Solving bound-state equations in QCD_2 with bosonic and fermionic quarks*

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Abstract: We investigate the bound-state equations (BSEs) in two-dimensional QCD in the $N_c \to \infty$ limit, viewed from both the infinite momentum frame (IMF) and the finite momentum frame (FMF). The BSE of a meson in the original 't Hooft model, *viz.*, spinor QCD_2 containing only fermionc quarks, has been extensively studied in literature. In this work, we focus on the BSEs pertaining to two types of "exotic" hadrons, a "tetraquark" which is composed of a bosonic quark and bosonic antiquark, and a "baryon" which is composed of a bosonic antiquark and a fermionic quark. Utilizing the Hamiltonian approach, we derive the corresponding BSEs for both types of "exotic" hadrons, from the perspectives of the light-front and equal-time quantization, and confirm the known results. The recently available BSEs for "tetraquark" in FMF has also been recovered with the aid of the diagrammatic approach. For the first time we also present the BSEs of a "baryon" in FMF in the extended 't Hooft model. By solving various BSEs numerically, we obtain the mass spectra pertaining to "tetraquark" and "baryon" and the corresponding bound-state wave functions of the lowest-lying states. It is numerically demonstrated that, when a "tetraquark" or "baryon" is continuously boosted, the forward-moving component of the bound-state wave function approaches the corresponding light-cone wave function, while the backward-moving component fades away.

Keywords: two-dimensional QCD, 1/Nc expansion, scalar quark, exotic hadrons, bound-state equations

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I. INTRODUCTION

Understanding the hadron structure from QCD is the central mission of the contemporary hadron physics. Due to our limited knowledge about the color confinement mechanism, we are still unable to write down, let alone to solve, the bound-state equations (BSEs) pertaining to any hadron in terms of the relativistic quark and gluons degrees of freedom. There are some influential and powerful nonperturbative approaches, such as Dyson-Schwinger (DS)/Bethe-Salpeter (BS) equations [1−4], as well as light-front (LF) quantization [5−8], which, under some approximation, enable one to numerically solve the BSEs of hadron in Minkowski spacetime. Unfortunately, at practical level, these approaches heavily depend on some unsystematized truncation, whose prediction is thus subject to certain amount of model dependence.

The limit of infinite number of color, *viz.*, $1/N_c$ expansion [9, 10], has proved to be a useful nonperturbative tool to help us to understand a number of some essential phenomena of QCD, such as Regge trajectory [10], $U(1)_A$ problem [11, 12], and so on. Unfortunately, due to the enormous complexity of nonabelian gauge theory in four spacetime dimension, at present we still do not know how to write down the BSEs for a hadron even in the large N_c limit, let alone to deduce the nonperturbative features of a hadron in a quantitative manner.

In 1974 't Hooft invented a solvable toy model of QCD, *i.e.*, QCD in two spacetime dimensions meanwhile in the $N_c \to \infty$ limit [13]. Thanks to the absence of transverse degree of freedom of the gauge field together with the planarity of Feynman diagrams, 't Hooft was able to write down the bound state equation of a meson in a closed form. The resulting discrete mesonic energy levels

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can be solved numerically, where the Regge trajectory becomes manifest. Soon it was also realized that 't Hooft model has also possesses some interesting properties like (naive) asymptotic freedom [14], nonvanishing quark condensate and "spontaneous" chiral symmetry breaking [15, 16]. Therefore, the 't Hooft model may be regarded as an instructive theoretical laboratory, which may help us to gain some insight into the nonperturbative aspects of QCD in the real world [14, 17−24].

The BSE of 't Hooft model was originally derived with the aid of the diagrammatic technique based on the DS/BS equations, also within the context of light-front quantization. Therefore, the *'t Hooft equation* is valid only in the infinite momentum frame (IMF), *viz.*, in which a meson moves with the speed of the light. It is worth noting that, an alternative approach to derive the 't Hooft equation is the operator approach based on bosonization of the light-front Hamiltonian, which has been extensively discussed in literature [25−31].

Poincaré invariance demands that the meson mass spectra must be identical in any inertial reference frame. One naturally wonders how the BSEs in 't Hooft model look like in the reference frame other than IMF. An important progress was made by Bars and Green in 1978 [32], who explicitly constructed a pair of coupled BSEs of mesons in QCD_2 in the finite momentum frame (FMF), in which a meson moves with a finite momentum, including static case. The resulting BSEs pertaining to FMF, dubbed *Bars-Green equation*, is much more involved than the 't Hooft equation pertaining to IMF. It was also formally demonstrated that the Poincaré algebra does hold in the color-singlet subspace [32]. Later Poincaré invariance of meson spectrum has been explicitly verified by numerically solving Bars-Green in FMFs (static meson [20] and moving meson [23]).

The advent of large momentum effective theory (LaMET) [33, 34] allows one to directly compute the partonic distributions of a hadron on the Euclidean lattice. A key element of LaMET is that, the quasi distributions, which are the matrix element of the equal-time, spacelike correlator sandwiched between a moving hadron state, under continuous Lorentz boost, will finally approach the light-cone parton distributions, which are the matrix element of the light-like correlator sandwiched between a hadron in IMF. 't Hooft model turns out to be a valuable theoretical laboratory to develop some intuition about the profiles of the quasi distributions. In the 't Hooft model, the light-cone distribution is simply linked with the 't Hooft wave function, and the quasi-distributions can be constructed in terms of the Bars-Green wave functions and Bogoliubov-chiral angle. It has been analytically and numerically verified that, a variety of quasi distributions in the 't Hooft model does converge to the light-cone distributions, as anticipated from LaMET [24, 35, 36].

Shortly after 't Hooft's original work, Shei and Tsao [37] in 1977 investigated the scalar QCD_2 which instead contains bosonic quarks. With the aid of the diagrammatic approach in the context of LF quantization, Shei and Tsao also derived the BSE for a meson composed of a bosnic quark and a bosonic antiquark. The *Shei-Tsao equation* looks very similar to the 't Hooft equation. In 1978 Tomaras rederived the Shei-Tsao equation from the angle of Hamiltonian approach, and elaborated on some subtlety pertaining to quark mass renormalization [38]. Utilizing the Hamiltonian approach in the context of the equal-time quantization, Ji, Liu and Zahed have recently also derived the BSEs in scalar QCD_2 pertaining to FMF [39], and showed that these BSEs do approach the Shei-Tsao equation when boosted to the IMF.

In the real world there is no bosonic quark. However, the notion of diquark turns out to be useful in baryon spectrum and structure, at least on the phenomenological ground [1)]. In 2008 Grinstein, Jora and Polosa investigated the mesonic mass spectra in scalar QCD_2 [49], who argued that, a bosonic quark may mimick a *diquark* to some extent, therefore the meson in scalar QCD_2 may be related to the tetraquark state in the real world, which is conjectured to consist of a compact diquark and antidiquark. It is expected that the study of the "tetraquark" spectrum in scalar QCD_2 [49] may shed some light on the tetraquark spectrum in the realistic QCD_4 [49].

It is difficult to investigate the BSE for a baryon in the original 't Hooft model, since a baryon would become infinitely heavy in the $N_c \to \infty$ limit. Nevertheless, once accepting the notion of the diquark, one may mimick a "baryon" by a bound state formed by a bosonic quark and fermionic antiquark in the extended 't Hooft model. The BSE for such a "baryon" state in IMF was first obtained with the aid of diagrammatic technique by Aoki [50].

It is the primary goal to derive the BSEs for a "baryon" state in the extended 't Hooft model in FMF, which constitute the counterparts of the Bars-Green equations of mesons in the original 't Hooft model. We also conduct a comprehensive numerical study of the "baryon" spectrum and the bound-state wave functions of the lowest-lying baryon. It is especially interesting to visualize how the wave functions in the FMF evolve to the light-cone wave

1) The diquark conjecture dates back to the early days of the quark model [40–41], by interpreting a baryon as a bound state composed of a diquark and another quark. The predictions based on the diquark ansatz reduce the number of levels in the baryon spectrum, which get closer to the experiment data than the three-body predictions in the quark model. In order for the lowest-lying octet and decuplet baryons to form a fully symmetric **56** representation of the spin-flavor $SU(6)$, the diquark in lowest-lying baryons must be in a $^3S_1$ state, which corresponds to a spin-1 boson. Intuitively, a stable diquark may be formed since the gluonic exchange between two quarks in the $\bar{\mathbf{3}}$ representation of color $SU(3)$ is attractive. Later some more sophisticated approaches [42−44], *e.g.*, a Faddeev-type formulation which preserves the Lorentz and chiral symmetries, also adopt the diquark picture Some recent development on diquark-based phenomenology can be found in Refs. [45−48].

functions when increasing the momentum of the "baryon".

We use the Hamiltonian approach in equal-time quantization with a "fermionization" procedure to derive the BSEs of "baryon" in FMF. For the sake of completeness, we also revisit the derivation of BSE of "baryon" in the IMF from the angle of the Hamiltonian approach in LF quantization. Since the "baryon" contains a bosonic quark, the BSEs of which are intimated connected with those of "tetraquark" composed entirely of bosonic quark and antiquark. Therefore, to facilitate a coherent reading, we feel it beneficial to give a self-contained treatment of both types of exotic hadrons. Therefore we decide to revisit the derivations of the BSEs of the "tetraquark in IMF and FMF using Hamiltonian approach, which were originally done in [38, 39]. Some subtle issue about the quark mass renormalization in LF and equal-time quantization in scalar QCD_2 is highlighted. Moreover, it is worth pointing out that the diagrammatic derivation of the BSEs of a "tetraquark" in FMF is much difficult than its counterpart in IMF, since the seagull vertex does not vanish even after taking the axial gauge. A novel outcome of this work is to successfully reproduce the BSEs of "tetraquark" from the angle of the diagrammatic DS/BS approaches.

The rest of the paper is organized as follows. In Sec. II, we define the extended 't Hooft model and set up some notations. In Sec. III, we revisit the derivation of BSEs of an "tetraquark" in both IMF and FMF, with some emphasis on the relation between the renormalized quark mass in LF and equal-time quantization. Sec. IV constitutes the main body of this work, where we derive the BSEs of a "baryon" in both IMF and FMF. The BSEs in FMF are obtained for the first time. In Sec. V we present the numerical results of the mass spectra of "tetraquark" and "baryon". In particular, we show the numerical profiles of the bound-state wave functions of the lowest-lying states, with different hadron velocities. We summarize in Sec. VI. We devote Appendix A to a diagrammatic derivation of the corresponding BSEs of "tetraquark" in FMF. In Appendix B, we present a detailed discussion on the connection between two renormalized quark masses introduced in LF quantization and equal-time quantization for the "tetraquark" case.

II. SETUP OF THE STAGE

The extended 't Hooft model contains both bosonic and fermionic quarks and gluons. For simplicity, we only consider a single species of bosonic quark and fermionic quark. The corresponding hybrid QCD_2 Lagrangian is dictated by the $SU(N_c)$ gauge invariance:

$$\mathcal{L}_{h\mathrm{QCD}_2} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\psi}(\mathrm{i}\not{D} - m_F)\psi + (D^\mu\phi)^\dagger D_\mu\phi - m_B^2\phi^\dagger\phi, \tag{1}$$

where ψ and ϕ denote the Dirac and complex scalar fields, m_F and m_B refer to the masses of the fermionic and bosonic quarks, and A^a_μ represents the gluon field with color index $a=1,2,\cdots,N_c^2-1$. $D_\mu = \partial_\mu - \mathrm{i}g_s A^a_\mu T^a$ signifies the color covariant derivative, and $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$ represents the gluonic field strength tensor.

The generators of $SU(N_c)$ group in the fundamental representation obey the following relation:

$$\mathrm{tr}(T^aT^b) = \frac{\delta^{ab}}{2}, \tag{2a}$$

$$\sum_a T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right). \tag{2b}$$

The Lorentz two-vector is defined as $x^\mu = \left(x^0, x^z\right)$, with the superscript 0 and z representing the time and spatial indices. The Dirac γ-matrices in two space-time dimensions are represented by

$$\gamma^0 = \sigma_1, \qquad \gamma^z = -\mathrm{i}\sigma_2, \qquad \gamma^5 \equiv \gamma^0\gamma^z = \sigma_3, \tag{3}$$

where σ^i $(i=1,2,3)$ signifies the Pauli matrices.

In LF quantization it is also convenient to adopt the light-cone coordinates, which are defined through $x^\pm = x_\mp = \left(x^0 \pm x^z\right)/\sqrt{2}$, with the light-front time denoted by x^+.

Throughout this work, we are interested in the large-N_c limit:

$$N_c \to \infty, \qquad \lambda \equiv \frac{g_s^2 N_c}{4\pi} \text{ fixed}, \tag{4}$$

with λ referring to the 't Hooft coupling constant. We are also tacitly working in the so-called weak coupling limit, where $m_F, m_B \gg g \sim 1/\sqrt{N_c}$ [15].

III. BSES FOR "TETRAQUARK" FROM HAMILTONIAN APPROACH

In this section we first revisit the derivation of BSE of a "tetraquark" in IMF, then revisit the derivation of the BSEs of a "tetraquark" in FMF. Some special attention is paid to the quark mass renormalization in both LF and equal-time quantization.

A. BSE of "tetraquark" in IMF

We start with rederivation of the BSE of "tetraquark" in the IMF using the operator approach [38]. For this purpose, it is most convenient to quantize the scalar QCD_2 in equal LF time.

1. *LF Hamiltonian of scalar* QCD_2

We express the scalar QCD_2 lagrangian in terms of light-cone coordinates. Similar to the original 't Hooft model, a great simplification can be achieved by imposing the light-cone gauge $A^{+a}=0$ [1)]:

$$\mathcal{L}_{\mathrm{sQCD}_2}=\frac{1}{2}\left(\partial_- A^{-a}\right)^2+\left(\partial_-\phi^\dagger\right)D_+\phi+\left(D_+\phi\right)^\dagger\partial_-\phi-m^2\phi^\dagger\phi. \tag{5}$$

The canonical conjugate momenta of bosonic quark fields are given by $\pi\equiv\dfrac{\partial\mathcal{L}}{\partial\left(\partial_+\phi^\dagger\right)}=\partial_-\phi$ and $\pi^\dagger=\partial_-\phi^\dagger$. After Legendre transformation, one arrives at the following LF Hamiltonian:

$$H_{\mathrm{LF}}=\int \mathrm{d}x^-\left(-\frac{1}{2}\left(\partial_- A^{-a}\right)^2+\mathrm{i}g_s A^{-a}\left(\pi^\dagger T^a\phi-\phi^\dagger T^a\pi\right)+m^2\phi^\dagger\phi\right). \tag{6}$$

Due to the absence of the light-front time derivative of the gluon field in (5), A^{-a} is no longer a dynamical variable, which is subject to the following constraint:

$$\partial_-^2 A^{-a}=g_s J^a, \tag{7}$$

with $J^a\equiv\mathrm{i}\left(\phi^\dagger T^a\pi-\pi^\dagger T^a\phi\right)$.

