Exact solution to two-dimensional isotropic charged harmonic oscillator in uniform magnetic field in non-commutative phase space*

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Abstract In this paper, the isotropic charged harmonic oscillator in uniform magnetic field is researched in the non-commutative phase space; the corresponding exact energy is obtained, and the analytic eigenfunction is presented in terms of the confluent hypergeometric function. It is shown that in the non-commutative space，the isotropic charged harmonic oscillator in uniform magnetic field has the similar behaviors to the Landau problem.

Key words non-commutative quantum mechanics, isotropic charged harmonic oscillator, exact solution

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1 Introduction

In the past few years, problems in non-commutative space have attracted much interest and attention[1—4]. This is mainly because of the study of open string attached to D-brane in the presence of background B-field inducing non-commutativity in its end points[5—8] and the research of Hall effect[9] presenting non-commutativity in the canonical coordinates and momentum. One way to deal with the non-commutative space is to construct a new kind of field theory, changing the standard product of the fields by the star product (Weyl-Moyal):

$$(f*g)(x)=\exp\left(\frac{\mathrm{i}}{2}\theta_{ij}\,\partial_i\,\partial_j\right)f(x)g(y)\big|_{x=y}\,. \qquad (1)$$

Here the constant parameter θ_{ij} which is the real and anti-symmetric matrix elements represents the non-commutativity of the space; f and g are the infinitely differentiable functions. In this theory some interesting results have been found[10]. Another approach is to assume the relation rules:

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\theta_{ij},\quad [\hat{x}_i,\hat{p}_j]=\mathrm{i}\hbar\delta_{ij},\quad [\hat{p}_i,\hat{p}_j]=0\,. \qquad (2)$$

Thus, a non-commutative quantum mechanics can be formulated, of which some relevant results have already been obtained[11—14] by the perturbation theory in non-commutative space.

However, most physics problems in non-commutative space are approximately solved by the perturbation methods. In the present paper, the Hamiltonian of isotropic charged harmonic oscillator in uniform magnetic field is presented in the non-commutative phase space; the corresponding exact energy is obtained, and the analytic eigenfunction is presented in terms of the confluent hypergeometric function. It is shown that the isotropic charged harmonic oscillator in uniform magnetic field in non-commutative phase space has the similar behaviors to the Landau problem.

This paper is organized as follows. For comparing result with the one in commutative space, in Section 2, the isotropic charged harmonic oscillator in uniform magnetic field is researched in commutative space. In Section 3, the isotropic charged harmonic oscillator in uniform magnetic field is exactly solved in non-commutative phase space; the corresponding

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exact energy and analytic eigenfunction are obtained respectively, meanwhile the non-commutative effect is discussed.

2 Isotropic charged harmonic oscillator in commutative space

Considering an isotropic charged harmonic oscillator with electric charge q and mass m moves in a two-dimensional plane under a uniform magnetic field B perpendicular to the plane and the vector potentials have the following form[15]:

$$A_1 = -\frac{1}{2}Bx_2, \quad A_2 = \frac{1}{2}Bx_1 . \tag{3}$$

The Hamiltonian of the system is

$$\begin{aligned} H = &\frac{1}{2m}\left[\left(p_1 + \frac{qB}{2c}x_2\right)^2 + \left(p_2 - \frac{qB}{2c}x_1\right)^2\right] + \\ &\frac{1}{2}m\omega^2(x_1^2 + x_2^2) = H_0 - \frac{qB}{2mc}L_z . \end{aligned} \tag{4}$$

With

$$H_0 = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega_0^2}{2}(x_1^2 + x_2^2),$$

$$\omega_0 = \sqrt{\frac{q^2B^2 + 4m^2\omega^2c^2}{4m^2c^2}}, \quad L_z = x_1p_2 - x_2p_1 .$$

Considering $[H_0, L_z] = 0$, the eigenfunction of H can take the collective eigenstate of (H_0, L_z). In order to solve the eigenfunction of Schrödinger equation conveniently, let us take the pole coordinates in our discussion.

