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裂变 F-P 方程的代数解法

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摘　　要

本文介绍了时间演化方程的 Lie 代数解法. 求出了谐振子势场中裂变 F-P 方程解析解的积分表式. 该方法容易推广到变系数情形.

当把可裂变原子核的变形运动看作布朗粒子的扩散运动时，在裂变形变坐标 x 和共轭速度 v 的两维相空间中,几率密度 $W(x,v,t)$ 满足 F-P 方程[1,2]:

$$\frac{\partial W(x,v,t)}{\partial t}=\left\{-v\frac{\partial}{\partial x}+\frac{\partial}{\partial v}\left(\gamma v-\frac{K(x)}{m}\right)+\frac{D}{m^2}\frac{\partial^2}{\partial v^2}\right\}W(x,v,t), \tag{1}$$

其中 γ 是粘滞系数, m 是惯性张量, $D=\gamma mkT$, kT 是核温度. 外力

$$K(x)=-\frac{\partial V(x)}{\partial x}, \tag{2}$$

$V(x)$ 是裂变位势. 当 $V(x)$ 取谐振子势

$$V(x)=\frac{1}{2}m\omega_0^2x^2 \tag{3}$$

时,方程(1)有解析解. 文献[2]给出了初始分布为高斯分布时谐振子势场中裂变 F-P 方程解析解的表式.

初始分布为非高斯分布时情形怎样?谐振子势场中的 ω_0 依赖于 t 时情况又是怎样? 应用 Lie 代数技巧,可以较方便地解决这些问题.

1. 时间演化方程的 Lie 代数解法

在很多物理分支中,要求解时间演化方程:

$$\begin{cases}\dfrac{dU(t)}{dt}=A(t)U(t), & \text{(4 a)}\\[2mm] U(0)=I. & \text{(4 b)}\end{cases}$$

当 A 不显含 t 时,时间演化算符

$$U(t)=\exp(At). \tag{5}$$

当 A 显含 t 时,平常用费曼展开方法求解,这时要引进编时算符.

六十年代初, Wei 和 Norman 提出了一种求解时间演化方程的 Lie 代数方法[3]

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八十年代，由于 Squeezing 现象的发现，Wei 和 Norman 方法重新受到人们的注意[4-8]。

设（4 a）右端的 $A(t)$ 为 Lie 代数G的元素，

$$A(t)=\sum_{i=1}^{M} a_i(t)T_i, \tag{6}$$

$\{T_1,T_2,\cdots T_N:N\geqslant M\}$ 为G的生成元,时间演化算符可以分解成

$$U(t)=\prod_{j=1}^{N}\exp\{g_j(t)T_j\}. \tag{7}$$

将上式代入(4)式可证 $g_j(t)(j=1—N)$ 满足非线性方程组

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{pmatrix}=\begin{pmatrix} \xi_{11} & \cdots & \xi_{1N} \\ \vdots & & \\ \xi_{M1} & \cdots & \xi_{MN} \end{pmatrix}\begin{pmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \vdots \\ \dot{g}_N \end{pmatrix} \tag{8 a}$$

和初始条件

$$g_j(0)=0.\ (j=1—N) \tag{8 b}$$

ξ_{kl} 为G的结构常数和 $g_j(t)(j=1—N)$ 的解析函数.

当G为可解 Lie 代数时,方程组(8)很容易积分。当G为一般的有限维 Lie 代数时，它可以表示为可解 Lie 代数R和半单 Lie 代数S的半直和:

$$G=R\oplus S. \tag{9}$$

与此相应，$A(t)$ 也可以写成两部分:

$$A(t)=A_R(t)+A_S(t), \tag{10}$$

其中 $A_R(t)\in R, A_S(t)\in S$.

$U(t)$ 可以写成

$$U(t)=U_S(t)U_R(t), \tag{11}$$

其中 $U_S(t)$ 满足

$$\begin{cases} \dfrac{dU_S(t)}{dt}=A_S(t)U_S(t), & \text{(12 a)} \\ U_S(0)=I. & \text{(12 b)} \end{cases}$$

$U_R(t)$ 满足

$$\begin{cases} \dfrac{dU_R(t)}{dt}=\{U_S^{-1}(t)A_R(t)U_S(t)\}U_R(t), & \text{(13 a)} \\ U_R(0)=I. & \text{(13 b)} \end{cases}$$

上述分解,显然可使问题简化.

