lost its identity. The reason is that its color has been distributed over three states, as illustrated in Fig. 2.

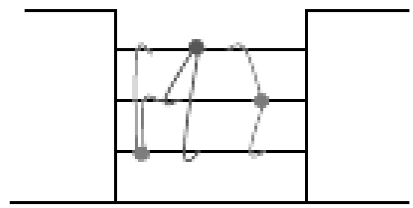

Fig. 1. In a three-quark color-singlet state, each of the three quarks occupies three states simultaneously.

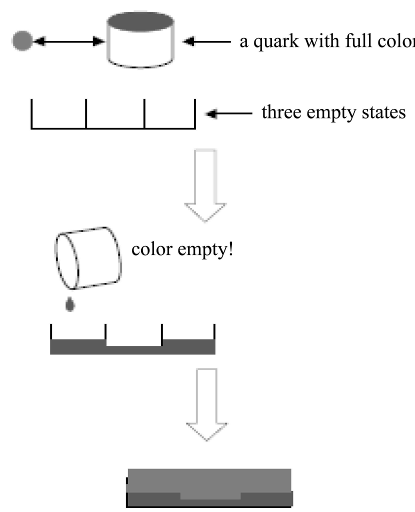


Fig. 2. In a three-quark color-singlet state, the three quarks are viewed as an entirety with three colors that cannot be separated. The color forment appears at arXiv: 0708.3538.

Similarly, the Hamiltonian method used in the study of the FQHE can also be applied to quarks. If the color of the non-local quark has been specified, the Dirac Eq. (7) will reduce to the same form as $(\mathrm{i}\,\partial\!\!\!/-g\,A\!\!\!/-m)\psi(x,x_0)=0$ with only two colors appearing in the wave function $\psi^{\mathrm{Transpose}}=(\psi^a,\psi^b)$ ($a$, $b$ denote the un-specified colors). For heavy quarks, the approximation based on considering $|\boldsymbol{p}|/m_{\mathrm{quark}}$ as a small quantity is always possible. Since in our FQHE gauge $A_\mu$ is projected onto a plane and its scalar part is vanishing, it can be shown by following the standard approximation procedure in QED[25] that the large components of the wave functions $\psi^a,\psi^b$ satisfy the following Schrödinger-Pauli equation,

$$\mathrm{i}\frac{\partial}{\partial t}\psi=-\frac{1}{2m}[\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}-\mathrm{i}g\boldsymbol{A})]^2\psi. \tag{14}$$

Here, the wave function $\psi$ has the same form as $\psi^{\text{Transpose}}=(\psi^a,\psi^b)$, but only the first two spinor components of each color remain; $\sigma^i$ are Pauli matrices for spins; and $A_{x,y}=A^a_{x,y}\tau^a$. Additionally, in the FQHE the spin effect in the Hamiltonian is suppressed[26]. In the heavy quark limit it is reasonable to assume that both spin components satisfy the same equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi^{1,2}=H\psi^{1,2},\quad H=-\frac{1}{2m}(\boldsymbol{\nabla}-\mathrm{i}g\boldsymbol{A})^2, \tag{15}$$

where $\psi^{a,b}=[(\psi^1,\psi^2)_{a,b}]^{\text{Transpose}}$. This Hamiltonian is just the same as that used in the FQHE, from which the energy gap for the ground state can be evaluated as $\Delta\sim\frac{e^2}{l_{\mathrm{B}}}$, where $l_{\mathrm{B}}$ is the magnetic length defined by $l_{\mathrm{B}}^2=\left|\frac{1}{q_{\mathrm{e}}B}\right|$[26]. In the case of quarks, the energy gap for the ground state is $\Delta\sim\frac{g^2}{l^a_{\mathrm{B}}}$, with $(l^a_{\mathrm{B}})^2=\left|\frac{1}{q_gF^a_{\mu\nu}}\right|$. This means that the field induced by the di-quark determines the energy gap for the three-quark-system.

## 3 Outlook and discussion

The analogy between the three-quark baryon states and the FHQE studied in this work is a natural extension to previous assumptions and perceptions[27—29]. We expect this work to open new windows for further investigation on nonperturbative properties of quarks. Here, we have dealt with the non-local quark and its planar motion under the heavy-quark approximation $|\boldsymbol{p}|/m_{\text{quark}}\ll 1$. This ensures the credibility of the model and its relation to QCD is comparable with quark potential models.

Our model is valid for a non-Abelian field with symmetry group $SU(3)$, but not for any other $SU(N)$ field of strong interactions. In this sense, the FQHE, which in 2-dimensions appears as an artificially designed condensed matter property, seems to be an intrinsic characteristics of quarks. The difference is that the FQHE in condensed matter is a cooperation involving a great number of particles whereas a hadron contains only a few constituent quarks. We stress that the correlation effect is a crucial point to understand the tri-quark confinement. Along this way, the implication of other fractional states of the FQHE will be instructive in understanding quark states of other possible types. Applying the chiral-symmetry of 2-dimensional graphene[30] to light hadrons may shed light on the nonperturbative properties of light quarks.

*H.J.W. is grateful to Prof. S. S. Wu and Prof. Yu-Xin Liu for encouraging discussions.*

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