第24卷 第8期 高能物理与核物理 Vol.24, No.8
2000年8月 HIGH ENERGY PHYSICS AND NUCLEAR PHYSICS Aug., 2000

Spin Operators in the Quantum Field Theory *

ZHANG PengFei[1)] RUAN TuNan

(CCAST (World Lab), Beijing 100080, China)

(National Synchrotron Radiation Laboratory and Department of Modern Physics, University of Science and Technology of China, Hefei 230029, China)

Abstract The concept of moving spin of the relativistic particle is further discussed in the quantum field theory. Two new operators, field quanta spin and moving spin of quantum field system, are introduced. We show that, in virtue of these two operators, some problem in the quantum field theory concerned spin can be neatly settled.

Key words quantum field theory, field quanta spin, moving spin of quantum field system

1 Field Quanta Spin

In the previous paper[1], we discussed the spin operator for a massive relativistic particle with nonzero spin. It has been known that the spin operator for the states of the nonrelativistic particle should be replaced by the moving spin (PMS). Now for the quantum field, therefore some modification to the spin operator is expected.

Consider a massive field with spin s. Suppose the independent canonical coordinates are φ_A ($A=1,2\cdots,n$); and the Lagrangian density is $\mathcal{L}$, hereby the conjugate fields are $\pi_A=\partial\mathcal{L}/\partial\varphi_A$. We may now define an operator as follows,

$$^{q}\boldsymbol{S}=\int \mathrm{d}^3x(-\mathrm{i}):\sum_{A,B}\pi_A(x)\hat{s}(\hat{p})_{A,B}\varphi_B(x):,\tag{1}$$

where $\hat{s}(\hat{p})$ is just the moving spin as defined by Eq. (10) in Ref. [1]. It is easy to check that

$$[P_\mu,{}^{q}S_j]=0,\qquad [J_i,{}^{q}S_j]=[{}^{q}S_i,{}^{q}S_j]=\mathrm{i}\epsilon_{ijk}{}^{q}S_k,\tag{2}$$

which manifests the properties of a spin operator for $^{q}\boldsymbol{S}$. Owing to what follows, we may call this new defined operator the field quanta spin (FQS). Accordingly the total angular momentum is now redivided as following,

$$\boldsymbol{J}={}^{q}\boldsymbol{L}+{}^{q}\boldsymbol{S},\tag{3}$$

where $^{q}\boldsymbol{L}$ may be called field quanta orbital angular momentum. While the following commutation rules are easy to check,

$$[{}^{q}L_i,{}^{q}S_j]=0,\quad [J_i,{}^{q}L_j]=[{}^{q}L_i,{}^{q}L_j]=\mathrm{i}\epsilon_{ijk}{}^{q}L_k,\quad [P_i,{}^{q}L_j]=[P_i,J_j]=\mathrm{i}\epsilon_{ijk}P_k.\tag{4}$$

For concrete, we come now to the Dirac field. The Lagrangian density is

$$\mathcal{L}=\bar{\psi}_\sigma(\mathrm{i}\boldsymbol{\gamma}\cdot\nabla+\mathrm{i}\gamma_0\partial_t-m)_{\sigma\rho}\psi_\rho=\bar{\psi}(\mathrm{i}\not{\partial}-m)\psi,\tag{5}$$

Received 9 October 1999, Revised 14 February 2000

* Supported by NSFC (19677102, 19775044, 19991484), Doctoral Program Foundation of the Institution of Higher Education of China (97035807), BEPC National Lab of China and Center for Theoretical Nuclear Physics of Lanzhou Heavy Ion Accelerator National Lab of China

1) E-mail: zhpf@ustc.edu.cn

775—779

where ψ_σ are the four components of the spinor, while $\bar{\psi}=\psi^+\gamma_0$. The conjugate fields are

$$\pi_\sigma=\frac{\partial \mathscr{L}}{\partial \dot{\psi}_\sigma}=\mathrm{i}\psi_\sigma^+ \qquad (\sigma=1,2,3,4). \tag{6}$$

The equation of motion is $(\mathrm{i}\,\not{p}-m)\psi=0$ with the following plane wave solution

$$\omega(\boldsymbol{p},E)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}-\mathrm{i}Et}=\omega(p)\mathrm{e}^{-\mathrm{i}p\cdot x}. \tag{7}$$