Solving A^{-a} in term of J^a in (7), and substituting back into (6), one then reduces the LF Hamiltonian to

$$H_{\mathrm{LF}}=\int \mathrm{d}x^-\left(m^2\phi^\dagger\phi-\frac{g_s^2}{2}J^a\frac{1}{\partial_-^2}J^a\right). \tag{8}$$

The LF Hamiltonian now becomes nonlocal. Note that the rigorous meaning of $1/\partial_-^2 J^a$ in (8) is

$$\frac{1}{\partial_-^2}J^a\left(x^-\right)=\int \mathrm{d}y^- G_\rho^{(2)}\left(x^- y^-\right)J^a\left(y^-\right), \tag{9}$$

where $G^{(2)}$ represents the Green function

$$\partial_-^2 G^{(2)}\left(x^-\right)=\delta\left(x^-\right). \tag{10}$$

The actual solution of the Green function turns out to be

$$G_\rho^{(2)}\left(x^- y^-\right)=-\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^+}{2\pi}\Theta\left(\left|k^+\right|-\rho\right)\frac{\mathrm{e}^{\mathrm{i}k^+\left(x^- y^-\right)}}{\left(k^+\right)^2}. \tag{11}$$

To render $G^{(2)}$ mathematically well-defined, we have introduced an infrared cutoff ρ to regularize the severe IR divergence pertaining to $k^+\to 0$. This parameter may also be viewed as an artificial gauge parameter. Needless to say, this fictitious parameter must disappear in the final expressions for any physical entities.

2. *Quantization and bosonization*

We impose the canonical quantization for the scalar QCD_2 in (8) in equal LF time. It is convenient to Fourier-expand the ϕ and π fields in terms of the quark and antiquark's annihilation and creation operators:

$$\phi^i\left(x^-\right)=\int_0^\infty\frac{\mathrm{d}k^+}{2\pi}\frac{1}{\sqrt{2k^+}}\left[a^i\left(k^+\right)\mathrm{e}^{-\mathrm{i}k^+x^-}+c^{i\dagger}\left(k^+\right)\mathrm{e}^{\mathrm{i}k^+x^-}\right], \tag{12a}$$

$$\pi^{j\dagger}\left(x^-\right)=\mathrm{i}\int_0^\infty\frac{\mathrm{d}k^+}{2\pi}\sqrt{\frac{k^+}{2}}\left[a^{j\dagger}\left(k^+\right)\mathrm{e}^{\mathrm{i}k^+x^-}-c^j\left(k^+\right)\mathrm{e}^{-\mathrm{i}k^+x^-}\right]. \tag{12b}$$

where $i,j=1,\cdots,N_c$ are color indices. The annihilation and creation operators are assumed to obey the standard commutation relations:

$$\left[a^i\left(k^+\right),a^{j\dagger}\left(p^+\right)\right]=\left[c^i\left(k^+\right),c^{j\dagger}\left(p^+\right)\right]=(2\pi)\delta\left(k^+-p^+\right)\delta^{ij}. \tag{13}$$

A useful trick to diagonalize the Hamiltonian is the bosonization technique [25−31]. One first introduces the following four compound color-singlet operators:

$$W\left(k^+,p^+\right)\equiv\frac{1}{\sqrt{N_c}}\sum_i c^i\left(k^+\right)a^i\left(p^+\right),$$
$$W^\dagger\left(k^+,p^+\right)\equiv\frac{1}{\sqrt{N_c}}\sum_i a^{i\dagger}\left(p^+\right)c^{i\dagger}\left(k^+\right), \tag{14a}$$

$$A\left(k^+,p^+\right)\equiv\sum_i a^{i\dagger}\left(k^+\right)a^i\left(p^+\right),$$
$$C\left(k^+,p^+\right)\equiv\sum_i c^{i\dagger}\left(k^+\right)c^i\left(p^+\right). \tag{14b}$$

It is straightforward to find the commutation rela-

1) For notational simplicity, we will use the symbol m instead of m_B to signify the bosonic quark mass in this section.

tions among these four compound operators:

$$\left[W\left(k_1^+,p_1^+\right),W^\dagger\left(k_2^+,p_2^+\right)\right] = (2\pi)^2\,\delta\left(k_1^+-k_2^+\right)\delta\left(p_1^+-p_2^+\right)+O\left(\frac{1}{N_\mathrm{c}}\right), \tag{15a}$$

$$\left[W\left(k_1^+,p_1^+\right),A\left(k_2^+,p_2^+\right)\right] = 2\pi\delta\left(p_1^+-k_2^+\right)W\left(k_1^+,p_2^+\right), \tag{15b}$$

$$\left[W\left(k_1^+,p_1^+\right),C\left(k_2^+,p_2^+\right)\right] = 2\pi\delta\left(k_1^+-k_2^+\right)W\left(p_2^+,p_1^+\right), \tag{15c}$$

$$\left[A\left(k_1^+,p_1^+\right),A\left(k_2^+,p_2^+\right)\right] = 2\pi\delta\left(p_1^+-k_2^+\right)A\left(k_1^+,p_2^+\right)-2\pi\delta\left(p_2^+-k_1^+\right)A\left(k_2^+,p_1^+\right), \tag{15d}$$

$$\left[C\left(k_1^+,p_1^+\right),C\left(k_2^+,p_2^+\right)\right] = 2\pi\delta\left(p_1^+-k_2^+\right)C\left(k_1^+,p_2^+\right)-2\pi\delta\left(p_2^+-k_1^+\right)C\left(k_2^+,p_1^+\right), \tag{15e}$$

$$\left[A\left(k_1^+,p_1^+\right),C\left(k_2^+,p_2^+\right)\right] = 0. \tag{15f}$$

Substituting (12) into the LF Hamiltonian in (8), and express everything in terms of the compound operator basis as specified in (14), we can break the light-front Hamiltonian into three pieces:

$$H_\mathrm{LF} = H_{\mathrm{LF};0} + :H_{\mathrm{LF};2}: + :H_{\mathrm{LF};4}: + O\left(\frac{1}{\sqrt{N_\mathrm{c}}}\right), \tag{16}$$

whose explicit expressions read

$$H_{\mathrm{LF};0} = N_c\int\frac{\mathrm{d}x^-}{2\pi}\left(\int_0^\infty\frac{m^2\mathrm{d}k^+}{2k^+}-\pi\lambda\int_0^\infty\frac{\mathrm{d}k_3^+}{2\pi}\int_0^\infty\frac{\mathrm{d}k_4^+}{2\pi}\frac{\left(k_3^+-k_4^+\right)^2}{\left(k_3^++k_4^+\right)^2k_3^+k_4^+}\Theta\left(|k_3^++k_4^+|-\rho\right)\right), \tag{17a}$$

$$\begin{aligned} :H_{\mathrm{LF};2}: = {} & m^2\int_0^\infty\frac{\mathrm{d}k^+}{2\pi 2k^+}\left[A\left(k^+,k^+\right)+C\left(k^+,k^+\right)\right]+\int_0^\infty\frac{\mathrm{d}k_1^+}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}k_2^+}{2\pi}\frac{2\pi\lambda}{k_1^+|k_2^+|}\left(\frac{k_1^++k_2^+}{k_2^+-k_1^+}\right)^2 \\ & \times\Theta(|k_2^+-k_1^+|-\rho)[A\left(k_1^+,k_1^+\right)+C\left(k_1^+,k_1^+\right)], \end{aligned} \tag{17b}$$

$$\begin{aligned} :H_{\mathrm{LF};4}: = {} & -\pi^3\lambda\int_0^\infty\frac{\mathrm{d}k_1^+}{2\pi\sqrt{k_1^+}}\int_0^\infty\frac{\mathrm{d}k_2^+}{2\pi\sqrt{k_2^+}}\int_0^\infty\frac{\mathrm{d}k_3^+}{2\pi\sqrt{k_3^+}}\int_0^\infty\frac{\mathrm{d}k_4^+}{2\pi\sqrt{k_4^+}}\frac{1}{\left(k_3^+-k_4^+\right)^2} \\ & \times\Theta\left(|k_3^+-k_4^+|-\rho\right)\left(k_1^++k_2^+\right)\left(k_4^++k_3^+\right)\times\left[W^\dagger\left(k_4^+,k_1^+\right)W\left(k_3^+,k_2^+\right)\delta\left(k_1^+-k_2^++k_4^+-k_3^+\right)\right. \\ & \left.+W^\dagger\left(k_2^+,k_3^+\right)W\left(k_1^+,k_4^+\right)\delta\left(k_2^+-k_1^++k_3^+-k_4^+\right)\right], \end{aligned} \tag{17c}$$

with : : denoting the normal ordering. $H_{\mathrm{LF};0}$ denotes the LF energy of the vacuum, which appears to be severely IR divergent.

The confinement characteristics of QCD indicates that all the physical excitation must be the color singlets. In the color-singlet subspace of states, the compound operators A and C are not independent operators, which, at lowest order in $1/N_\mathrm{c}$, actually can be expressed as the convolution between the color-singlet quark-antiquark pair creation/annihilation operators W and $W^\dagger$ in (14a):

$$A(k^+,p^+)\to\int_0^\infty\frac{\mathrm{d}q^+}{2\pi}W^\dagger(q^+,k^+)W(q^+,p^+), \tag{18a}$$

$$C(k^+,p^+)\to\int_0^\infty\frac{\mathrm{d}q^+}{2\pi}W^\dagger(k^+,q^+)W(p^+,q^+). \tag{18b}$$

Plugging (18) into (17b) and (17c), and relabelling the momenta in $W^{(\dagger)}(k^+,p^+)$ by $p^+=xP^+$, $k^+=(1-x)P^+$, we can rewrite the $:H_{\mathrm{LF};2}:$ and $:H_{\mathrm{LF};4}:$ pieces at the lowest order in $1/N_\mathrm{c}$ as

$$
\begin{aligned}
:H_{\mathrm{LF};2}:=&\int_0^\infty\frac{\mathrm{d}P^+P^+}{(2\pi)^2}\int_0^1\mathrm{d}xW^\dagger\left((1-x)P^+,xP^+\right)W\left((1-x)P^+,xP^+\right)\left[\frac{m^2}{2xP^+}+\frac{m^2}{2(1-x)P^+}\right.\\
&\left.+\frac{\lambda}{8xP^+}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{|y|}\frac{(x+y)^2}{(y-x)^2}\Theta\left(|(y-x)P^+|-\rho\right)+\frac{\lambda}{8(1-x)P^+}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{|1-y|}\frac{(2-x-y)^2}{(y-x)^2}\Theta\left(|(y-x)P^+|-\rho\right)\right],
\end{aligned}
\tag{19a}
$$

$$
\begin{aligned}
:H_{\mathrm{LF};4}:=&-\frac{\lambda}{2(2\pi)^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^\infty\frac{\mathrm{d}P^+(P^+)^2}{2\pi}W^\dagger\left((1-x)P^+,xP^+\right)W\left((1-y)P^+,yP^+\right)\\
&\times\frac{1}{\sqrt{xP^+}}\frac{1}{\sqrt{yP^+}}\frac{1}{\sqrt{(1-y)P^+}}\frac{1}{\sqrt{(1-x)P^+}}\frac{1}{[(x-y)P^+]^2}\Theta\left(|(x-y)P^+|-\rho\right)\left[(1-x)P^++(1-y)P^+\right]\left(yP^++xP^+\right).
\end{aligned}
\tag{19b}
$$

3. *Diagonalization of LF Hamiltonian*

Our strategy of deriving the BSE is by enforcing the light-front Hamiltonian (19) in a diagonalize form. For this purpose, we introduce an infinite tower of *tetraquark* annihilation/creation operators: $w_n(P^+)/w_n^\dagger(P^+)$, with n and P^+ indicating the principal quantum number and the light-cone momentum of the "meson" in the physical spectrum. We assume that the $w_n(P^+)/w_n^\dagger(P^+)$ operator basis can be transformed into the color-singlet quark-antquark pair creating/annihilation operator basis in the following fashion:

$$
W\left((1-x)P^+,xP^+\right)=\sqrt{\frac{2\pi}{P^+}}\sum_{n=0}^{\infty}\chi_n(x)w_n\left(P^+\right),
\tag{20a}
$$

$$
w_n\left(P^+\right)=\sqrt{\frac{P^+}{2\pi}}\int_0^1\mathrm{d}x\chi_n(x)W\left((1-x)P^+,xP^+\right),
\tag{20b}
$$

with the coefficient function $\chi_n(x)$ interpreted as the light-cone wave function of the n-th "tetraquark".

It is desirable to demand that the "tetraquark" annihilation and creation operators obey the standard commutation relations:

$$
\left[w_n\left(P_1^+\right),w_m^\dagger\left(P_2^+\right)\right]=2\pi\delta\left(P_1^+-P_2^+\right)\delta_{nm},
\tag{21}
$$

consequently the light-cone wave function $\chi_n(x)$ must satisfy the following orthogonality and completeness conditions:

$$
\int_0^1\mathrm{d}x\chi_n(x)\chi_m(x)=\delta_{nm},
\tag{22a}
$$

$$
\sum_n\chi_n(x)\chi_n(y)=\delta(x-y).
\tag{22b}
$$

The n-th "tetraquark" state can be constructed via

$$
|P_n^-,P^+\rangle=\sqrt{2P^+}\,w_n^\dagger(P^+)|0\rangle,
\tag{23}
$$

where $P_n^-=M_n^2/(2P^+)$ denotes the LF energy of the n-th excited "tetraquark" state, with M_n the respective tetraquark mass.

In the $N_c\to\infty$ limit, the scalar QCD_2 is composed of an infinite number of non-interacting mesons. To account for this fact, one anticipates that the LF Hamiltonian can be recast into a simple diagonal form in terms of the "tetraquark" annihilation/creation operators:

$$
H_{\mathrm{LF}}=H_{\mathrm{LF};0}+\int_0^\infty\frac{\mathrm{d}P^+}{2\pi}P_n^-\sum_n w_n^\dagger\left(P^+\right)w_n\left(P^+\right).
\tag{24}
$$

In order to arrive at the desired form (24), all the non-diagonal terms in (19) after transformed in the $w_n/w_n^\dagger$ basis, exemplified by $w_n^\dagger w_m$ $(m\neq n)$, $w^\dagger w^\dagger$, ww, $\cdots$, must vanish. This condition imposes some nontrivial constraint on the light-cone wave function $\chi_n(x)$, which can be cast into an integral equation:

$$
\begin{aligned}
&\left(\frac{m^2}{x}+\frac{m^2}{1-x}+\frac{\lambda}{4x}⨍_{-\infty}^{\infty}\frac{\mathrm{d}y}{|y|}\frac{(y+x)^2}{(y-x)^2}\right.\\
&\left.+\frac{\lambda}{4(1-x)}⨍_{-\infty}^{\infty}\frac{\mathrm{d}y}{|1-y|}\frac{(2-x-y)^2}{(y-x)^2}\right)\chi_n(x)\\
&-\frac{\lambda}{2}⨍_0^1\frac{\mathrm{d}y}{(x-y)^2}\frac{(2-x-y)(x+y)}{\sqrt{x(1-x)y(1-y)}}\chi_n(y)\\
&=M_n^2\chi_n(x).
\end{aligned}
\tag{25}
$$

Reassuringly, the potential IR divergence as $y\to x$ is tamed by the principal value (PV) prescription, denoted by the symbol $⨍$. Note the occurrence of the PV arises from taking the vanishing limit of the artificial IR regulator ρ first introduced in (11). Here we show two PV prescriptions defined in term of the IR regulator ρ [17, 51, 52]:

$$\rlap{-}\int \mathrm{d}y \frac{f(y)}{(x-y)^2} \equiv \lim_{\rho\to 0^+} \int \mathrm{d}y \frac{f(y)}{2}\left[\frac{1}{(x-y+i\rho)^2}+\frac{1}{(x-y-i\rho)^2}\right]$$
$$= \lim_{\rho\to 0^+} \int \mathrm{d}y \Theta(|x-y|-\rho)\frac{f(y)}{(x-y)^2} - \frac{2f(x)}{\rho}. \tag{26}$$

4. *Quark mass renormalization and the renormalized BSE*

Though the IR divergence is cured by the PV prescription, the BSE in scalar QCD_2, (25), is still plagued with logarithmic ultraviolet divergences, which arise as $y\to 0$ or $y\to\pm\infty$ in the first integral, and also arise as $y\to 1$ or $y\to\pm\infty$ in the second integral in (25).