Letting

$$\varphi(\rho, \phi) = \chi(\rho)\mathrm{e}^{\mathrm{i}m_1\varphi}, \quad m_1 = 0, \pm 1, \pm 2, \cdots \tag{5}$$

and substituting Eqs. (4), (5) into $H\varphi = E\varphi$, by separating variables, one can get the radial equations as follows:

$$\begin{aligned} &\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\right) + \frac{1}{2}m\omega_0^2\rho^2\right] - \\ &\frac{qB\hbar}{2mc}m_1\chi(\rho) = E\chi(\rho). \end{aligned} \tag{6}$$

Eq. (6) is the famous Schrödinger radial equation of Landau problem. The energy of the system can be given as follows[16]

$$E_{n_\rho m_1} = \hbar\omega_0(2n_\rho + |m_1| + 1) + m_1\hbar\frac{qB}{2mc} , \tag{7}$$

with $n_\rho = 0, 1, 2, \cdots, m_1 = 0, \pm 1, \pm 2, \cdots$.

The corresponding radial eigenfunction is given by (unnormalized)

$$\chi(\rho) \sim \rho^{|m_1|}F(-n_\rho, |m_1| + 1, \beta^2\rho^2)\exp[-\beta^2\rho^2/2], \tag{8}$$

where $F(-n_\rho, |m_1| + 1, \beta^2\rho^2)$ is the confluent hypergeometric function and $\beta^2 = \frac{qB}{2\hbar c}$.

Then the eigenfunction of the system is

$$\begin{aligned} \varphi(\rho, \phi) = &N\rho^{|m_1|}F(-n_\rho, |m_1| + 1, \beta^2\rho^2)\cdot \\ &\exp[-\beta^2\rho^2/2]\mathrm{e}^{\mathrm{i}m_1\phi} , \\ &n_\rho = 0, 1, 2, \cdots ,\ m_1 = 0, \pm 1, \pm 2, \cdots , \end{aligned} \tag{9}$$

where N is the normalized constant.

3 Isotropic charged harmonic oscillator in non-commutative phase space

Although in the string theory only the coordinate's space exhibits a non-commutative structure, considering the momentum is the partial derivatives of the action with respect to the non-commutative spatial coordinates, naturally, the momentum's space also exhibits a non-commutative structure[17]. In order to describe a non- commutative phase space, referring to Ref. [1] the commutation relations in Eq. (2) should be changed as follows:

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = \mathrm{i}\bar{\theta}_{ij} , \tag{10}$$

with θ_{ij} and $\bar{\theta}_{ij}$ being the anti-symmetric matrixes with very small elements representing the non-commutative property of space in the non-commutative phase space.

In the following discussion, $\hat{F}$ denotes the variables in the non-commutative phase space in order to distinguish the variables F in commutative space. According to Ref. [1], one possible way of implementing algebra Eq. (10) is to construct the non-commutative variables $\{\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2\}$ from the commutative variables $\{x_1, p_1, x_2, p_2\}$ by the following means of linear transformations:

$$\hat{x}_i = \alpha x_i - \frac{1}{2\alpha\hbar}\theta_{ij}p_j, \quad \hat{p}_i = \alpha p_i + \frac{1}{2\alpha\hbar}\bar{\theta}_{ij}x_j . \tag{11}$$

In order to maintain the Bose-Einstein statistics, parameters θ, $\bar{\theta}$ and α must satisfy the relation as follows:

$$\bar{\theta} = 4\hbar^2\alpha^2(1 - \alpha^2)/\theta . \tag{12}$$

Now considering an isotropic charged harmonic oscillator in a two-dimensional non-commutative phase space, the Schrödinger equation can be written as

$$\begin{aligned} \hat{H}(\hat{x}, \hat{p})\psi = &\hat{H}\left(\alpha x_i - \frac{1}{2\alpha\hbar}\theta_{ij}p_j,\ \alpha p_i + \right. \\ &\left.\frac{1}{2\alpha\hbar}\bar{\theta}_{ij}x_j\right)\psi = E\psi . \end{aligned} \tag{13}$$