In the particle rest frame, $\boldsymbol{p}=0$, $p=\mathring{p}$, $E=\pm m$; $\omega(p)$ can be expanded in term of the following base,

$$u_\lambda(\mathring{p})=\begin{pmatrix}\xi_\lambda\\0\end{pmatrix},\quad v_\lambda(\mathring{p})=\begin{pmatrix}0\\\xi_\lambda\end{pmatrix}\quad(\lambda=\pm1/2). \tag{8}$$

If $\boldsymbol{p}\neq0, E=\pm\sqrt{|\boldsymbol{p}|^2+m^2}=\pm E_p$; we may take the following canonical frame

$$u_\lambda(p)=\mathrm{e}^{\theta\boldsymbol{\alpha}\cdot\boldsymbol{n}/2}u_\lambda(\mathring{p}),\qquad v_\lambda(p)=\mathrm{e}^{\theta\boldsymbol{\alpha}\cdot\boldsymbol{n}/2}v_\lambda(\mathring{p}), \tag{9}$$

where $\boldsymbol{n}=\boldsymbol{p}/|\boldsymbol{p}|$, while θ satisfies $\tanh\frac{\theta}{2}=|\boldsymbol{p}|/(E_p+m)$. It can be check that

$$\hat{s}(\mathring{p})u_\lambda(p)\mathrm{e}^{-\mathrm{i}p\cdot x}=\frac{1}{2}(\boldsymbol{\sigma})_{\lambda'\lambda}u_{\lambda'}(p)\mathrm{e}^{-\mathrm{i}p\cdot x},\quad \hat{s}(\mathring{p})v_\lambda(p)\mathrm{e}^{\mathrm{i}p\cdot x}=\frac{1}{2}(\boldsymbol{\sigma})_{\lambda'\lambda}v_{\lambda'}(p)\mathrm{e}^{\mathrm{i}p\cdot x}, \tag{10}$$

in which $\hat{s}(\mathring{p})$ is the PMS for Dirac particle.

Now expand $\psi,\bar{\psi}$ in the momentum space,

$$\begin{aligned}\psi(\boldsymbol{x},t)&=\int\frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\sum_{\lambda=\pm1/2}\sqrt{\frac{m}{E_p}}\{c_\lambda(p)u_\lambda(p)\mathrm{e}^{-\mathrm{i}p\cdot x}+d_\lambda^+(p)v_\lambda(p)\mathrm{e}^{\mathrm{i}p\cdot x}\},\\ \bar{\psi}(\boldsymbol{x},t)&=\int\frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\sum_{\lambda=\pm1/2}\sqrt{\frac{m}{E_p}}\{c_\lambda^+(p)\bar{u}_\lambda(p)\mathrm{e}^{\mathrm{i}p\cdot x}+d_\lambda(p)\bar{v}_\lambda(p)\mathrm{e}^{-\mathrm{i}p\cdot x}\},\end{aligned} \tag{11}$$

where $c_\lambda(p)$, $c_\lambda^+(p)$ and $d_\lambda(p)$, $d_\lambda^+(p)$ are known as the annihilation and creation operators for particle and antiparticle respectively with the following anticommutators

$$\{c_\lambda(p),\ c_{\lambda'}^+(p')\}=\{d_\lambda(p),\ d_{\lambda'}^+(p')\}=\delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'), \tag{12}$$

with other anticommutators eliminated.

In the following we go over to the field quanta spin. One may substitute the expansions of Eq.(11) into Eq.(1). Noticing Eq.(10), one can get after direct calculations

$${}^{\mathrm{q}}S^i=\int\mathrm{d}^3p\sum_{\lambda,\lambda'=\pm1/2}s_{\lambda\lambda'}^{(i)}\{c_\lambda^+(p)c_{\lambda'}(p)-d_{\lambda'}^+(p)d_\lambda(p)\}. \tag{13}$$

Next we consider the states of the Dirac field.

(a) Vacuum state: $|0\rangle$. One may take the normalization as $\langle0|0\rangle=1$. The vacuum state should be invariant under space time transformation, i.e., $\boldsymbol{J}|0\rangle=0$, $\boldsymbol{P}|0\rangle=0$.