As first pointed out by Shei and Tsao [38], it is essential to renormalize the quark mass m in order to eliminate the UV divergence. Concretely speaking, one introduce the renormalized quark mass m_r according to

$$m_r^2 = m^2 + \frac{\lambda}{2}\int_\delta^\Lambda \frac{\mathrm{d}y}{y}, \tag{27}$$

where the mass counterterm logarithmically depends on the UV cutoffs $\Lambda \gg \sqrt{\lambda}$, and $\delta\to 0^+$.

Replacing the integration boundaries of the first two integrals on the left side of (25) with $\int_{-\Lambda}^{-\delta}+\int_\delta^\Lambda$ and $\int_{-\Lambda}^{1-\delta}+\int_{1+\delta}^\Lambda$, respectively, and working out the integrals, it is straightforward to find that they diverge in the form of $\frac{\lambda}{2}\ln\Lambda/\delta$, which are canceled exactly by the quark mass counterterm, leaving out a finite remnant -2λ. Consequently the BSE becomes UV regular, which entails the renormalized quark mass only [38]:

$$\left(\frac{m_r^2-2\lambda}{x}+\frac{m_r^2-2\lambda}{1-x}\right)\chi_n(x)$$
$$-\frac{\lambda}{2}\rlap{-}\int_0^1 \frac{\mathrm{d}y}{(x-y)^2}\frac{(2-x-y)(x+y)}{\sqrt{x(1-x)y(1-y)}}\chi_n(y)$$
$$= M_n^2\chi_n(x). \tag{28}$$

Note that this BSE is similar to, albeit slightly more involved than, the celebrated 't Hooft's equation in spinor QCD_2.

B. BSE of "tetraquark" in FMF

It is advantageous to derive the BSE in QCD_2 in FMF in the familiar equal-time quantization. The BSE in scalar QCD_2 was recently derived with the aid of the operator approach in equal-time quantization [39]. The goal of this subsection is essentially to revisit the derivation in [39], with some new elements added. For instance, we present a new way of deriving the mass gap equation from the variational perspective, as well as elaborate on the subtlety pertaining to the quark mass renormalization. Moreover, we also for the first time employ the diagrammatic technique to derive the BSE of a tetraquark in FMF. We devote Appendix A to a detailed explanation of deriving the BSE of "tetraquark" in FMF from the perspective of DS/BS equations.

1. *The Hamiltonian in the axial gauge*

Similar to the treatment of the spinor QCD_2 [32], it is most convenient to choose the axial gauge $A^{az}=0$ to quantize the scalar QCD_2 in equal time. The Lagrangian then reduces to

$$\mathcal{L}_{\mathrm{sQCD}_2} = \frac{1}{2}(\partial_z A_0^a)^2 + (D_0\phi)^\dagger D_0\phi - (\partial_z\phi^\dagger)\partial_z\phi - m^2\phi^\dagger\phi. \tag{29}$$

The conjugate momenta are $\pi = D_0\phi$, $\pi^\dagger = (D_0\phi)^\dagger$. The Hamiltonian is obtained through the Legendre transformation:

$$H = \int \mathrm{d}z\left(|\partial_z\phi|^2 + m^2|\phi|^2 + \pi^\dagger\pi\right) + \frac{g_s^2}{2}\int \mathrm{d}z\left(J^a\frac{-1}{\partial_z^2}J^a\right), \tag{30}$$

with $J^a = \mathrm{i}\left(\phi^\dagger T^a\pi - \pi^\dagger T^a\phi\right)$.

Similar to (9), one can express $\frac{1}{\partial_z^2}J^a$ as the convolution between J^a and the Green function $\widetilde{G}^{(2)}$, which is defined through

$$\widetilde{G}_\rho^{(2)}(z) = -\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\Theta(|k|-\rho)\frac{\mathrm{e}^{\mathrm{i}kz}}{k^2}. \tag{31}$$

Analogous to (11), we again introduce an IR regulator ρ to make the Green function well-defined.

2. *Dressed quark basis, mass-gap equation, and quark mass renormalization*

One can conduct the equal-time quantization for the Hamiltonian as specified in (30). It is convenient to Fourier-expand the ϕ and $\pi^\dagger$ fields in the basis of the quark/antiquark's annihilation and creation operators:

$$\phi^i(z) = \int \frac{\mathrm{d}k}{2\pi\sqrt{2E_k}}\mathrm{e}^{\mathrm{i}kz}\left[a^i(k)+c^{i\dagger}(-k)\right], \tag{32a}$$

$$\pi^{j\dagger}(z) = \mathrm{i}\int\frac{\mathrm{d}k}{2\pi}\sqrt{\frac{E_k}{2}}\mathrm{e}^{-\mathrm{i}kz}\left[a^{j\dagger}(k)-c^j(-k)\right], \tag{32b}$$

where E_k denotes the energy of a *dressed* quark, whose concrete dispersion relation will be determined by the mass-gap equation in the following. The commutation relations between the quark annihilation and creation operators are the same as (13), except all the $+$-components are replaced with the z-components.

Analogous to the bosonizaion procedure adopted in the LF case in Sec. III.A.2, we introduce the following four color-singlet compound operators:

$$W(p,q) \equiv \frac{1}{\sqrt{N_c}} \sum_i c^i(-p) a^i(q),$$

$$W^\dagger(p,q) \equiv \frac{1}{\sqrt{N_c}} \sum_i a^{i\dagger}(q) c^{i\dagger}(-p), \tag{33a}$$

$$A(p,q) \equiv \sum_i a^{i\dagger}(p) a^i(q),$$

$$C(p,q) \equiv \sum_i c^{i\dagger}(-p) c^i(-q). \tag{33b}$$

The commutation relations among these compound operators are identical to (15), except all the + component are replaced with the z-components.

Analogous to (16), we decompose the Hamiltonian (30) into three pieces:

$$H = H_0 + :H_2: + :H_4: + O\left(\frac{1}{\sqrt{N_c}}\right), \tag{34}$$

whose explicit expressions read

$$H_0 = N_c \int \mathrm{d}z \left(\int \frac{\mathrm{d}k(k^2+m^2)}{2\pi(2E_k)} + \int \frac{\mathrm{d}k E_k}{4\pi} + \frac{\pi\lambda}{4} \int \frac{\mathrm{d}k_1}{2\pi} \int \frac{\mathrm{d}k_2}{2\pi} \frac{\left(E_{k_2} - E_{k_1}\right)^2}{(k_1-k_2)^2} \frac{1}{E_{k_1} E_{k_2}} \right), \tag{35a}$$

$$:H_2: = \int \frac{\mathrm{d}k}{2\pi} \widetilde{\Pi}^+(k) \left(A(k,k) + C(k,k)\right) + \sqrt{N_c} \int \frac{\mathrm{d}k}{2\pi} \widetilde{\Pi}^-(k) \left(W(k,k) + W^\dagger(k,k)\right), \tag{35b}$$

$$\begin{aligned} :H_4: = \frac{\lambda}{32\pi^2} \iiiint \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \frac{\delta(k_2-k_1+k_4-k_3)}{(k_4-k_3)^2} \Theta(|k_4-k_3|-\rho) \times \Big[-2 f_+(k_1,k_2) f_+(k_3,k_4) W^\dagger(k_1,k_4) W(-k_2,-k_3) \\ + f_-(k_1,k_2) f_-(k_3,k_4) W^\dagger(k_1,k_4) W^\dagger(k_3,k_2) + f_-(k_1,k_2) f_-(k_3,k_4) W(k_1,k_4) W(k_3,k_2) \Big], \end{aligned} \tag{35c}$$

with $\widetilde{\Pi}^\pm(k)$ and $f_\pm$ defined by

$$\widetilde{\Pi}^\pm(k) = \frac{1}{2}\left(\frac{k^2+m^2}{E_k} \pm E_k\right) + \frac{\lambda}{4} \int \mathrm{d}k_1 \frac{\frac{E_{k_1}}{E_k} \pm \frac{E_k}{E_{k_1}}}{(k+k_1)^2} \Theta(|k+k_1|-\rho), \tag{36a}$$

$$f_\pm(k_1,k_2) = \sqrt{\frac{E_{k_2}}{E_{k_1}}} \pm \sqrt{\frac{E_{k_1}}{E_{k_2}}}. \tag{36b}$$

It is desirable to put the $:H_2:$ piece, which governs the dressed quark energy, into a diagonalized form. For this purpose, the coefficient of the off-diagonal $W+W^\dagger$ term in $:H_2:$ is demanded to vanish. The constraint $\widetilde{\Pi}^- = 0$ then leads to a constraint for E_k [39]:

$$\frac{k^2+m^2}{E_k} - E_k + \frac{\lambda}{2} \int \mathrm{d}k_1 \left(\frac{E_{k_1}}{E_k} - \frac{E_k}{E_{k_1}}\right) \frac{1}{(k+k_1)^2} = 0. \tag{37}$$

This integral equation can be solved numerically to determine the dispersion relation. Following the spinor QCD_2 case [32], we also refer to this equation as the mass-gap equation.

Here we provide an alternative route to derive the mass-gap equation (37). The physical requirement is that the vacuum energy H_0 in (35a), albeit being severely divergent, should be minimized with respect to all possible functional forms of E_k. This requirement leads to a variational equation:

$$\frac{\delta H_0[E_k]}{\delta E_p} = 0, \tag{38}$$

which leads to

$$-\int \frac{\mathrm{d}k}{2\pi} \frac{k^2+m^2}{E_k^2} \delta(k-p) + \int \frac{\mathrm{d}k}{2\pi} \delta(k-p) + \pi\lambda \int \frac{\mathrm{d}k_1}{2\pi} \int \frac{\mathrm{d}k_2}{2\pi} \frac{1}{(k_1-k_2)^2} \left(\frac{1}{E_{k_2}} - \frac{E_{k_2}}{E_{k_1}^2}\right) \delta(k_1-p) = 0. \tag{39}$$

Conducting the integration, we then recover the mass-gap equation in (37) [39].

At first sight, the mass-gap equation (37) suffers from both IR and UV divergences. Actually, the potential IR divergence with $k_1 \to -k$ can be tamed by the PV prescription (26). However, as $|k_1| \to \infty$, one can ascertain that $E_{k1} \to |k_1|$, and the integral in (37) exhibits a logarithmic UV divergence. Fortunately, the UV divergence can be absorbed in the quark mass through the renormalization procedure. One may follow [39] to introduce the renormalized quark mass:

$$m_{\tilde{r}}^2 = m^2 + \frac{\lambda}{2} ⨍ \mathrm{d}k_1 \frac{E_{k_1}}{k_1^2}. \tag{40}$$

Slightly differing from [39], we have imposed the PV prescription to circumvent the IR divergence arising from the $k_1 \to 0$ region.

Plugging (40) into (37), we then obtain the renormalized mass-gap equation:

$$\frac{k^2 + m_{\tilde{r}}^2}{E_k} - E_k + \frac{\lambda}{2} ⨍ \mathrm{d}k_1 \left[\left(\frac{E_{k_1}}{E_k} - \frac{E_k}{E_{k_1}} \right) \frac{1}{(k+k_1)^2} - \frac{E_{k_1}}{E_k} \frac{1}{k_1^2} \right] = 0, \tag{41}$$

which is free from logarithmic UV divergence.

Since the prescribed renormalization scheme in equal-time quantization differs from that in the LF quantization, the renormalized mass $m_{\tilde{r}}^2$ is not necessarily equal to m_r introduced in (27). We devote Appendix B to a detailed discussion on the connection between these two renormalized quark masses.

3. *Bogoliubov transformation and diagonalization of Hamiltonian*

Following the same line of reasoning that leads to (18), the confinement feature of QCD implies that, at the lowest order in $1/N_c$, the compound operators A and C defined in (33) can be expressed as

$$A(k_1,k_2) \to \int \frac{\mathrm{d}p}{2\pi} W^\dagger(p,k_1) W(p,k_2),$$
$$C(k_1,k_2) \to \int \frac{\mathrm{d}p}{2\pi} W^\dagger(k_1,p) W(k_2,p). \tag{42}$$

With this replacement, the Hamiltonian (34) can be built solely out of the color-singlet compound operators W and $W^\dagger$:

$$H = \int \frac{\mathrm{d}p\mathrm{d}q}{4\pi^2} \left(\widetilde{\Pi}^+(p) + \widetilde{\Pi}^+(q) \right) W^\dagger(p,q) W(p,q) - \frac{\lambda}{32\pi^2} \int \mathrm{d}P \iint \mathrm{d}k\mathrm{d}p \frac{O+Q}{(p-k)^2} \Theta(|p-k|-\rho), \tag{43}$$

where

$$O = 2S_+(p,k,P) W^\dagger(P-p,p) W(P-k,k), \tag{44a}$$

$$Q = S_-(p,k,P) \Big(W^\dagger(p,P-p) W^\dagger(P-k,k) + W(p,P-p) W(P-k,k) \Big), \tag{44b}$$

with

$$S_\pm(p,k,P) = f_\pm(P-p,P-k) f_\pm(p,k). \tag{45}$$

To put the Hamiltonian (43) in a diagonalized form, it is advantageous to employ the Bogoliubov transformation [22]. The key is to introduce a new set of color-singlet "tetraquark" operators represented by w and $w^\dagger$. Schematically, two sets of color-singlet operators are connected through the following Bogoliubov transformation:

$$w = \mu W + \nu W^\dagger,$$
$$w^\dagger = \mu W^\dagger + \nu W,$$
$$\mu^2 - \nu^2 = 1. \tag{46}$$

The coefficient μ and ν can be determined such that the Hamiltonian gets diagonalized in the new operator basis.