And the Hamiltonian of the system takes the form

$$\begin{aligned}\hat{H} = &\frac{1}{2m}(\hat{p}_1^2+\hat{p}_2^2)+\frac{q^2B^2+4m^2\omega^2c^2}{8mc^2}(\hat{x}_1^2+\hat{x}_2^2)-\frac{qB}{2mc}(\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1)=\\ &\frac{1}{2m}\left[\left(\alpha p_1+\frac{\bar\theta}{2\alpha\hbar}x_2\right)^2+\left(\alpha p_2-\frac{\bar\theta}{2\alpha\hbar}x_1\right)^2\right]+\frac{q^2B^2+4m^2\omega^2c^2}{8mc^2}\left[\left(\alpha x_1-\frac{\theta}{2\alpha\hbar}p_2\right)^2+\left(\alpha x_2+\frac{\theta}{2\alpha\hbar}p_1\right)^2\right]-\\ &\frac{qB}{2mc}\left[\left(\alpha x_1-\frac{\theta}{2\alpha\hbar}p_2\right)\left(\alpha p_2-\frac{\bar\theta}{2\alpha\hbar}x_1\right)-\left(\alpha x_2+\frac{\theta}{2\alpha\hbar}p_1\right)\left(\alpha p_1+\frac{\bar\theta}{2\alpha\hbar}x_2\right)\right]=\\ &\left[\frac{\alpha^2}{2m}+\frac{qB\theta}{4mc\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta^2}{32m\alpha^2\hbar^2c^2}\right](p_1^2+p_2^2)+\left[\frac{\alpha^2(q^2B^2+4m^2\omega^2c^2)}{8mc^2}+\frac{qB\bar\theta}{4mc\hbar}+\frac{\bar\theta^2}{8m\alpha^2\hbar^2}\right](x_1^2+x_2^2)-\\ &\left[\frac{qB}{2mc}+\frac{\bar\theta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right](x_1p_2-x_2p_1)\,. \end{aligned}\tag{14}$$

In order to calculate conveniently, letting

$$\begin{cases}\tilde{m}=\left[\dfrac{\alpha^2}{m}+\dfrac{qB\theta}{2mc\hbar}+\dfrac{(q^2B^2+4m^2\omega^2c^2)\theta^2}{16m\alpha^2\hbar^2c^2}\right]^{-1}\\ \tilde{\omega}^2=\left[\dfrac{\alpha^2(q^2B^2+4m^2\omega^2c^2)}{4mc^2}+\dfrac{qB\bar\theta}{2mc\hbar}+\dfrac{\bar\theta^2}{4m\alpha^2\hbar^2}\right]\Big/\tilde{m}\end{cases},\tag{15}$$

Eq. (14) takes the form as the following

$$\begin{aligned}\hat{H} = &\frac{1}{2\tilde m}(p_1^2+p_2^2)+\frac{1}{2}\tilde m\tilde\omega^2(x_1^2+x_2^2)-\left[\frac{qB}{2mc}+\frac{\bar\theta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right](x_1p_2-x_2p_1)=\\ &H_0'-\left[\frac{qB}{2mc}+\frac{\bar\theta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right]L_z\,,\end{aligned}\tag{16}$$

with $H_0'=\frac{1}{2\tilde m}(p_1^2+p_2^2)+\frac{1}{2}\tilde m\tilde\omega^2(x_1^2+x_2^2)$. Observing the similarity between the Hamiltonians (4) and (16), the solution of eigenvalue problem can be worked out in a manner to the one used in the previous section.

Letting

$$\psi(\rho,\phi)=\chi(\rho)\mathrm{e}^{\mathrm{i}m_l\varphi},\quad m_l=0,\pm1,\pm2,\cdots.\tag{17}$$

Substituting Eqs. (16), (17) into (13), by separating variables, in the non-commutative phase space, one can get the radial equations as follows

$$\begin{aligned}&\left\{\left[-\frac{\hbar^2}{2\tilde m}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right)+\frac{1}{2}\tilde m\tilde\omega^2\rho^2\right]-\right.\\ &\left.\left[\frac{qB}{2mc}+\frac{\bar\theta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right]m_l\hbar\right\}\chi(\rho)=E\chi(\rho)\,,\end{aligned}\tag{18}$$

Eq. (18) is the famous Schrödinger radial equation of Landau problem. The energy of the system can be given as follows

$$\begin{aligned}E_{n_\rho m_l}=&\hbar\tilde\omega(2n_\rho+|m_l|+1)+\\ &m_l\hbar\left[\frac{qB}{2mc}+\frac{\bar\theta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right].\end{aligned}\tag{19}$$