(b) One particle state

$$|(p,\lambda;+)\rangle=\sqrt{2E_p}c_\lambda^+(p)|0\rangle,|(p,\lambda;-)\rangle=2\lambda\sqrt{2E_p}d_{-\lambda}^+(p)|0\rangle, \tag{14}$$

where '+' and '−' in the notation of the states stands for particle and anti particle respectively; the phase convention for the antiparticle states are adopted according to the charge conjugation transformation.

Via the transformations of the annihilation and creation operators, one can get after some computation

$$U[a,L]\,|(p,\lambda;\pm)\rangle=\mathrm{e}^{\mathrm{i}Lp\cdot a}\sum D_{\lambda'\lambda}^{\frac{1}{2}}(\bar{R}(L,p))^{\cdot}|(Lp,\lambda';\pm)\rangle,$$

which indicates that $|(p,\lambda;\pm)\rangle$ are the canonical states.

We can now check the action of ${}^{\mathrm{q}}\boldsymbol{S}$ on the states with Eq.(13). First,

$${}^{\mathrm{q}}\boldsymbol{S}|0\rangle=0, \tag{16}$$

i.e., the vacuum state of Dirac field is the eigen state of FQS with zero eigenvalue. Up to now in the text books on the quantum field theory, however, the spin operator is $\boldsymbol{S}$ and the vacuum state is not its eigen state. This had been remained as a problem in the quantum field theory. We know now that this problem is well settled if we adopt FQS as the spin operator of the the quantum field.

Next, for the one particle states $|(p,\lambda;\pm)\rangle$, from Eqs. (13, 14) we can get

$$ {}^{q}S^{i}\,|(p,\lambda;\pm)\rangle = s^{(i)}_{\lambda',\lambda}\,|(p,\lambda';\pm)\rangle, \tag{17}$$

especially, ${}^{q}S^{z}|(p,\lambda;\pm)\rangle = \lambda|(p,\lambda;\pm)\rangle$. Eq. (17) shows that ${}^{q}S^{i}$ are exactly the spin operator for the one particle states. In texts on the quantum field theory, the spin problem has been discussed[2—4]. However, the problem is now discussed more neatly in virtue of FQS. For this reason, and also for the defect of the spin operator $\boldsymbol{S}$ on vacuum state, FQS are more proper as a spin operator for the quantum field theory.

2 Moving Spin of Quantum Field System

In the above, the spin operator for the vacuum and one-particle states of the quantum field is clarified. As for the arbitrary particle states of the quantum field, such an operator is also found.

For the arbitrary N-particle states, one can construct the N-particle canonical states $|p_{12\cdots N}\lambda;wj\eta\rangle_c$ (where η stands for the degenerate indices), which is the eigenstate of the momentum operator P,

$$ P|p_{12\cdots N}\lambda;wj\eta\rangle_c = p_{12\cdots N}|p_{12\cdots N}\lambda;wj\eta\rangle_c. \tag{18}$$

We found that for these states there exists such an operator $\boldsymbol{S}(P)$ which satisfies

$$ \boldsymbol{S}(P)|p_{12\cdots N}\lambda;wj\eta\rangle_c = \boldsymbol{j}_{\Lambda'\Lambda}|p_{12\cdots N}\lambda;wj\eta\rangle_c, \tag{19}$$

where $\boldsymbol{j}_{\Lambda'\Lambda}$ is the spin representation matrices for spin-j, as $s^{(i)}$ in Eq. (3) in Ref. [1].

The explicit expression of $\boldsymbol{S}(P)$ reads as following,

$$ \boldsymbol{S}(P) = (P^2)^{-\frac{1}{2}}\{P_0\boldsymbol{J} - (\sqrt{P^2}+P_0)^{-1}\boldsymbol{P}(\boldsymbol{P}\cdot\boldsymbol{J}) - \boldsymbol{P}\times\boldsymbol{K}\}, \tag{20}$$

in which $\boldsymbol{J}$ and $\boldsymbol{K}$ are the generators for rotation and boost respectively, while $P^2 = P_\mu P^\mu = P_0^2 - |\boldsymbol{P}|^2$. In the above expression only P, J, K appear, and this indicates an universal meaning of $\boldsymbol{S}(P)$.

Eq. (19) manifest the property of $\boldsymbol{S}(P)$ as a spin operator for the various canonical states. On the other hand, from the procedure of the construction of the N-particle canonical states, it is known that the canonical states are the complete base for the space of arbitrary N-particle states. Hence, $\boldsymbol{S}(P)$ is an universal spin operator for various states of the quantum field. Accordingly we may name it as moving spin of quantum field system (FSMS).