In the case of scalar QCD_2, we introduce two infinite towers of color-singlet "tetraquark" annihilation and creation operators, w_n and $w_n^\dagger$, which are linear combinations of the W and $W^\dagger$ operators through Bogoliubov transformation. Inversely, we can express the W operators in terms of infinite sum of w_n and $w_n^\dagger$ operators:

$$W(q-P,q) = \sqrt{\frac{2\pi}{|P|}} \sum_{n=0}^{\infty} \Big[w_n(P) \chi_n^+(q,P) - w_n^\dagger(-P) \chi_n^-(q-P,-P) \Big]. \tag{47}$$

The operators $w_n(P)$ and $w_n^\dagger(P)$ bear clear physical meaning, which represent the annihilation and creation operators for the nth "tetraquark" state carrying momentum P. The coefficient functions $\chi_n^\pm$ can be interpreted as the forward/backward-moving "tetraquark" wave functions, playing the role of the Bogoliubov coefficients μ and ν in (46). The operators w_n and $w_n^\dagger$ are anticipated to obey the standard commutation relations:

$$\left[w_n(P), w_m^\dagger(P') \right] = 2\pi \delta_{nm} \delta\left(P - P'\right), \tag{48a}$$

$$\left[w_n(P), w_m(P')\right] = \left[w_n^{\dagger}(P), w_m^{\dagger}(P')\right] = 0. \tag{48b}$$

To fulfill these commutation relations, the "tetraquark" wave functions $\chi_n^{\pm}$ must satisfy the following orthogonality and completeness conditions

$$\int_{-\infty}^{+\infty} \mathrm{d}p \left[\chi_+^n(p,P)\chi_+^m(p,P) - \chi_-^n(p,P)\chi_-^m(p,P)\right] = |P|\delta^{nm}, \tag{49a}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}p \left[\chi_+^n(p,P)\chi_-^m(p-P,-P) - \chi_-^n(p,P)\chi_+^m(p,P)\right] = 0, \tag{49b}$$

$$\sum_{n=0}^{\infty} \left[\chi_+^n(p,P)\chi_+^n(q,P) - \chi_-^n(p-P,-P)\chi_-^n(q-P,-P)\right] = |P|\delta(p-q), \tag{49c}$$

$$\sum_{n=0}^{\infty} \left[\chi_+^n(p,P)\chi_-^n(q,P) - \chi_-^n(p-P,-P)\chi_+^n(q-P,-P)\right] = 0. \tag{49d}$$

The physical vacuum is defined by $w_n(P)|\Omega\rangle = 0$, Note $|\Omega\rangle$ differs from the dressed quark vacuum $|0\rangle$, which is defined by minimizing $:H_2:$. The n-th mesonic state can be constructed by

$$|P_n^0, P\rangle = \sqrt{2P_n^0}\, w_n^{\dagger}(P)|\Omega\rangle, \tag{50}$$

where $P_n^0 = \sqrt{M_n^2 + P^2}$ denotes the energy of the n-th mesonic state.

Plugging (47) into (43), we desire to put the Hamiltonian in a diagonalized form in terms of the "tetraquark" creation and annihilation operators:

$$H = H_0' + \int \frac{\mathrm{d}P}{2\pi} \sum_n P_n^0 w_n^{\dagger}(P) w_n(P) + O\left(\frac{1}{\sqrt{N_c}}\right), \tag{51}$$

where H_0' is the shifted vacuum energy.

Similar to the recipe leading to (24) in the light-front case, we enforce that all the non-diagonal operators (43) in the new $w_n/w_n^{\dagger}$ basis, such as $w_n^{\dagger} w_m$ $(m \neq n)$, $w^{\dagger} w^{\dagger}$, ww, $\cdots$, ought to vanish. This criterion imposes quite nontrivial constraints on the mesonic wave function $\chi_n^{\pm}(x)$. As a matter of fact, such constraint can be cast into two coupled integral equations for $\chi_n^{\pm}$:

$$\begin{aligned}
&\left(\Pi^+(p) + \Pi^+(P-p) \mp P_n^0\right)\chi_n^{\pm}(p,P) \\
&= \frac{\lambda}{4} \fint \frac{\mathrm{d}k}{(p-k)^2} \times \Big(S_+(p,k,P)\chi_n^{\pm}(k,P) \\
&\quad - S_-(p,k,P)\chi_n^{\mp}(k,P)\Big),
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
\Pi^+(k) = \widetilde{\Pi}^+(k) - \frac{\lambda}{\rho} &= \frac{1}{2}\left(\frac{k^2+m^2}{E_k} + E_k\right) \\
&+ \frac{\lambda}{4} \fint \mathrm{d}k_1 \frac{\dfrac{E_{k_1}}{E_k} + \dfrac{E_k}{E_{k_1}}}{(k+k_1)^2},
\end{aligned} \tag{53}$$

which agree with the "tetraquark" BSEs in scalar QCD$_2$ in FMF recently derived by Ji, Liu and Zahed [39].

Some remarks are in order. As proved in the spinor QCD$_2$ case by Bars and Green [32], when boosted to IMF, *i.e.*, taking $P \to \infty$ limit, the backward-moving mesonic wave function dies out, whereas the forward-moving mesonic wave function approaches the light-cone wave function, consequently Bars-Green equations will reduce to 't Hooft equation. This pattern perfectly fits into the tenet of LaMET, and one naturally anticipates the same story will repeat itself for scalar QCD$_2$. Indeed, as formally proved in [39], when boosted to IMF, the backward-moving "tetraquark" wave function χ_n^- does fade away, while the forward-moving "tetraquark" wave function χ_n^+ approaches the light-cone wave function $\chi_n(x)$ with $x \equiv p/P$. As a consequence, in the IMF the equal-time BSEs (52) descend to the Shei-Tsao equation in (28). In Sec. V we will provide numerical evidence for the aforementioned pattern, *viz.*, the forward-moving "tetraquark" wave function indeed tends to converge to its light-cone counterpart with the increasing "tetraquark" momentum.

IV. BSES FOR "BARYON" FROM HAMILTONIAN APPROACH

As mentioned in Introduction, if the bosonic quark can be interpreted as the diquark, the bound state formed by the fermionic quark and bosonic antiquark in the hybrid QCD$_2$ may bear some resemblance with the ordinary baryon in the real world. The goal of this section is to derive the BSEs of such a "baryon" in both IMF and FMF.

A. BSE of "baryon" in IMF

The BSE of "baryon" in the hybrid QCD2 was first obtained using diagrammatic approach in LF quantization by Aoki in 1993 [50]. Shortly after the "baryon" mass spectra were also studied by Aoki and Ichihara [53]. Note this BSE of "baryon" is valid only in the IMF. In this subsection, we will revise of the derivation of the "baryon" BSE in IMF [50], yet instead starting from the Hamiltonian approach.

1. *LF Hamiltonian of hybrid* QCD$_2$

We start from the hybrid QCD$_2$ Lagrangian which contains both scalar and spinor quarks. Imposing the light-cone gauge $A^{+a}=0$ and adopting the light-front coordinates, equation (1) reduces to

$$\mathcal{L}_{\rm hQCD_2} = \frac{1}{2}\left(\partial_- A^{-a}\right)^2 + i\left(\psi_R^\dagger D_+\psi_R + \psi_L^\dagger \partial_- \psi_L\right) - \frac{m_F}{\sqrt{2}}\left(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L\right) + \left(\partial_-\phi^\dagger\right) D_+\phi + (D_+\phi)^\dagger \partial_-\phi - m_B^2\phi^\dagger\phi. \tag{54}$$

with m_B and m_F signifying the masses of the bosonic and fermionic quarks, respectively. Two sets of canonical momenta are $\pi_\phi = \partial_-\phi, \pi_\psi = \psi_R$. Note ψ_L is a constrained rather than canonical variable. After Legendre transformation, we obtain the following LF Hamiltonian:

$$H_{\rm LF} = \int {\rm d}x^- \left[\frac{m_F}{\sqrt{2}}\psi_R^\dagger\psi_L + m_B^2\phi^\dagger\phi - \frac{g_s^2}{2} J^a \frac{1}{\partial_-^2} J^a\right], \tag{55}$$

with

$$J^a \equiv i\left(\phi^\dagger T^a\pi - \pi^\dagger T^a\phi\right) + \psi_R^\dagger T^a\psi_R. \tag{56}$$

2. *Compound operators and fermionization*

The bosonic quark field has been Fourier-expanded in the quark annihilation/creation operator basis in (12). The fermionic quark field can be Fourier-expanded accordingly,

$$\psi_R^i(x^-) = \int_0^\infty \frac{{\rm d}k^+}{2\pi}\left[b^i\left(k^+\right){\rm e}^{-{\rm i}k^+x^-} + {\rm d}^{i^\dagger}\left(k^+\right){\rm e}^{{\rm i}k^+x^-}\right]. \tag{57}$$

Following the bosonization procedure in the preceding section, here we introduce a set of color-singlet compound operators composed of the bosonic and fermionic quark annihilation and creation operators. Since the system we are studying is the "baryon", we refer to this procedure as *fermionization*.

Besides the bosonic compound operators already introduced in (14), we enumerate some new color-singlet compound operators [1]:

$$B\left(k^+,p^+\right) \equiv \sum_i b^{i\dagger}\left(k^+\right) b^i\left(p^+\right),$$
$$D\left(k^+,p^+\right) \equiv \sum_i {\rm d}^{i\dagger}\left(k^+\right){\rm d}^i\left(p^+\right), \tag{58a}$$

$$K\left(k^+,p^+\right) \equiv \frac{1}{\sqrt{N_{\rm c}}}\sum_i b^i\left(p^+\right)c^i\left(k^+\right),$$
$$\overline{K}\left(k^+,p^+\right) \equiv \frac{1}{\sqrt{N_{\rm c}}}\sum_i {\rm d}^i\left(k^+\right)a^i\left(p^+\right). \tag{58b}$$

Note the compound operator K annihilates a fermionic quark and a bosonic antiquark, whereas the compound operator $\overline{K}$ annihilates a fermionic antiquark and a bosonic quark. The anti-commutation relations among the fermionic compound operators K, $K^\dagger$, $\overline{K}$ and $\overline{K}^\dagger$ become

$$\left\{K\left(k_1^+,p_1^+\right), K^\dagger\left(k_2^+,p_2^+\right)\right\} = (2\pi)^2\delta\left(k_1^+-k_2^+\right)\delta\left(p_1^+-p_2^+\right) + O(1/N_c), \tag{59a}$$

$$\left\{\overline{K}\left(k_1^+,p_1^+\right), \overline{K}^\dagger\left(k_2^+,p_2^+\right)\right\} = (2\pi)^2\delta\left(k_1^+-k_2^+\right)\delta\left(p_1^+-p_2^+\right) + O(1/N_c), \tag{59b}$$

$$\left\{K\left(k_1^+,p_1^+\right), \overline{K}\left(k_2^+,p_2^+\right)\right\} = \left\{K^\dagger\left(k_1^+,p_1^+\right), \overline{K}\left(k_2^+,p_2^+\right)\right\} = 0, \tag{59c}$$

$$\left\{K\left(k_1^+,p_1^+\right), \overline{K}^\dagger\left(k_2^+,p_2^+\right)\right\} = \left\{K^\dagger\left(k_1^+,p_1^+\right), \overline{K}^\dagger\left(k_2^+,p_2^+\right)\right\} = 0. \tag{59d}$$

We are particularly interested in the interaction term in (55) that couples the bosonic quark sector with the fermionic sector, which can be expressed in terms of the product of the fermionic compound operators:

$$\begin{aligned} &-\left[{\rm i}g_s\left(\phi^\dagger T^a\pi - \pi^\dagger T^a\phi\right)\right]\frac{1}{\partial_-^2}\left(g_s\psi_R^\dagger T^a\psi_R\right) \\ &= -4\pi\lambda\int_0^\infty \frac{{\rm d}k_1^+}{2\pi\sqrt{2k_1^+}}\int_0^\infty \frac{{\rm d}k_2^+}{2\pi\sqrt{2k_2^+}}\int_0^\infty \frac{{\rm d}k_3^+}{2\pi} \\ &\times\int_0^\infty \frac{{\rm d}k_4^+}{2\pi}\frac{\left(k_1^++k_2^+\right)}{\left(k_3-k_4\right)^2}\Theta\left(|k_3-k_4|-\rho\right) \\ &\times 2\pi\delta\left(k_2^+-k_1^++k_3^+-k_4^+\right)\left[\overline{K}^\dagger\left(k_4^+,k_1^+\right)\overline{K}\left(k_3^+,k_2^+\right)\right. \\ &\left.+K^\dagger\left(k_2^+,k_3^+\right)K\left(k_1^+,k_4^+\right)\right]. \end{aligned} \tag{60}$$

1) Note here the normalization convention for the compound operators B and D follows that of [36], where they are scaled by a factor of $\sqrt{N_{\rm c}}$ compared to those defined in [24].

We then break the full LF Hamiltonian (55) into three pieces:

$$H_{\rm LF} = H_{\rm LF;0} + :H_{\rm LF;2}: + :H_{\rm LF;4}: + O(1/N_{\rm c}), \tag{61}$$

with

$$H_{\rm LF;0} = N_c \int \frac{{\rm d}x^-}{2\pi}\left(\int_0^\infty \frac{m_B^2 {\rm d}k^+}{2k^+} + \pi\lambda \int_0^\infty \frac{{\rm d}k_3^+}{2\pi}\int_0^\infty \frac{{\rm d}k_4^+}{2\pi}\frac{\left(k_3^+-k_4^+\right)\left(k_4^+-k_3^+\right)}{\left(k_3^++k_4^+\right)^2 k_3^+k_4^+}\times\Theta\left(|k_3^++k_4^+|-\rho\right)+\frac{\lambda}{2}+\frac{\lambda-m_F^2}{2}\int_\rho^\infty\frac{{\rm d}k^+}{k^+}\right), \tag{62a}$$

$$\begin{aligned}:H_{\rm LF;2}: :=& \int\frac{{\rm d}x^-}{2\pi}\int_0^\infty\frac{{\rm d}k^+}{2k^+}\left[m_B^2 A\left(k^+,k^+\right)+m_B^2 C\left(k^+,k^+\right)+m_F^2 B\left(k^+,k^+\right)+m_F^2 D\left(k^+,k^+\right)\right]\\ &+\lambda\int_0^\infty\frac{{\rm d}k^+}{2\pi}\left(\frac{1}{\rho}-\frac{1}{k^+}\right)\left[B\left(k^+,k^+\right)+D\left(k^+,k^+\right)\right]\\ &+\frac{\lambda}{4}\int_{-\infty}^\infty {\rm d}x^-\iint\frac{{\rm d}k_1^+\cdot{\rm d}k_2^+}{(2\pi)k_1^+k_2^+}\left(\frac{k_1^++k_2^+}{k_2^+-k_1^+}\right)^2\Theta(|k_2-k_1|-\rho)\left[A\left(k_1^+,k_1^+\right)+C\left(k_2^+,k_2^+\right)\right],\end{aligned} \tag{62b}$$

$$\begin{aligned}:H_{\rm LF;4}: :=& -g_s^2N_{\rm c}\int_0^\infty\frac{{\rm d}k_1^+}{2\pi\sqrt{2k_1^+}}\int_0^\infty\frac{{\rm d}k_2^+}{2\pi\sqrt{2k_2^+}}\int_0^\infty\frac{{\rm d}k_3^+}{2\pi}\int_0^\infty\frac{{\rm d}k_4^+}{2\pi}\frac{\left(k_1^++k_2^+\right)}{(k_3-k_4)^2}\Theta(|k_3-k_4|-\rho)\\ &\times 2\pi\delta\left(k_2^+-k_1^++k_3^+-k_4^+\right)\left[\overline{K}^\dagger\left(k_4^+,k_1^+\right)\overline{K}\left(k_3^+,k_2^+\right)+K^\dagger\left(k_2^+,k_3^+\right)K\left(k_1^+,k_4^+\right)\right].\end{aligned} \tag{62c}$$