2. 常系数谐振子势中裂变 F-P 方程的解析解

当方程(1)和裂变位势(3)中的系数为常数时,裂变方程可写成

$$\frac{\partial W(x,v,t)}{\partial t}=\left\{-v\frac{\partial}{\partial x}+\gamma+\gamma v\frac{\partial}{\partial v}+\omega_0^2 x\frac{\partial}{\partial v}+\frac{D}{m^2}\frac{\partial^2}{\partial v^2}\right\}W(x,v,t). \tag{14}$$

令

$$W(x,v,t)=U(t)W(x,v,0), \tag{15}$$

则时间演化算符满足方程

$$\begin{cases}\dfrac{\mathrm{d}U(t)}{\mathrm{d}t}=\left\{-v\dfrac{\partial}{\partial x}+\gamma+\gamma v\dfrac{\partial}{\partial v}+\omega_0^2x\dfrac{\partial}{\partial v}+\dfrac{D}{m^2}\dfrac{\partial^2}{\partial v^2}\right\}U(t), & (16\text{ a})\\ U(0)=I. & (16\text{ b})\end{cases}$$

令

$$K_0=\frac{1}{2}\left(x\frac{\partial}{\partial x}-v\frac{\partial}{\partial v}\right),\qquad K_+=x\frac{\partial}{\partial v},\qquad K_-=v\frac{\partial}{\partial x},$$

$$N=\frac{1}{2}\left(x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}\right),\quad a^2=\frac{\partial^2}{\partial x^2},\quad ab=\frac{\partial^2}{\partial x\,\partial v},\quad b^2=\frac{\partial^2}{\partial v^2};$$

(16 a) 可以写成

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=\left\{(\omega_0^2K_+-K_--\gamma K_0)+\left(\gamma+\gamma N+\frac{D}{m^2}b^2\right)\right\}U(t). \tag{17}$$

与(17)式相应的 Lie 代数G为

$$G=\{K_+,K_-,K_0;1,N,a^2,ab,b^2\}, \tag{18}$$

其中半单的部分

$$S=\{K_+,K_-,K_0\}, \tag{19}$$

这是一个 $SU(2)$ 代数. 可解的部分

$$R=\{1,N,a^2,ab,b^2\}. \tag{20}$$

$U_S(t)$ 满足方程

$$\begin{cases}\dfrac{\mathrm{d}U_S(t)}{\mathrm{d}t}=\{\omega_0^2K_+-K_--\gamma K_0\}U_S(t), & (21\text{ a})\\ U_S(0)=I. & (21\text{ b})\end{cases}$$

令

$$U_S(t)=\exp\{g_1(t)K_+\}\exp\{g_0(t)K_0\}\exp\{g_{-1}(t)K_-\}, \tag{22}$$

$g_1(t)$、$g_0(t)$、$g_{-1}(t)$ 满足方程组

$$\begin{cases}\dot g_1-g_1\dot g_0-g_1^2\exp(-g_0)\dot g_{-1}=\omega_0^2, & (23\text{ a})\\ \dot g_0+2g_1\exp(-g_0)\dot g_{-1}=-\gamma, & (23\text{ b})\\ \dot g_{-1}\exp(-g_0)=-1, & (23\text{ c})\end{cases}$$

和初始条件

$$g_i(0)=0.\ (i=1,0,-1) \tag{24}$$

方程组(23)可以化为

$$\begin{cases}\dot g_1+\gamma g_1-g_1^2=\omega_0^2, & (25\text{ a})\\ \dot g_0=2g_1-\gamma, & (25\text{ b})\\ \dot g_{-1}=-\exp(g_0), & (25\text{ c})\end{cases}$$