As a fact, it can be verified that $\boldsymbol{S}(P)$ acts on the vacuum state and one-particle state just in the same way as ${}^{q}\boldsymbol{S}$.

Considering the infinitesimal rotary transformation of the canonical states, one can found that the total angular of the field appears as

$$ \boldsymbol{J} = -\mathrm{i}\boldsymbol{P}\times\frac{\partial}{\partial\boldsymbol{P}} + \boldsymbol{S}(P). \tag{21}$$

We may define $\boldsymbol{L}(P) = -\mathrm{i}\boldsymbol{P}\times\partial/\partial\boldsymbol{P}$, then $\boldsymbol{L}(P)$, $\boldsymbol{S}(P)$ are the orbital angular momentum and 'spin' operator respectively. Some relations concerned $\boldsymbol{L}(P), \boldsymbol{S}(P)$ are listed as follows.

$$[J_i, L(P)_j] = [L(P)_i, L(P)_j] = \mathrm{i}\epsilon_{ijk}L(P)_k,$$
$$[J_i, S(P)_j] = [S(P)_i, S(P)_j] = \mathrm{i}\epsilon_{ijk}S(P)_k,$$
$$[L(P)_i, P_j] = [J_i, P_j] = \mathrm{i}\epsilon_{ijk}P_k,$$
$$[S(P)_i, P_j] = [S(P)_i, P_0] = [L(P)_i, S(P)_j] = [L(P)_i, P_0] = 0,$$
$$\boldsymbol{L}(P)\cdot\boldsymbol{P} = \boldsymbol{P}\cdot\boldsymbol{L}(P) = 0.$$

The more detailed account is to be appeared elsewhere.

3 Conclusion Remarks

In the preceding paper[1] and this paper, we present a systematic discussion of the spin operators for various relativistic states. For a relativistic particle, the moving spin (PMS) instead of the ordinary spin operator should be adopted for describing its spin. The further discussion in the quantum field theory leads to the two new operators FQS and FSMS. The explicit expression of FQS in momentum space and FSMS in terms of the generators of Poincaré group are obtained. With the aid of FQS and FSMS, problems of the quantum field concerned spin can be neatly settled. It is reasonable that FQS and FSMS are the proper spin operators for relativistic quantum field. We hope that all these results will be helpful in our endeavor to the spin crisis.

The authors are grateful to Profs. ZHU YuCan and ZHENG ZhiPeng in IHEP for their support and useful suggestions. We also thank Prof. S. U. Chung in BNL for his kind help on the research materials.

References

1 ZHANG PengFei, RUAN TuNan. High Energy Phys. and Nucl. Phys. (in Chinese), 2000, **24**(6): 585
(张鹏飞,阮图南.高能物理与核物理,2000,**24**(6): 585)

2 David Lurié. Particles and Fields. John Wiley & Sons, 1968

3 Bjorken J D, Drell S D. Relativistic Quantum Fields. McGraw-Hill Book Company, 1965

4 Bogoliubov N N, Shirkov D V. Quantum Fields. The Benjamin/Cummings Publishing Company, 1983

Appendix

Generators of $\mathscr{P}$ of Quantum Field

Suppose that the field variables are $\varphi_A\ (A=1,2\cdots,n)$, with Lagrangian density $\mathscr{L}$. The conjugate fields are $\pi_A=\partial\mathscr{L}/\partial\dot{\varphi}_A$. By Nöther's theorem, the momentum of the field is

$$P_\mu = \int d^3x : \mathscr{P}_\mu(x) : , \tag{22}$$

where $\mathscr{P}_\mu(x) = \sum_A \pi_A \partial_\mu \varnothing_A - \eta_{\mu 0}\mathscr{L}(x)$ is the energy-momentum density. While the angular momentum is

$$J_{\mu\nu} = \int d^3x : [x_\mu \mathscr{P}_\nu - x_\nu \mathscr{P}_\mu - i\sum_{A,B}\pi_A(\hat{s}_{\mu\nu})_{A,B}\varphi_B] : = L_{\mu\nu} + S_{\mu\nu}, \tag{23}$$

with $$L_{\mu\nu} = \int d^3x : [x_\mu \mathscr{P}_\nu - x_\nu \mathscr{P}_\mu] : , \quad S_{\mu\nu} = \int d^3x : (-i)\sum_{A,B}\pi_A(\hat{s}_{\mu\nu})_{A,B}\varphi_B : \tag{24}$$

where $\hat{s}_{\mu\nu}$ is the spin tensor for one particle.