As dictated by the confinement property of QCD, the bosonic color-singlet operators A, D, B and C are not independent, but can be replaced by the convolution of the following fermionic compound operators:

$$A\left(p^+,q^+\right)\to\int_0^\infty\frac{{\rm d}r^+}{2\pi}\overline{K}^+\left(r^+,p^+\right)\overline{K}\left(r^+,q^+\right), \tag{63a}$$

$$D\left(p^+,q^+\right)\to\int_0^\infty\frac{{\rm d}r^+}{2\pi}\overline{K}^+\left(p^+,r^+\right)\overline{K}\left(q^+,r^+\right), \tag{63b}$$

$$C\left(p^+,q^+\right)\to\int_0^\infty\frac{{\rm d}r^+}{2\pi}K^+\left(p^+,r^+\right)K\left(q^+,r^+\right), \tag{63c}$$

$$B\left(p^+,q^+\right)\to\int_0^\infty\frac{{\rm d}r^+}{2\pi}K^+\left(r^+,p^+\right)K\left(r^+,q^+\right). \tag{63d}$$

Substituting these relations to (62b) and (62c), relabelling the fermionic antiquark and a bosonic quark of the light-cone momenta by $p^+ = xP^+$ and $r^+ = (1-x)P^+$, and only retaining the leading-order terms in $1/N_{\rm c}$, the LF Hamiltonian can be solely built out of the fermionic compound operators K, $K^\dagger$, $\overline{K}$ and $\overline{K}^\dagger$:

$$\begin{aligned}:H_{\rm LF;2}: :=& \int_0^\infty\frac{{\rm d}P^+}{(2\pi)^2}\int_0^1{\rm d}xK^\dagger((1-x)P^+,xP^+)K((1-x)P^+,xP^+)\times\left[\frac{m_B^2}{2(1-x)}+\frac{m_F^2}{2x}+\frac{P^+\lambda}{\rho}-\frac{\lambda}{x}\right.\\ &\left.+\frac{1}{8(1-x)}\int_{-\infty}^\infty\frac{{\rm d}y}{|y|}\frac{(2-x-y)^2}{(y-x)^2}\Theta\left(|y-x|-\frac{\rho}{P^+}\right)\right]+(K\to\overline{K}),\end{aligned} \tag{64a}$$

$$\begin{aligned}:H_{\rm LF;4}: :=& -\frac{\lambda}{8\pi^2}\int_0^\infty{\rm d}P^+\iint_0^1{\rm d}x{\rm d}y\overline{K}^\dagger\left((1-x)P^+,xP^+\right)\overline{K}\left((1-y)P^+,yP^+\right)\\ &\times\frac{1}{\sqrt{(1-x)(1-y)}}\frac{2-x-y}{(y-x)^2}\Theta\left(|y-x|-\frac{\rho}{P^+}\right)+(K\to\overline{K}),\end{aligned} \tag{64b}$$

where the charge conjugation symmetry has been invoked to condense the expression.

3. *Diagonalization of LF Hamiltonian*

Our goal is to diagonalize the LF Hamiltonian in (61). For this purpose, it is advantageous to introduce a new set of operators $k_n(P^+)/k_n^\dagger(P^+)$ (together with $\bar{k}_n(P^+)/\bar{k}_n^\dagger(P^+)$), which annihilates/creates the n-th "baryon" ("anti-baryon") state, with n indicating the principal quantum number, and P^+ denotes the light-cone momentum of the corresponding "baryon" ("anti-baryon"). We hypothesize that the K basis and the k_n basis are connected through the following relation [1)]:

$$K\left((1-x)P^+,xP^+\right)=\sqrt{\frac{2\pi}{P^+}}\sum_{n=0}^{\infty}\Phi_n(x)k_n\left(P^+\right), \tag{65}$$

where the coefficient function $\Phi_n(x)$ stands for the light-cone wave function of the n-th excited "baryon" state.

If we demand that the "baryon" annihilation/creation operators $k_n/k_n^\dagger$ obey the standard anti-commutation relation:

$$\left\{k_n\left(P_1^+\right),k_m^\dagger\left(P_2^+\right)\right\}=2\pi\delta\left(P_1^+-P_2^+\right)\delta_{nm}, \tag{66}$$

the light-cone wave functions $\Phi_n(x)$ must satisfy the following orthogonality and completeness conditions:

$$\int_0^1 \mathrm{d}x\,\Phi_n(x)\Phi_m^*(x)=\delta_{nm}, \tag{67a}$$

$$\sum_n \Phi_n(x)\Phi_n^*(y)=\delta(x-y). \tag{67b}$$

The n-th "baryon" state can be constructed via

$$|P_n^-,P^+\rangle=\sqrt{2P^+}k_n^\dagger(P^+)|0\rangle, \tag{68}$$

with the light-cone energy $P_n^-=M_n^2/(2P^+)$, and M_n represents the mass of the n-th excited "baryon" state.

In order to ultimately put the LF Hamiltonian in the desired diagonalized form:

$$H_{\rm LF}=H_{\rm LF;0}+\int\frac{\mathrm{d}P^+}{2\pi}P_n^-\sum_n\left[k_n^\dagger(P^+)k_n(P^+)+\bar{k}_n^\dagger(P^+)\bar{k}_n(P^+)\right]. \tag{69}$$

all the non-diagonal terms in (64), after transformed into the $k_n/k_n^\dagger$ basis, such as $k_n^\dagger k_m$ $(m\neq n)$, $k^\dagger k^\dagger$, kk, $\cdots$, must vanish. This criterion imposes nontrivial constraint on the light-cone wave function $\Phi_n(x)$:

$$\begin{aligned}&\left(\frac{m_F^2-2\lambda}{x}+\frac{m_{B,r}^2-2\lambda}{1-x}\right)\Phi_n(x)\\&-\lambda⨍_0^1\frac{\mathrm{d}y}{\sqrt{(1-x)(1-y)}}\frac{2-x-y}{(x-y)^2}\Phi_n(y)\\&=M_n^2\Phi_n(x).\end{aligned} \tag{70}$$

with the renormalized bosonic quark mass $m_{B,r}$ introduced in (27).

Equation (70) is the desired BSE of "baryon" in the IMF, which agrees with what is originally derived by Aoki via diagrammatic approach [53].

B. BSEs of "baryon" in FMF

We proceed to derive the BSEs for the "baryon" in hybrid QCD$_2$ in FMF, based on the Hamiltonian approach in the context of the equal-time quantization. To the best of our knowledge, this set of BSEs has never been known before.

1. *Hamiltonian in the axial gauge*

Imposing the axial gauge $A^{az}=0$ in (1), the hybrid QCD$_2$ lagrangian reduces to

$$\begin{aligned}\mathcal{L}_{\rm hQCD_2}=&\frac{1}{2}\left(\partial_z A_0^a\right)^2+(D_0\phi)^\dagger D_0\phi-(\partial_z\phi^\dagger)\partial_z\phi\\&-m_B^2\phi^\dagger\phi+\mathrm{i}\psi^\dagger\left(D_0+\gamma^5\partial_z\right)\psi-m_F\bar{\psi}\psi.\end{aligned} \tag{71}$$

After Legendre transformation, we obtain

$$\begin{aligned}H=\int\mathrm{d}z\Big[&\pi^\dagger\pi+|\partial_z\phi|^2+m_B^2|\phi|^2+\psi^\dagger(-\mathrm{i}\gamma^5\partial_z+m_F\gamma^0)\psi\\&-\frac{g_s^2}{2}J^a\frac{1}{\partial_z^2}J^a\Big],\end{aligned} \tag{72}$$

with

$$J^a=\psi^\dagger T^a\psi-\mathrm{i}\left(\pi^\dagger T^a\phi-\phi^\dagger T^a\pi\right). \tag{73}$$

2. *Dressed quark basis and Fermionization*

The bosonic quark field can be Fourier-expanded in terms of the annihilation and creation operators in accordance with (32). The fermionic quark field is Fourier-expanded as [24]

1) Obviously the charge conjugation invariance indicates that $\overline{K}$ basis and the $\bar{k}_n$ basis are connected through the same relation, but with the coefficient functions interpreted as the light-cone wave functions of the "anti-baryon". Since the "baryon" and "anti-baryon" degrees of freedom decouple in the LF quantization, as is evident in (64), we will concentrate on the "baryon" BSE in this subsection. The "anti-baryon" BSE can be straightforwardly obtained via charge conjugation invariance.

$$\psi^i(z) = \int \frac{\mathrm{d}p}{2\pi} \frac{1}{\sqrt{2\widetilde{E}(p)}} \left[b^i(p)u(p) + \mathrm{d}^{i\dagger}(-p)v(-p)\right] \mathrm{e}^{\mathrm{i}pz}, \tag{74}$$

where the unregularized dispersion relation of the dressed fermionic quark $\widetilde{E}(p)$ is given by [24]

$$\widetilde{E}(p) = m_F \cos\theta(p) + p\sin\theta(p) + \frac{\lambda}{2}\int \frac{\mathrm{d}k}{(p-k)^2}\Theta(|k-p|-\rho)\cos[\theta(p)-\theta(k)], \tag{75}$$

with $\theta(p)$ representing the Bogoliubov-chiral angle. Note that the dispersion relation depends on an artificial IR regulator, which can also be viewed as a gauge artifact, since the energy of an colored object like QCD cannot be a physical quantity.

The Bogoliubov-chiral angle can be determined through the mass-gap equation in spinor QCD_2 [32]:

$$p\cos\theta(p) - m_F\sin\theta(p) = \frac{\lambda}{2} ⨍_{-\infty}^{+\infty} \frac{\mathrm{d}k}{(p-k)^2} \sin\left[\theta(p)-\theta(k)\right]. \tag{76}$$

Like the mass gap equation in scalar QCD_2, (37), this mass gap equation can also be deduced via variational approach, *viz.*, by enforcing the vacuum energy to be minimized.

Analogous to the fermionization procedure employed in Sec. IV.A.2, we introduce the following color-singlet compound operators:

$$B(p,q) \equiv \sum_i b^{i\dagger}(p)b^i(q), \qquad D(p,q) \equiv \sum_i \mathrm{d}^{i\dagger}(-p)\mathrm{d}^i(-q), \tag{77a}$$

$$M(p,q) \equiv \sum_i d_i(-p)b_i(q), \qquad M^\dagger(p,q) \equiv \sum_i b_i^\dagger(q)d_i^\dagger(-p), \tag{77b}$$

$$K(p,q) \equiv \frac{1}{\sqrt{N_\mathrm{c}}}\sum_i b^i(q)c^i(-p), \qquad K^\dagger(p,q) \equiv \frac{1}{\sqrt{N_\mathrm{c}}}\sum_i b^{i\dagger}(q)c^{i\dagger}(p), \tag{77c}$$

$$\overline{K}(p,q) \equiv \frac{1}{\sqrt{N_\mathrm{c}}}\sum_i \mathrm{d}^i(-p)a^i(q), \qquad \overline{K}^\dagger(p,q) \equiv \frac{1}{\sqrt{N_\mathrm{c}}}\sum_i \mathrm{d}^{i\dagger}(-p)a^{i\dagger}(q). \tag{77d}$$

Note that the compound operator K annihilates a fermionic quark and a bosonic antiquark, and the compound operator $\overline{K}$ annihilates a fermionic antiquark and a bosonic quark. The anticommutation relations among four fermionic compound color-singlet operators K, $K^\dagger$, $\bar{K}$ and $\overline{K}^\dagger$ become

$$\left\{K(k_1,p_1), K^\dagger(k_2,p_2)\right\} = (2\pi)^2\delta(k_1-k_2)\delta(p_1-p_2) + O(1/N_c), \tag{78a}$$

$$\left\{\overline{K}\left(k_1^+,p_1^+\right), \overline{K}^\dagger\left(k_2^+,p_2^+\right)\right\} = (2\pi)^2\delta(k_1-k_2)\delta(p_1-p_2) + O(1/N_c), \tag{78b}$$

$$\left\{K(k_1,p_1), \overline{K}(k_2,p_2)\right\} = \left\{K^\dagger(k_1,p_1), \overline{K}(k_2,p_2)\right\} = 0, \tag{78c}$$

$$\left\{K(k_1,p_1), \overline{K}^\dagger(k_2,p_2)\right\} = \left\{K^\dagger(k_1,p_1), \overline{K}^\dagger(k_2,p_2)\right\} = 0. \tag{78d}$$

Since we are solely interested in the "baryon" state, the interaction term in (72) that couples the bosonic quark sector with the fermionic sector is of pivotal importance. It can be expressed in terms of the products of the fermionic compound operators:

$$\begin{aligned} g_s\psi^\dagger T^a\psi\frac{1}{\partial_z^2}\left[g_s i\left(\pi^\dagger T^a\phi - \phi^\dagger T^a\pi\right)\right] = &-\pi\lambda\int\frac{\mathrm{d}p_1}{2\pi}\int\frac{\mathrm{d}p_2}{2\pi}\int\frac{\mathrm{d}k_1}{2\pi}\int\frac{\mathrm{d}k_2}{2\pi}\frac{1}{(k_2-k_1)^2}\Theta(|k_2-k_1|-\rho)2\pi\delta(p_2-p_1+k_2-k_1) \\ &\times\left\{f_+(k_1,k_2)\cos\frac{\theta(p_1)-\theta(p_2)}{2}\left((K^\dagger(k_2,p_1)K(k_1,p_2) + \overline{K}^\dagger(p_2,k_1)\overline{K}(p_1,k_2)\right)\right. \\ &\left. + f_-(k_1,k_2)\sin\frac{\theta(p_1)-\theta(p_2)}{2}\left(K^\dagger(k_2,p_1)\overline{K}^\dagger(p_2,k_1) + \overline{K}(p_1,k_2)K(k_1,p_2)\right)\right\}, \end{aligned} \tag{79}$$

where $f_\pm$ is given in (36b).