由此可见,只要解出 g_1,g_0 和 g_{-1} 很容易求得. (25 a) 为一 Riccati 方程. 解之得

$$g_1(t) = \frac{\omega_0^2 \text{sh}\omega_1 t}{\omega_1 \text{ch}\omega_1 t + \frac{\gamma}{2}\text{sh}\omega_1 t}, \tag{26}$$

$$g_0(t) = -2\ln\left(\text{ch}\omega_1 t + \frac{\gamma}{2\omega_1}\text{sh}\omega_1 t\right), \tag{27}$$

$$g_{-1}(t) = -\frac{\text{sh}\omega_1 t}{\omega_1 \text{ch}\omega_1 t + \frac{\gamma}{2}\text{sh}\omega_1 t}. \tag{28}$$

这里，

$$\omega_1 = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}. \tag{29}$$

计算 $U_S^{-1}(t)\left(\gamma + \gamma N + \frac{D}{m^2}b^2\right)U_S(t)$ 后可知，$U_R(t)$ 满足

$$\begin{cases} \frac{dU_R(t)}{dt} = \left\{\gamma + \gamma N + \frac{D}{m^2}\exp(-g_0)(b^2 + 2g_{-1}ab + g_{-1}^2 a^2)\right\}U_R(t), & (30\text{ a}) \\ U_R(0) = I. & (30\text{ b}) \end{cases}$$

令

$$U_R(t) = \exp\{\alpha_5(t)\}\exp\{\alpha_4(t)N\}\exp\{\alpha_3(t)b^2\}\exp\{\alpha_2(t)ab\}\exp\{\alpha_1(t)a^2\}, \tag{31}$$

代入 (30 a)、(30 b) 可知 $\alpha_i(t)(i = 1—5)$ 满足

$$\begin{cases} \dot{\alpha}_1 = \frac{D}{m^2}\exp(\gamma t)g_{-1}^2\exp(-g_0), & (32\text{ a}) \\ \dot{\alpha}_2 = \frac{2D}{m^2}\exp(\gamma t)g_{-1}\exp(-g_0), & (32\text{ b}) \\ \dot{\alpha}_3 = \frac{D}{m^2}\exp(\gamma t)\exp(-g_0), & (32\text{ c}) \\ \dot{\alpha}_4 = \gamma, & (32\text{ d}) \\ \dot{\alpha}_5 = \gamma, & (32\text{ e}) \end{cases}$$

$$\alpha_i(0) = 0. \ (i = 1—5) \tag{33}$$

解之得

$$\alpha_1(t) = -\frac{D}{2\gamma m^2\omega_0^2} + \frac{D}{2m^2\omega_0^2}\exp(\gamma t)\left(\frac{1}{\gamma} + \frac{\gamma}{2\omega_1^2}\text{sh}^2\omega_1 t - \frac{1}{\omega_1}\text{sh}\omega_1 t\,\text{ch}\omega_1 t\right), \tag{34}$$

$$\alpha_2(t) = -\frac{D}{m^2\omega_1^2}\exp(\gamma t)\text{sh}^2\omega_1 t, \tag{35}$$

$$\alpha_3(t) = -\frac{D}{2\gamma m^2} + \frac{D}{2m^2}\exp(\gamma t)\left(\frac{1}{\gamma} + \frac{\gamma}{2\omega_1^2}\text{sh}^2\omega_1 t + \frac{1}{\omega_1}\text{sh}\omega_1 t\text{ch}\omega_1 t\right), \tag{36}$$

$$\alpha_4(t) = \gamma t, \tag{37}$$

$$\alpha_5(t) = \gamma t. \tag{38}$$

与方程(14)相应的时间演化算符为

$$U(t) = U_S(t)U_R(t)$$

$$= \exp\{\gamma t\}\exp\left\{g_1 x\frac{\partial}{\partial v}\right\}\exp\left\{\frac{1}{2}(\gamma t+g_0)x\frac{\partial}{\partial x}+\frac{1}{2}(\gamma t-g_0)v\frac{\partial}{\partial v}\right\}$$

$$\cdot\exp\left\{g_{-1}v\frac{\partial}{\partial x}\right\}\exp\left\{\alpha_1\frac{\partial^2}{\partial x^2}+\alpha_2\frac{\partial^2}{\partial x\,\partial v}+\alpha_3\frac{\partial^2}{\partial v^2}\right\}. \tag{39}$$