The corresponding radial eigenfunction is given by (unnormalized)

$$\chi(\rho)\sim\rho^{|m_l|}F(-n_\rho,|m_l|+1,\ \tilde\beta^2\rho^2)\cdot\exp[-\tilde\beta^2\rho^2/2],\tag{20}$$

where

$$\tilde\beta^2=\frac{qB}{2mc}+\frac{\bar\theta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\,.$$

Finally, eigenfunction of the system in non-commutative phase space is

$$\begin{aligned}\psi(\rho,\phi)\ =\ &\tilde N\rho^{|m_l|}F(-n_\rho,\ |m_l|+1,\ \tilde\beta^2\rho^2)\cdot\\ &\exp[-\tilde\beta^2\rho^2/2]\mathrm{e}^{\mathrm{i}m_l\phi}\,,\\ &n_\rho=0,1,2,\cdots,\ m_l=0,\pm1,\pm2,\cdots,\end{aligned}\tag{21}$$

where $\tilde N$ is the normalized constant.

In order to see the non-commutative effect explicitly, let us do some discussions.

First, let us do some discussions about the energy.

(1) When the space-space and momentum-momentum are all non-commutative, namely, $\bar\theta\neq0$, $\theta\neq0$, from Eqs. (7) and (19), we can easily find that the energy shift caused by the space-space and momentum-momentum non-commutativivity can be

given by

$$\Delta E = \hbar(2n_\rho + |m_l| + 1)(\tilde{\omega} - \omega_0) + m_l\hbar\left[\frac{\bar{\theta}}{\hbar} + \frac{(q^2B^2 + 4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right], \quad (22)$$

with $\tilde{\omega}$, ω_0 is defined by Eqs. (15) and (4), respectively.

(2) When the space-space is non-commutative, and the momentum-momentum is commutative, namely, $\alpha = 1$, $\bar{\theta} = 0$, $\theta \neq 0$, the energy shift is

$$\Delta E = \hbar(2n_\rho + |m_l| + 1)(\tilde{\omega} - \omega_0) + m_l\hbar\frac{(q^2B^2 + 4m^2\omega^2c^2)\theta}{8m\hbar c^2},$$

with

$$\tilde{\omega} = \sqrt{\frac{(q^2B^2 + 4m^2\omega^2c^2)}{4\tilde{m}mc^2}},$$

$$\tilde{m} = \frac{1}{\dfrac{1}{m} + \dfrac{qB\theta}{2mc\hbar} + \dfrac{(q^2B^2 + 4m^2\omega^2c^2)\theta^2}{16m\alpha^2\hbar^2c^2}}$$

and ω_0 is defined by Eq. (4).

(3) When the space-space and momentum-momentum are all commutative, namely, $\alpha = 1$, $\bar{\theta} = 0$, $\theta = 0$, then $\tilde{m} = m$, $\tilde{\omega} = \omega_0$, $\Delta E = 0$, the energy return to the case of general quantum mechanics.

Second Comparing Eq. (9) with Eq. (21), we can find the eigenfunction in non-commutative phase space has the same form as the one in commutative space. They are all the confluent hypergeometric functions, but the parameter $\tilde{\beta}$ of confluent hypergeometric functions in non-commutative phase space is different from the corresponding parameter β in commutative space.

(1) When the space-space is non-commutative, and momentum-momentum is commutative, namely $\alpha = 1$, $\bar{\theta} = 0$, $\theta \neq 0$, then

$$\tilde{\beta}^2 = \frac{qB}{2mc} + \frac{(q^2B^2 + 4m^2\omega^2c^2)\theta}{8m\hbar c^2}.$$

(2) When the space-space and momentum-momentum are all commutative, namely, $\alpha = 1$, $\bar{\theta} = 0$, $\theta = 0$, We have $\tilde{\beta}^2 = \beta^2$; the eigenfunction (21) returns to the case of general quantum mechanics.

4 Conclusion

In conclusion, we have researched the Hamiltonian for the isotropic charged harmonic oscillator in uniform magnetic field in the non-commutative phase space. It is shown that in non-commutative phase space, the isotropic charged harmonic oscillator in uniform magnetic field can been seen as the Landau problem. Thus, the corresponding exact energy has been obtained, and the analytic eigenfunction presented in terms of the confluent hypergeometric function; meanwhile the non-commutative effect has been discussed carefully.

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