Expressing everything in terms of the color-singlet compound operators, we then break the full Hamiltonian (72) into three pieces:

$$H = H_0 + : H_2 : + : H_4 : + O(1/N_c) \tag{80}$$

with

$$H_0 = N_c \int \mathrm{d}z \Big\{ \int \frac{\mathrm{d}k(k^2+m_1^2)}{2\pi(2E_k)} + \int \frac{\mathrm{d}kE_k}{4\pi} + \frac{\pi\lambda}{2}\int\frac{\mathrm{d}k_1}{2\pi}\int\frac{\mathrm{d}k_2}{2\pi}\frac{\left(E_{k_2}-E_{k_1}\right)^2}{(k_1-k_2)^2}\frac{1}{E_{k_1}E_{k_2}}$$
$$+ \frac{\mathrm{d}p_1}{2\pi}\mathrm{Tr}\left[\left(p_1\gamma^5+m_2\gamma^0\right)\Lambda_-(p_1)+\frac{\lambda}{2}\int\frac{\mathrm{d}p_2}{(p_2-p_1)^2}\Theta(|p_2-p_1|-\rho)\Lambda_+(p_1)\Lambda_-(p_2)\right]\Big\}, \tag{81a}$$

$$: H_2 := \int \frac{\mathrm{d}k}{2\pi}\widetilde{\Pi}^+(k)\left(A(k)+C(k)\right)+\int\frac{\mathrm{d}p}{2\pi}\widetilde{E}(p)\left(B(p,p)+D(p,p)\right), \tag{81b}$$

$$: H_4 := -\pi\lambda\int\frac{\mathrm{d}p_1}{2\pi}\int\frac{\mathrm{d}p_2}{2\pi}\int\frac{\mathrm{d}k_1}{2\pi}\int\frac{\mathrm{d}k_2}{2\pi}\frac{1}{(k_2-k_1)^2}\Theta(|k_2-k_1|-\rho)2\pi\delta(p_2-p_1+k_2-k_1)$$
$$\times\Big\{f_+(k_1,k_2)\cos\frac{\theta(p_1)-\theta(p_2)}{2}\left((K^\dagger(k_2,p_1)K(k_1,p_2)+\overline{K}^\dagger(p_2,k_1)\overline{K}(p_1,k_2)\right)$$
$$+f_-(k_1,k_2)\sin\frac{\theta(p_1)-\theta(p_2)}{2}\left(K^\dagger(k_2,p_1)\overline{K}^\dagger(p_2,k_1)+\overline{K}(p_1,k_2)K(k_1,p_2)\right)\Big\}, \tag{81c}$$

where

$$\Lambda_\pm(k)=T(k)\frac{1\pm\gamma^0}{2}T^\dagger(k), \qquad T(k)=\exp\left[-\frac{1}{2}\theta(k)\gamma^z\right]. \tag{82}$$

Demanding that the $: H_2 :$ piece to bear a diagonalized form separately for the dressed bosonic quarks and fermionic quarks, we can obtain the mass-gap equations for both types of quarks, (37) and (76).

3. *Bogoliubov transformation, diagonalization of Hamiltonian*

Like in the preceding subsection, the color confinement characteristic of QCD indicates that the bosonic color-singlet compound operators A, B, C and D can not be independent, yet at lowest order in $1/N_c$ can be expressed in terms of the convolutions of the following fermionic color-singlet compound operators:

$$A(k_1,k_2)\to\int\frac{\mathrm{d}p}{2\pi}\overline{K}^\dagger(p,k_1)\overline{K}(p,k_2),$$
$$D(k_1,k_2)\to\int\frac{\mathrm{d}p}{2\pi}\overline{K}^\dagger(k_1,p)\overline{K}(k_2,p), \tag{83a}$$

$$C(k_1,k_2)\to\int\frac{\mathrm{d}p}{2\pi}K^\dagger(k_1,p)K(k_2,p),$$
$$B(k_1,k_2)\to\int\frac{\mathrm{d}p}{2\pi}K^\dagger(p,k_1)K(p,k_2). \tag{83b}$$

Making these replacements in (80), and only retaining the leading terms in $1/N_c$, the Hamiltonian now only depends on the fermionic compound operators K, $\overline{K}$ and their Hermitian conjugates:

$$: H_2 := \iint\frac{\mathrm{d}P\mathrm{d}q}{(2\pi)^2}\left(\widetilde{\Pi}^+(P-q)+\widetilde{E}(q)\right)\overline{K}^\dagger(q,q-P)\overline{K}(q,q-P)+\left(\widetilde{\Pi}^+(P-q)+\widetilde{E}(q)\right)K^\dagger(q-P,q)K(q-P,q), \tag{84a}$$

$$: H_4 := -\frac{\lambda}{8\pi^2}\int\mathrm{d}P\iint\frac{\mathrm{d}q\mathrm{d}k}{(k-q)^2}\Theta(|k-q|-\rho)\Big\{f_+(q-P,k-P)\cos\frac{\theta(k)-\theta(q)}{2}$$
$$\times\left(\overline{K}^\dagger(q,q-P)\overline{K}(k,k-P)+K^\dagger(k-P,k)K(q-P,q)\right)$$
$$+f_-(q-P,k-P)\sin\frac{\theta(k)-\theta(q)}{2}\left(\overline{K}(k,k-P)K(q-P,q)+K^\dagger(k-P,k)\overline{K}^\dagger(q,q-P)\right)\Big\}. \tag{84b}$$

To diagonalize the Hamiltonian, we invoke the Bogoliubov transformation (46), *viz.*, by expressing the fermionic color-singlet compound operators K, $\overline{K}$ in terms of the annihilation/creation operators of the "baryon" and "anti-baryon":

$$K(q-P,q) = \sqrt{\frac{2\pi}{|P|}}\sum_{n=0}^{\infty}\Big[k_n(P)\,\Phi_n^+(q,P) + \bar{k}_n^\dagger(-P)\,\overline{\Phi}_n^-(q-P,-P)\Big], \tag{85a}$$

$$\overline{K}(q,q-P) = \sqrt{\frac{2\pi}{|P|}}\sum_{n=0}^{\infty}\Big[\bar{k}_n(-P)\overline{\Phi}_n^+(q-P,-P) + k_n^\dagger(P)\,\Phi_n^-(q,P)\Big]. \tag{85b}$$

where k_n annihilates the n-th excited "baryon" state and $\bar{k}_n$ annihilates the n-th excited "anti-baryon" state. The Bogoliubov coefficient functions $\Phi_\pm^n$ can be interpreted as the forward-moving/backward-moving wave functions of the n-th excited "baryon" state, whereas the Bogoliubov coefficient functions $\overline{\Phi}_\pm^n$ can be interpreted as the forward-moving/backward-moving wave functions of the n-th "anti-baryon" state.

It is natural to anticipate that these "baryon/anti-baryon" annihilation and creation operators obey the standard anti-commutation relations:

$$\{k_n(P),k_m^\dagger(P')\} = 2\pi\delta_{nm}\delta(P-P'),\quad \{\bar{k}_n(P),\bar{k}_m^\dagger(P')\} = 2\pi\delta_{nm}\delta(P-P'), \tag{86a}$$

$$\{k_n(P),\bar{k}_m(P')\} = \{k_n^\dagger(P),\bar{k}_m^\dagger(P')\} = 0. \tag{86b}$$

The physical vacuum is defined by $k_n(P)|\Omega\rangle = \bar{k}_n(P)|\Omega\rangle = 0$ for any P and n. One then constructs a single "baryon" and "anti-baryon" states as

$$|P_n^0,P\rangle = \sqrt{2P_n^0}k_n^\dagger(P)\,|\Omega\rangle, \qquad |P_n^0,P\rangle = \sqrt{2P_n^0}\bar{k}_n^\dagger(P)\,|\Omega\rangle, \tag{87}$$

where $P_n^0 = \sqrt{M_n^2+P^2}$ with M_n denoting the mass of the n-th "baryon" state.

To be compatible with (86), the "baryon" and "anti-baryon" wave functions, $\Phi_n^\pm$ and $\overline{\Phi}_n^\pm$, must obey the following orthogonality and completeness conditions:

$$\int_{-\infty}^{+\infty} \mathrm{d}p\,\Big[\Phi_n^+(p,P)\,\Phi_m^+(p,P) + \Phi_n^-(p,P)\,\Phi_m^-(p,P)\Big] = |P|\delta^{nm}, \tag{88a}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}p\,\Big[\overline{\Phi}_n^-(p-P,-P)\,\overline{\Phi}_m^-(p-P,-P) + \overline{\Phi}_n^+(p-P,-P)\overline{\Phi}_m^+(p-P,-P)\Big] = |P|\delta^{nm}, \tag{88b}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}p\,\Big[\Phi_n^+(p,P)\,\overline{\Phi}_m^-(p-P,-P) + \Phi_n^-(p,P)\,\overline{\Phi}_m^+(p-P,-P)\Big] = 0, \tag{88c}$$

$$\sum_{n=0}^{\infty}\Big[\overline{\Phi}_n^+(p-P,-P)\,\overline{\Phi}_n^+(q-P,-P) + \Phi_n^-(p,P)\Phi_n^-(q,P)+\Big] = |P|\delta(p-q), \tag{88d}$$

$$\sum_{n=0}^{\infty}\Big[\Phi_n^+(p,P)\,\Phi_n^+(q,P) + \Phi_n^-(p-P,-P)\,\Phi_n^-(q-P,-P)\Big] = |P|\delta(p-q), \tag{88e}$$

$$\sum_{n=0}^{\infty}\Big[\Phi_n^+(p,P)\,\Phi_n^-(q,P) + \overline{\Phi}_n^+(q-P,-P)\,\overline{\Phi}_n^-(p-P,-P)\Big] = 0. \tag{88f}$$

Switching to k_n and $\bar{k}_n$ basis, we anticipate that all the non-diagonal terms in the Hamiltonian in (80) vanish, and end up with the desired diagonalized form:

$$H = H_0' + \int\frac{\mathrm{d}P}{2\pi}\sum_n P_n^0\Big[k_n^\dagger(P)k_n(P) + \bar{k}_n^\dagger(P)\bar{k}_n(P)\Big] + O(1/\sqrt{N_c}). \tag{89}$$

Demanding all the non-diagonalized terms to vanish, we find that the "baryon" wave functions $\Phi^n_\pm$ must satisfy the following coupled integral equations:

$$\begin{aligned}&\left(\Pi^+(P-p)+E(p)-P^0_n\right)\Phi^+_n(p,P)\\&=\frac{\lambda}{2}⨍\frac{\mathrm{d}k}{(k-p)^2}\left[f_+(k-P,p-P)\cos\frac{\theta(p)-\theta(k)}{2}\Phi^+_n(k,P)\right.\\&\left.+f_-(k-P,p-P)\sin\frac{\theta(p)-\theta(k)}{2}\Phi^-_n(k,P)\right],\end{aligned}\tag{90a}$$

$$\begin{aligned}&\left(\Pi^+(P-p)+E(p)+P^0_n\right)\Phi^-_n(p,P)\\&=\frac{\lambda}{2}⨍\frac{\mathrm{d}k}{(k-p)^2}\left[f_+(k-P,p-P)\cos\frac{\theta(p)-\theta(k)}{2}\Phi^-_n(k,P)\right.\\&\left.-f_-(k-P,p-P)\sin\frac{\theta(p)-\theta(k)}{2}\Phi^+_n(k,P)\right],\end{aligned}\tag{90b}$$

where the regularized dressed quark energy $E(p)$ is defined by [32]:

$$\begin{aligned}E(p)=\widetilde{E}(p)-\frac{\lambda}{\rho}&=m_F\cos\theta(p)+p\sin\theta(p)\\&+\frac{\lambda}{2}⨍\frac{\mathrm{d}k}{(p-k)^2}\cos[\theta(p)-\theta(k)].\end{aligned}\tag{91}$$

Equations (90) represent the BSEs for a "baryon" in hybrid QCD_2 in FMF, which constitute the main new results of this paper.

When boosted to the IMF, one can show that the BSEs (90) reduce to its LF counterpart, (70). In the other words, as the "baryon" momentum is increasing, the backward-moving wave function Φ^n_- quickly fades away, while the forward-moving wave function Φ^n_+ approaches the light-front wave function $\Phi^n(x)$.

V. NUMERICAL RESULTS

The numerical recipe of solving the BSEs in 't Hooft model has been adequately discussed in literature. We follow the approach outlined in [19] to solve the BSEs of the "tetraquark" and "baryon" in IMF, and follow the approach based on Hermite function expansion [21, 23] to solve the BSEs of the "extotic" hadrons in FMF.

The dimensionful 't Hooft coupling is usually taken to be $\sqrt{2\lambda}=340$ MeV, which is close to the value of string tension in QCD_4. For simplicity, we use $\sqrt{2\lambda}$ as the unit of the mass in the rest of this section. The renormalized mass of bosonic quark in equal-time quantization is taken to be $m_B^{\rm ET}\equiv m_{B,\tilde{r}}=1.5$, which is equivalent to the one defined in LF quantization, $m_{\rm B}^{\rm LC}\equiv m_{B,r}=1.69$ [1]. As for fermionic quarks, we choose the strange quark mass as $m_F=0.749$ and charm quark mass as $m_F=4.19$ [23].

In Fig. 1, the mass spectra of "tetraquark" and "baryon" with different quark species are plotted against the principal quantum number n. One can observe the tendency that the squared hadron mass linearly grows with n when the principle quantum number gets large. This pattern is identical with the Regge trajectory observed in the original 't Hooft model, which is also somewhat analogous to the Regge trajectories observed in the real world, where the squared mass of the excited hadronic state linearly grows with the spin.

In passing we mention a technical nuisance, *i.e.*, it is found that, when numerically solving the Shei-Tsao equation, the renormalized bosonic quark mass $m_{\rm B}^{\rm LC}$ can not be chosen less than unity. As pointed out in [54], such case may corresponds to a strongly coupled regime, where the the LF wave function near the endpoints is no longer real-valued.

In Fig 2, we plot the equal-time and light-front (LF) meson wave functions pertaining to "tetraquark" and "baryon", including both the ground states and the first excited states. For a "tetraquark" consisting of a single flavor of bosonic quark, the LF wave functions with even/odd n are symmetric/antisymmetric under the exchange $x\leftrightarrow(1-x)$ due to the charge conjugation symmetry. The LF wave functions always vanish in the endpoints $x=0,1$.

It is interesting to see how the bound-state wave functions obtained in equal-time quantization evolve with the hadron's momentum. As is evident in Fig 2, as the hadron momentum increases, the forward-moving wave functions of "tetraquark", χ_+, and of "baryon", Φ_+, rapidly converge to the corresponding light-cone wave functions. We have also numerically verified that, the backward-moving wave functions, χ_- and Φ_-, do quickly fade away with increasing hadron momentum. This finding is identical with what is observed in the numerical study of the original 't Hooft model [23], which is compatible with the tenet of LaMET.

VI. SUMMARY

In this work, we have made a comprehensive study of the extended 't Hooft model, *viz.*, two-dimensional QCD including both fermionic and bosonic quarks in $N_c\to\infty$ limit. We focus on derivations of the bound-state equations pertaining to two types of hadrons, a "tetraquark" composed of a bosonic quark and a bosonic antiquark, and a "baryon" composed of a fermionic quark and bo-

1) We mention that the connection of two different renormalized bosonic quark masses is given in (B7), since we follow [39] to impose the bosonic quark renormalization condition in accordance with (40). Of course, we could have chosen a different renormalization scheme as specified in (B8), such that $m_{B,\tilde{r}}=m_{B,r}$.