几率密度 $W(x,v,t)$ 可通过下列步骤求出:

1. $W_1(x,v,t)$

$$= \exp\left\{\alpha_1\frac{\partial^2}{\partial x^2}+\alpha_2\frac{\partial^2}{\partial x\,\partial v}+\alpha_3\frac{\partial^2}{\partial v^2}\right\}W(x,v,0)$$

$$= \frac{1}{4\pi^2}\iiiint_{R_4} W(\xi,\eta,0)\exp\{-\alpha_1 y^2-\alpha_2 yu-\alpha_3 u^2\}$$

$$\cdot\exp\{i[(x-\xi)y+(v-\eta)u]\}d\xi d\eta dy du, \tag{40a}$$

2. $W_2(x,v,t)$

$$= \exp\left\{g_{-1}v\frac{\partial}{\partial x}\right\}W_1(x,v,t)$$

$$= W_1(x+g_{-1}v,v,t), \tag{40 b}$$

3. $W_3(x,v,t)$

$$= \exp\left\{\frac{1}{2}(\gamma t+g_0)x\frac{\partial}{\partial x}+\frac{1}{2}(\gamma t-g_0)v\frac{\partial}{\partial v}\right\}W_2(x,v,t)$$

$$= W_2\left(x\exp\left\{\frac{1}{2}(\gamma t+g_0)\right\}, v\exp\left\{\frac{1}{2}(\gamma t-g_0)\right\}, t\right), \tag{40 c}$$

4. $W(x,v,t)$

$$= \exp\{\gamma t\}\exp\left\{g_1 x\frac{\partial}{\partial v}\right\}W_3(x,v,t)$$

$$= \exp(\gamma t)W_3(x,v+g_1 x,t). \tag{40 d}$$

当初始分布为高斯分布

$$W(x,v,0)=\frac{1}{\pi a_x a_v}\exp\left\{-\frac{x^2}{a_x^2}-\frac{v^2}{a_v^2}\right\}, \tag{41}$$

可以求得

$$W(x,v,t)$$

$$= \frac{\exp(\gamma t)}{\pi B}\exp\left\{-\frac{\beta_1}{B^2}\exp(\gamma t)x^2\right\}\exp\left\{-\frac{\beta_2}{B^2}\exp(\gamma t)v^2\right\}$$

$$\cdot\exp\left\{-\frac{\beta_3}{B^2}\exp(\gamma t)xv\right\}, \tag{42}$$

其中

$$B=\sqrt{(4\alpha_1+a_x^2)(4\alpha_3+a_v^2)-4\alpha_2^2}, \tag{43 a}$$

$$\beta_1=(4\alpha_3+a_v^2)[\exp(g_0)+2g_1g_{-1}+g_1^2g_{-1}^2\exp(-g_0)]$$

$$+(4\alpha_1+a_x^2)g_1^2\exp(-g_0)-4\alpha_2[g_1+g_1^2g_{-1}\exp(-g_0)], \tag{43 b}$$

$$\beta_2=[(4\alpha_3+a_v^2)g_{-1}^2-4\alpha_2 g_{-1}+(4\alpha_1+a_x^2)]\exp(-g_0), \tag{43 c}$$

$$\beta_3=2(4\alpha_3+a_v^2)[g_{-1}+g_1g_{-1}^2\exp(-g_0)]$$

$$-4\alpha_2[1+2g_1g_{-1}\exp(-g_0)]+2(4\alpha_1+a_x^2)g_1\exp(-g_0). \tag{43 d}$$

将 $g_i(t)$ 与 $\alpha_i(t)$ 的表式代入,即可得文献[2]中的结果.

当裂变谐振子势中的角频率 ω_0 为时间 t 的函数时,只要解出 Riccati 方程 (25 a),亦可得到解析解表式. 应用群约缩方法,亦可处理偏离谐振子势的情况.

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The Algebraic Method to Solve the Fission F-P Equation

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ABSTRACT

Using Lie algebraic method to solve the time evolution equation. The integration formula of the fission F-P equation in the harmonic potential is obtained. This method can be used to deal with the variable coefficients cases.