Impose the quantization condition to (φ,π), either commutation rules for Boson field or anticommutation rules for Fermion field, and from Eqs. (22,23,24), we may get by direct calculations

$$\begin{aligned}
&[P_\mu, P_\nu] = [P_\mu, S_{\sigma\rho}] = [L_{\mu\nu}, S_{\sigma\rho}] = 0,\\
&[P_\mu, L_{\sigma\rho}] = [P_\mu, J_{\sigma\rho}] = i(\eta_{\mu\sigma}P_\rho - \eta_{\mu\rho}P_\sigma),\\
&[L_{\mu\nu}, L_{\sigma\rho}] = -i(\eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma}),\\
&[S_{\mu\nu}, S_{\sigma\rho}] = -i(\eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\sigma}S_{\mu\rho} + \eta_{\nu\rho}S_{\mu\sigma} - \eta_{\mu\rho}S_{\nu\sigma}),\\
&[J_{\mu\nu}, T_{\sigma\rho}] = -i(\eta_{\mu\sigma}T_{\nu\rho} - \eta_{\nu\sigma}T_{\mu\rho} + \eta_{\nu\rho}T_{\mu\sigma} - \eta_{\mu\rho}T_{\nu\sigma}) \quad (T=S,L,J).
\end{aligned} \tag{25}$$

All these Poincaré algebra are just the same as that in the Appendix in Ref[1]. And similarly, the Pauli-Lubanski

$= \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}P^\nu J^{\sigma\rho}$, with the commutations

$$[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\sigma\rho}P^\sigma W^\rho, \quad [W_\mu, P_\nu] = 0. \quad [W_\mu, J_{\sigma\rho}] = i(\eta_{\mu\sigma}W_\rho - \eta_{\mu\rho}W_\sigma). \tag{26}$$

The two Casimir operators are $C_1 = P_\nu P^\nu$ and $C_2 = W_\nu W^\nu$, which satisfy $[C_i, P_\nu] = [C_i, J_{\mu\nu}] = 0 (i=1,2)$.

From the Lorentz tensors $J_{\mu\nu}$, $L_{\mu\nu}$ and $S_{\mu\nu}$, we may define the following space vector operators,

$$L_i=\frac{1}{2}\epsilon_{ijk}L^{jk}=\frac{1}{2}\epsilon_{ijk}L_{jk},\quad S_i=\frac{1}{2}\epsilon_{ijk}S^{jk}=\frac{1}{2}\epsilon_{ijk}S_{jk},\quad J_i=L_i+S_i;$$
$$Z_i=L_{0i}=-L^{0i},\quad T_i=S_{0i}=-S^{0i},\quad K_i=Z_i+T_i. \tag{27}$$

We should point out here that on the definition of $\hat{L}_i$ and $\hat{s}_i$ in Eq. (18) in the Appendix of Ref. [1] the factor $\frac{1}{2}$ is unwarily missed.

量子场论中的自旋算符*

张鹏飞[1)]　阮图南

(中国高等科学技术中心(世界实验室)　北京　100080)

(中国科学技术大学国家同步辐射实验室和近代物理系　合肥　230029)

摘要　从量子场论的角度对相对论粒子的运动自旋概念作了进一步深入研究.构造了场量子自旋以及场系统运动自旋两个新算符.给出了场量子自旋动量空间的显式表达式以及用 Poincaré 群生成元表示的场系统运动自旋的显式表达式.借助这两个算符,可以干净地解决有关场自旋的问题,表明它们才是场自旋的恰当的算符.

关键词　量子场论　场量子自旋　场系统运动自旋

1999－10－09 收稿,2000－02－14 收修改稿

* 国家自然科学基金(19677102,19775044,19991484),高等学校博士学科点专项科研基金(97035807),北京正负电子对撞机国家实验室和兰州重离子加速器国家实验室原子核理论研究中心资助

1)E-mail: zhpf@ustc. edu. cn