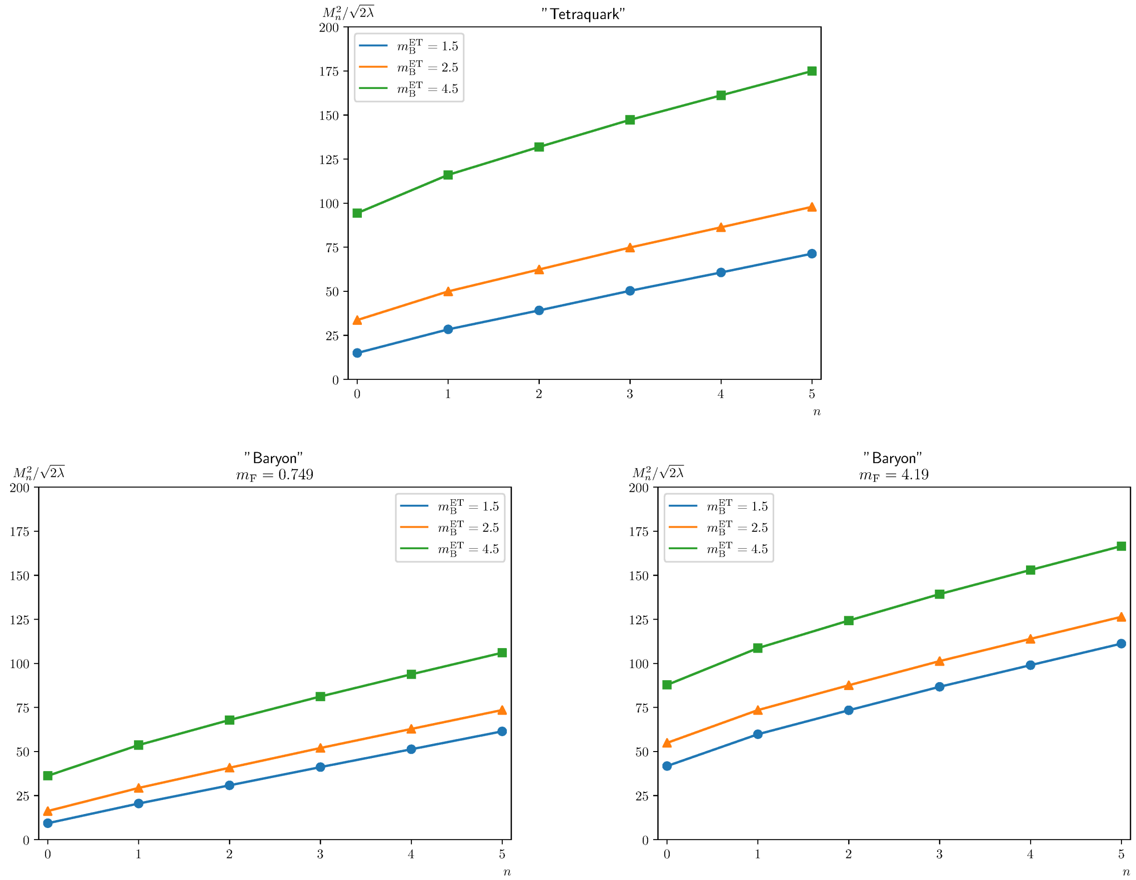


Fig. 1. (color online) Mass spectra of the "tetraquark" and "baryon" with some different set of quark masses.

sonic antiquark. Using the Hamiltonian approach, we derive these BSEs from the perspectives of light-front and equal-time quantization, which are associated with the IMF and FMF, respectively. We confirm the known results, such as those BSEs of "tetraquark" in IMF [37] and in FMF [39]. We have paid special attention to the issue concerning quark mass renormalization. It is found that, the renormalized bosonic quark mass in scalar QCD_2 in light-front quantization may not coincide with that in equal-time quantization, and the relationship between these two types of renormalized quark mass is established. Moreover, for the first time we also present a diagrammatic derivation of the "tetraquark" BSEs in FMF.

We have also confirmed the BSE of "baryon" in the extended 't Hooft model in IMF [50]. The main new result of this work is to derive, for the first time, the BSEs of "baryon" in FMF in the context of equal-time quantization.

We have also conducted a comprehensive numerical study of mass spectra of "tetraquark" and "baryon". The Regge trajectories are explicitly demonstrated. We have also obtained the profiles of the wave functions of the ground and first excited states of "tetraquark" and "baryon", viewed from different FMFs. We have numerically verified that, when the "tetraquark" and "baryon" are boosted to IMF, the forward-moving components of the bound-state wave functions approach the corresponding light-cone wave functions, and the backward-moving components fade away.

APPENDIX A: DIAGRAMMATIC DERIVATION OF THE BOUND-STATE EQUATION FOR SCALAR QCD_2 IN EQUAL-TIME QUANTIZATION

In this Appendix, we employ the diagrammatic technique to rederive the bound-state equations of "tetraquark" in scalar QCD_2 in the context of equal-time quantization.

A1. Feynman rules

To quantize scalar QCD_2 in equal-time, we adopt the axial gauge $A^z=0$ and proceed by expanding the Lagrangian as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{sQCD}_2} &= \mathrm{tr}\left(\partial_z A^0\right)^2+\left(\partial^\mu\phi^\dagger\right)\partial_\mu\phi-m^2\phi^\dagger\phi\\ &\quad -ig_s\left(\partial_0\phi^\dagger\right)A^0\phi+ig_s\phi^\dagger A^0\partial_0\phi+g_s^2\phi^\dagger A^0A^0\phi.\end{aligned} \tag{A1}$$

The Feynman rules can be directly derived from (A1)

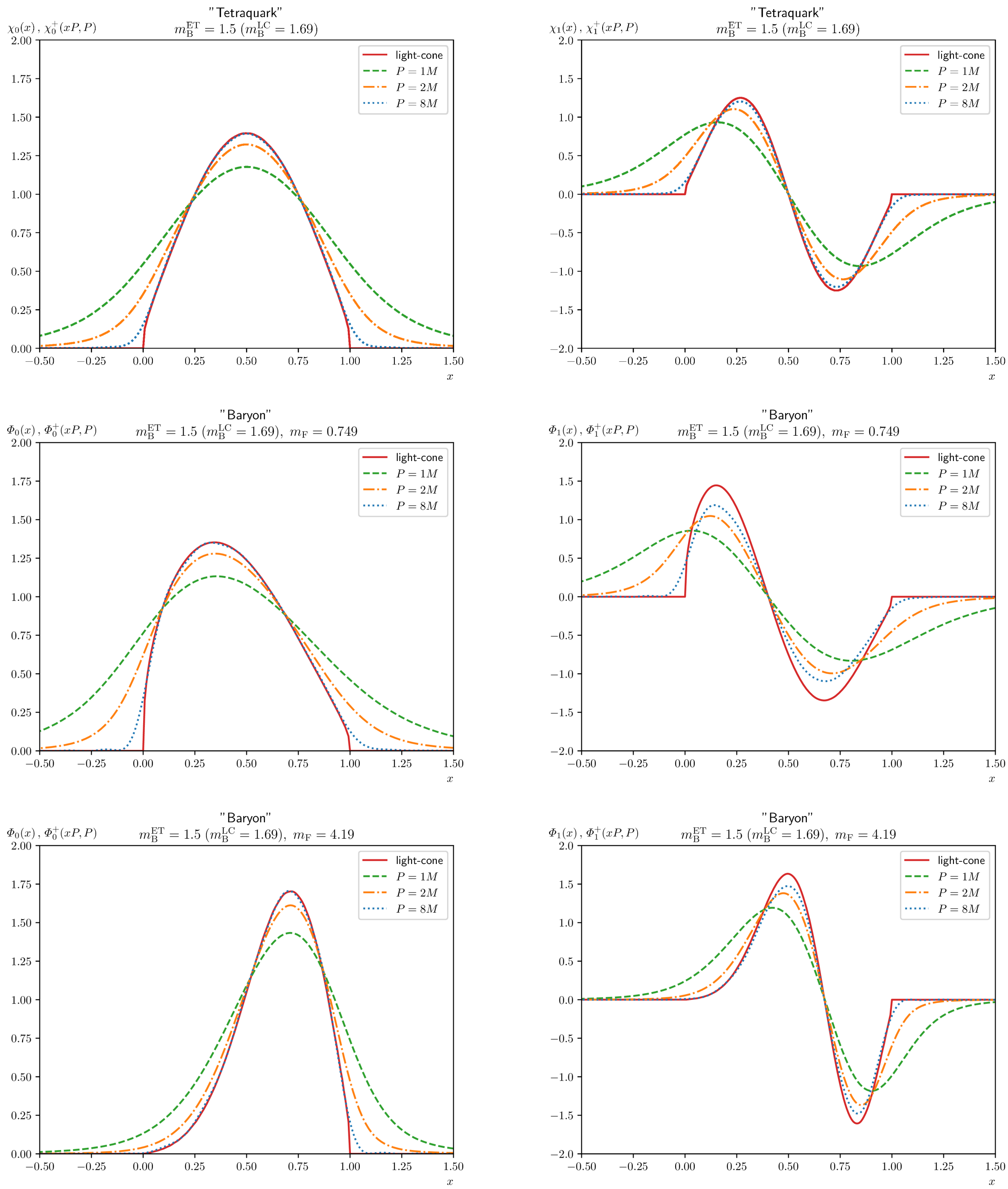
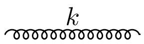
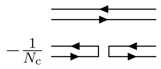
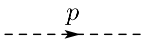
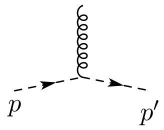
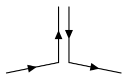
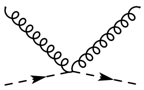
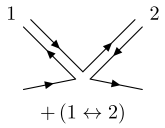


Fig. 2. (color online) Profiles of the forward-moving components of the wave functions of "tetraquark" and "baryons", viewed from different finite momentum frames. The wave functions of the ground state are shown in the left column, while those of the first excited state are shown in the right column. The solid curves represent the corresponding light-cone wave functions.

and are summarized in Table A1. A key challenge in equal-time quantization of scalar QCD$_2$, compared to its spinor counterpart, lies in the presence of seagull vertices. These vertices disrupt the rainbow-ladder topology even at leading order in the $1/N_c$ expansion.

The solution proceeds in a straightforward manner: we decompose each seagull vertex into two $q\bar{q}g$ vertices, and reorganize its contribution into a new term of the quark propagator between these two vertices. However, this cannot be done naively, since the Feynman rule for the $q\bar{q}g$ vertex depends on the momenta of the two adjacent quark lines, while a term in the quark propagator only accounts for the momentum flowing through it. To address this, we must also split the $q\bar{q}g$ vertex into different types and distribute the quark momenta into the two adjacent quark propagators separately. This process is illustrated in Fig. A1. To fully absorb the seagull vertex, it becomes necessary to introduce four distinct types of

Table A1. Original Feynman rules for sQCD_2 in the axial gauge

Building blocks	Double lines	Feynman rules
		$\frac{i}{(k^z)^2}$
		$\frac{i}{p^2-m^2+i\epsilon}$
		$\frac{ig_s}{\sqrt{2}}\left(p^0+p'^0\right)$
		$\frac{ig_s^2}{2}$

quark propagators, which are shown in Fig. A2. In retrospect, these propagators emerge quite naturally, and can, to some extent, be associated with the propagators from $\phi^\dagger(y), -\mathrm{i}\pi^\dagger(y)$ to $\phi(x), \mathrm{i}\pi(x)$ within the Hamiltonian formalism. The situation here is somewhat analogous though not identical to the relationship between canonical quantization and path integral quantization for the massive vector field. The contact term arising from the seagull vertex cancels out the $\left(p^0\right)^2$ numerator in the quark propagator, yielding the correct residue for p^0. For convenience, we define $\omega_p^2 \equiv (p^z)^2 + m^2 - \mathrm{i}\epsilon$.

To streamline the calculation, we express the four quark propagators (Fig. A2) in the form of a 2×2 matrix:

$$D^{(0)}(p) = \frac{\mathrm{i}\left(p^0\sigma_1 + \mathcal{P}_+ + \omega_p^2\mathcal{P}_-\right)}{p^2-m^2+\mathrm{i}\epsilon}, \tag{A2}$$

where the projectors $\mathcal{P}_\pm \equiv (1\pm\sigma_3)/2$ satisfy the orthogonality and completeness relations:

$$\mathcal{P}_\pm^2 = \mathcal{P}_\pm, \quad \mathcal{P}_\pm\mathcal{P}_\mp = 0, \quad \mathcal{P}_+ + \mathcal{P}_- = 1, \tag{A3}$$

and the algebra:

$$\begin{aligned} &\sigma_1\mathcal{P}_\pm = \mathcal{P}_\mp\sigma_1, \\ &(X\sigma_1 + Y\mathcal{P}_+ + Z\mathcal{P}_-)(X\sigma_1 - Z\mathcal{P}_+ - Y\mathcal{P}_-) = X^2 - YZ, \end{aligned} \tag{A4}$$

for scalar quantities X, Y, Z. It is important to emphasize that the introduction of the Pauli σ-matrices here is unrelated to spinors. In fact, the mass dimensions of the matrix elements in (A2) are not homogeneous. However, this particular choice of matrix multiplication ensures that only terms of uniform dimension will be added within the matrix elements. The modified Feynman rules are summarized in Table A2.

A2. Dyson-Schwinger equation

Let the contribution of the one-particle irreducible (1PI) diagrams to the self-energy correction of quark propagator be denoted by $-\mathrm{i}\Sigma(p)$. In the large-N_c limit, the dressed quark propagator $D(p)$ is determined by the rainbow diagrams (Fig. A3). It satisfies the following recursive equations:

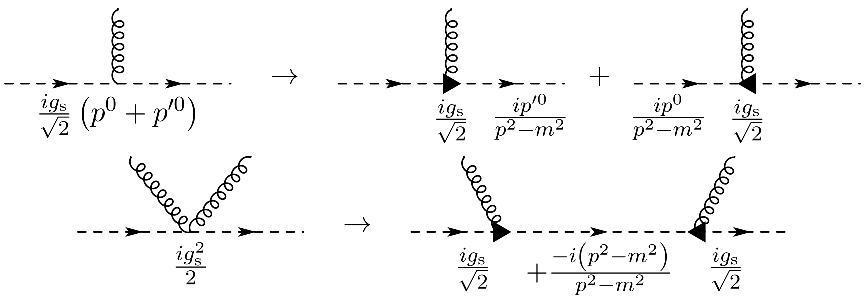


Fig. A1. Modification of the Feynman rules to remove the seagull vertex.

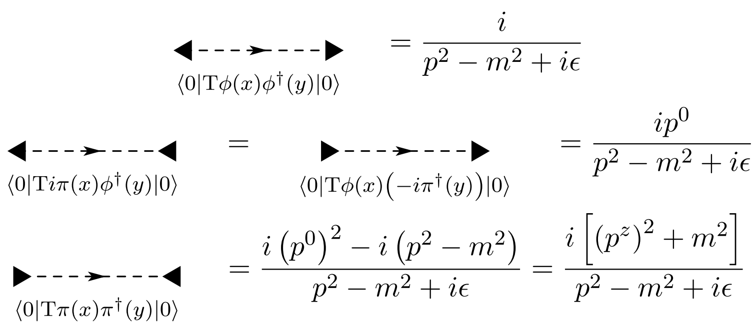
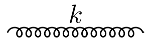
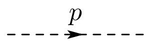
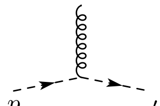


Fig. A2. Modification of the quark propagators.

Table A2. Modified Feynman rules for sQCD$_2$ in the axial gauge.

Building blocks	Feynman rules
 k	$\dfrac{i}{(k^z)^2}$
 p	$\dfrac{i}{p^0\sigma_1-\omega_p^2\mathcal{P}_+-\mathcal{P}_-}$
 p, p'	$\dfrac{ig_s}{\sqrt{2}}\sigma_1$

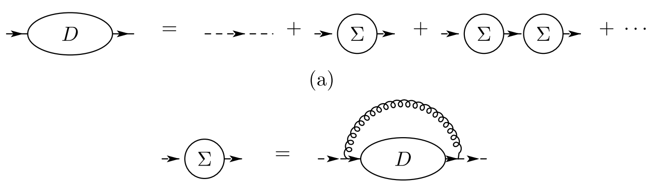


Fig. A3. Dressed quark propagator in the large-N_c limit.

$$D(p)=\frac{\mathrm{i}}{p^0\sigma_1-\omega_p^2\mathcal{P}_+-\mathcal{P}_--\Sigma(p)},\tag{A5a}$$

$$-\mathrm{i}\Sigma(p)=\frac{-\mathrm{i}g_s^2N_c}{2}\fint\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\sigma_1D(k)\sigma_1}{(p^z-k^z)^2},\tag{A5b}$$

where the following expansion has been employed:

$$\frac{1}{X-Y}=X^{-1}+X^{-1}YX^{-1}+X^{-1}YX^{-1}YX^{-1}+\cdots.\tag{A6}$$

Apparently $\Sigma(p)$ is independent of p^0. The most general form is

$$\Sigma(p)=A(p^z)\mathcal{P}_++B(p^z)\mathcal{P}_-.\tag{A7}$$

The σ_1 term does not appear in (A7) because we take the principal value of the k^0 integral in (A5b) at $k^0=\infty$, i.e., we take the average of the contour closed at the upper half-plane and the contour closed at the lower half-plane. The dressed quark propagator now becomes

$$D(p)=\mathrm{i}\frac{p^0\sigma_1+(1+B)\mathcal{P}_++\left(\omega_p^2+A\right)\mathcal{P}_-}{(p^0)^2-(1+B)\left(\omega_p^2+A\right)}.\tag{A8}$$

Introducing the following magic variables,

$$E_p^2\equiv\frac{\omega_p^2+A}{1+B},\quad F_p^2\equiv(1+B)\left(\omega_p^2+A\right),\tag{A9}$$

their advantage shows up immediately:

$$\begin{aligned}D(p)&=\mathrm{i}\frac{p^0\sigma_1+\frac{F_p}{E_p}\mathcal{P}_++E_pF_p\mathcal{P}_-}{\left(p^0-F_p\right)\left(p^0+F_p\right)}\\&=\frac{\mathrm{i}}{2}\left(\frac{\sigma_1+\frac{1}{E_p}\mathcal{P}_++E_p\mathcal{P}_-}{p^0-F_p+\mathrm{i}\epsilon}+\frac{\sigma_1-\frac{1}{E_p}\mathcal{P}_+-E_p\mathcal{P}_-}{p^0+F_p-\mathrm{i}\epsilon}\right),\end{aligned}\tag{A10}$$

where we have made assumptions that $\omega_p^2+A>0$, $1+B>0$. Substituting (A10) into (95b), we find

$$A(p^z)\mathcal{P}_++B(p^z)\mathcal{P}_-=\frac{\lambda}{2}\fint\mathrm{d}k^z\frac{E_k\mathcal{P}_++\frac{1}{E_k}\mathcal{P}_-}{(p^z-k^z)^2}.\tag{A11}$$

Matching the coefficients of $\mathcal{P}$ and utilizing (A9), we find the equations for E_p and F_p:

$$\begin{aligned}F_p&=\frac{1}{E_p}\left(\omega_p^2+\frac{\lambda}{2}\fint\mathrm{d}k^z\frac{E_k}{(p^z-k^z)^2}\right)\\&=E_p\left(1+\frac{\lambda}{2}\fint\mathrm{d}k^z\frac{1}{(p^z-k^z)^2}\frac{1}{E_k}\right).\end{aligned}\tag{A12}$$

(From (A12) one can identify E_p with E_p in the Hamiltonian approach and F_p with $\Pi^+(p)$.)

Explicit calculation shows that the numerators of (A10) can be decomposed into outer products of two vectors:

$$\frac{\sigma_1+\dfrac{1}{E_p}\mathcal{P}_++E_p\mathcal{P}_-}{2}=\xi(p)\tilde{\xi}(p)\sigma_1,\tag{A13a}$$

$$\frac{\sigma_1-\dfrac{1}{E_p}\mathcal{P}_+-E_p\mathcal{P}_-}{2}=\eta(p)\tilde{\eta}(p)\sigma_1,\tag{A13b}$$

where

$$\xi(p)=\frac{1}{\sqrt{2}}\left(1,E_p\right)^{\mathrm{T}},\quad\tilde{\xi}(p)=\frac{1}{\sqrt{2}}\left(1,1/E_p\right),\tag{A14a}$$

$$\eta(p)=\frac{1}{\sqrt{2}}\left(1,-E_p\right)^{\mathrm{T}},\quad\tilde{\eta}(p)=\frac{1}{\sqrt{2}}\left(1,-1/E_p\right).\tag{A14b}$$

The dressed quark propagator can then be written as

$$D(p)=\frac{\mathrm{i}\xi(p)\tilde{\xi}(p)\sigma_1}{p^0-F_p+\mathrm{i}\epsilon}+\frac{\mathrm{i}\eta(p)\tilde{\eta}(p)\sigma_1}{p^0+F_p-\mathrm{i}\epsilon}.\tag{A15}$$

3. Bethe-Salpeter equation

In the large-N_c limit, The meson-$q\bar{q}$ vertex $\Gamma(p;q)$ satisfies the following Bethe-Salpeter equation (Fig. A4):

$$\Gamma(p;q) = \frac{-\mathrm{i}g_s^2 N_c}{2} ⨍ \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(p^z-k^z)^2}\sigma_1 \times D(k)\Gamma(k;q)D(k-q)\sigma_1. \tag{A16}$$

$\Gamma(p;q)$ is independent of p^0, so we can perform the k^0 integration:

$$\mathcal{I}(k;q) \equiv \int \frac{\mathrm{d}k^0}{2\pi} D(k)\Gamma(k;q)D(k-q). \tag{A17}$$

The interpretation of $\mathcal{I}(k;q)$ at the matrix-element level is

$$\mathcal{I}(k;q) = \int \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}k^z z}\langle\Omega|\Phi(0,z)\Phi^\dagger(0,0)|M_n(q)\rangle, \tag{A18}$$

where $\Phi(x) = (\phi(x), \mathrm{i}\pi(x))^{\mathrm{T}}$ and M_n is the n-th excitation of the bound states. Among the four projections of $D(k)$ and $D(k-q)$, only two of them give nonzero residues. We arrive at

$$\begin{aligned}\mathcal{I}(k;q) &= \frac{\xi(k)\left[-\mathrm{i}\tilde{\xi}(k)\sigma_1\Gamma(k;q)\eta(k-q)\right]\tilde{\eta}(k-q)\sigma_1}{q_n^0 - F(k-q) - F(k) + \mathrm{i}\epsilon} + \frac{\eta(k)\left[-\mathrm{i}\tilde{\eta}(k)\sigma_1\Gamma(k;q)\xi(k-q)\right]\tilde{\xi}(k-q)\sigma_1}{-q_n^0 - F(k-q) - F(k) + \mathrm{i}\epsilon} \\ &\equiv -\chi_n^+(k^z;q^z)\sqrt{\frac{E_{k-q}}{E_k}}\xi(k)\tilde{\eta}(k-q)\sigma_1 - \chi_n^-(k^z;q^z)\sqrt{\frac{E_{k-q}}{E_k}}\eta(k)\tilde{\xi}(k-q)\sigma_1.\end{aligned} \tag{A19}$$

From now on we drop the superscript z in spacial momenta for simplicity. Substituting (A16) into (A19), we get

$$\left[q_n^0 - F(p-q) - F(p)\right]\chi_n^+(p;q) = -\lambda ⨍ \frac{\mathrm{d}k}{(p-k)^2}\sqrt{\frac{E_p E_{k-q}}{E_k E_{p-q}}} \times \left[\tilde{\xi}(p)\xi(k)\tilde{\eta}(k-q)\eta(p-q)\chi_n^+(k;q) + \tilde{\xi}(p)\eta(k)\tilde{\xi}(k-q)\eta(p-q)\chi_n^-(k;q)\right], \tag{A20a}$$

$$\left[-q_n^0 - F(p-q) - F(p)\right]\chi_n^-(p;q) = -\lambda ⨍ \frac{\mathrm{d}k}{(p-k)^2}\sqrt{\frac{E_p E_{k-q}}{E_k E_{p-q}}} \times \left[\tilde{\eta}(p)\eta(k)\tilde{\xi}(k-q)\xi(p-q)\chi_n^-(k;q) + \tilde{\eta}(p)\xi(k)\tilde{\eta}(k-q)\xi(p-q)\chi_n^+(k;q)\right]. \tag{A20b}$$

Using the fact that

$$\tilde{\xi}(p)\xi(k) = \tilde{\eta}(p)\eta(k) = \frac{E_p + E_k}{2E_p}, \tag{A21a}$$

$$\tilde{\xi}(p)\eta(k) = \tilde{\eta}(p)\xi(k) = \frac{E_p - E_k}{2E_p}, \tag{A21b}$$

and defining

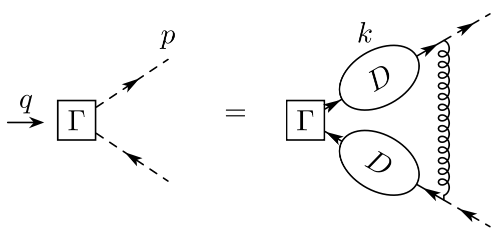


Fig. A4. Meson-$q\bar{q}$ vertex in the large-N_c limit.

$$f_\pm(p,k) \equiv \frac{E_k \pm E_p}{\sqrt{E_p E_k}}, \tag{A22a}$$

$$S_\pm(p,k;q) \equiv f_\pm(p,k) f_\pm(p-q,k-q), \tag{A22b}$$

the bound state equations can be written as

$$\left[q_n^0 - F(p-q) - F(p)\right]\chi_n^+(p;q) = -\frac{\lambda}{4} ⨍ \frac{\mathrm{d}k}{(p-k)^2} \times \left[S_+(p,k;q)\chi_n^+(k;q) - S_-(p,k;q)\chi_n^-(k;q)\right], \tag{A23a}$$

$$\left[-q_n^0 - F(p-q) - F(p)\right]\chi_n^-(p;q) = -\frac{\lambda}{4} ⨍ \frac{\mathrm{d}k}{(p-k)^2} \times \left[S_+(p,k;q)\chi_n^-(k;q) - S_-(p,k;q)\chi_n^+(k;q)\right]. \tag{A23b}$$

These equations are equivalent to (52) with $F \equiv \Pi^+$.

APPENDIX B: CONNECTION OF THE MASS RENORMALIZATION SCHEMES BETWEEN EQUAL-TIME AND LIGHT-FRONT QUANTIZATION FOR SCALAR QCD_2

In this appendix, we discuss some subtle issues about the mass renormalization in the scalar QCD_2, in both equal-time and light-front quantization. We start from the equal-time quantization, and examine the asymptotic behavior of E_k and Π^+ in the large momentum limit.

The integrals appearing in $\Pi^\pm$ defined in (36) as $k\to\infty$ can be calculated with the method of regions [55]. There are two regions of the integration variable k_1: 1) $m_{\tilde{r}} \sim |k_1| \ll k$; 2) $m_{\tilde{r}} \ll |k_1| \sim k$. We divide the integration regions, expand the integrand according to the different scaling of each region:

$$\begin{aligned}&\fint \mathrm{d}k_1 \left(\frac{\frac{E_{k_1}}{E_k} \pm \frac{E_k}{E_{k_1}}}{(k+k_1)^2} - \frac{E_{k_1}}{E_k}\frac{1}{k_1^2} \right) \\ &\approx \frac{1}{k}\fint_{|k_1|<\Lambda} \mathrm{d}k_1 \left(\pm\frac{1}{E_{k_1}} - \frac{E_{k_1}}{k_1^2}\right) \\ &\quad + \fint_{|k_1|>\Lambda} \mathrm{d}k_1 \left(\frac{\frac{|k_1|}{k} \pm \frac{k}{|k_1|}}{(k+k_1)^2} - \frac{1}{k|k_1|}\right), \end{aligned} \tag{B1}$$

where $m_{\tilde{r}} \ll \Lambda \ll k$. For Π^+, the bounds of each region can be extended to the whole momentum space since the asymptotic λ dependence of each region cancels each other:

$$\begin{aligned}&\fint \mathrm{d}k_1 \left(\frac{\frac{E_{k_1}}{E_k} + \frac{E_k}{E_{k_1}}}{(k+k_1)^2} - \frac{E_{k_1}}{E_k}\frac{1}{k_1^2} \right) \\ &\approx \frac{1}{k}\left[\fint \mathrm{d}k_1 \left(\frac{1}{E_{k_1}} - \frac{E_{k_1}}{k_1^2}\right) - 4\right]. \end{aligned} \tag{B2}$$

For Π^-, the integral is divergent for each region, but the divergences cancel each other in the sum of two regions:

$$\begin{aligned}&\fint \mathrm{d}k_1 \left(\frac{\frac{E_{k_1}}{E_k} - \frac{E_k}{E_{k_1}}}{(k+k_1)^2} - \frac{E_{k_1}}{E_k}\frac{1}{k_1^2} \right) \\ &\approx \lim_{\Lambda/k_1\to\infty} \frac{1}{k}\left[\fint_{-\Lambda}^{\Lambda} \mathrm{d}k_1 \left(-\frac{1}{E_{k_1}} - \frac{E_{k_1}}{k_1^2}\right) - 4\ln\frac{k}{\Lambda}\right]. \end{aligned} \tag{B3}$$

Define k_0 as

$$4\ln k_0 \equiv \lim_{\Lambda/k_1\to\infty} \fint_{-\Lambda}^{\Lambda} \mathrm{d}k_1 \left(-\frac{1}{E_{k_1}} - \frac{E_{k_1}}{k_1^2}\right) + 4\ln\Lambda. \tag{B4}$$

The asymptotic behavior of E_k can be derived from (41) as

$$E_k \approx k + \frac{1}{k}\left(\frac{m_{\tilde{r}}^2}{2} - \lambda \ln\frac{k}{k_0}\right). \tag{B5}$$

The value of E_k still can not be determined solely from (B5) since k_0 depends recursively on the functional form of E_{k_1}. Fortunately, however, when substituting (B5) into Π^+, the k_0 dependence cancels and we arrive at

$$\Pi^+(k) \approx k + \frac{m_{\tilde{r}}^2}{2k} + \frac{\lambda}{4k}\left[\fint \mathrm{d}k_1 \left(\frac{1}{E_{k_1}} - \frac{E_{k_1}}{k_1^2}\right) - 4\right], \tag{B6}$$

By relabeling the momenta $k = xP$ in (52) and taking the large P limit, one finds that (52) approaches (28). The relation between the renormalized masses m_r and $m_{\tilde{r}}$ can be determined by matching the two equations:

$$m_{\tilde{r}}^2 + \frac{\lambda}{2}\fint \mathrm{d}k_1 \left(\frac{1}{E_{k_1}} - \frac{E_{k_1}}{k_1^2}\right) = m_r^2. \tag{B7}$$

It is important to note that the relation between the equal-time renormalized mass and the light-front renormalized mass is scheme-dependent. Had we instead chosen another equal-time renormalized mass

$$m_{\tilde{r}}^2 = m^2 + \frac{\lambda}{2}\int \mathrm{d}k_1 \frac{1}{E_{k_1}}, \tag{B8}$$

we would have found that $m_{\tilde{r}} = m_r$.

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