Tensor formalism for the partial wave analysis of reactions with resonances decaying into four pseudoscalar mesons*

M.A. Matveev A.T. Sitnikov A.V. Sarantsev†

NRC "Kurchatov Institute", Petersburg Nuclear Physica Institute, Gatchina 188300, Russia

Abstract: We construct a formalism that describes the resonances decaying into four pseudoscalar meson final states. This method is fully covariant and can be directly applied to the partial-wave analysis of high statistical data. Two topologies of the process are considered: two intermediate resonances, each decaying into two final mesons, and cascade decay via three meson intermediate states. In particular, we consider the production of such states in the central collision reactions and in radiative J/Ψ decay.

Keywords: partial-wave analysis, radiative J/ψ decay, four-meson final state, glueballs, resonance, strong QCD

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I. INTRODUCTION

The main information about meson states comes from πN scattering with high energy pion beams, antiproton-nucleon annihilation, decay of relatively stable, heavy hadrons, and production of mesons in the central collision. In most of the reactions analyzed, resonances decay into final states with two stable particles (see, for example, [1−5]), or the reactions can be considered as quasi-two-particle scattering (for example, antiproton-proton annihilation at rest into three pseudoscalar mesons [6−8]). We note that some developments for multichannel decays have been made for the analysis of radiative decay J/Ψ [9], decay of heavy mesons [10−12], and antiproton-proton annihilation at rest into five pions [13]. However, no systematic formalism for the analysis of data in which meson resonances decay to a four-meson final state has been developed. Nevertheless, such a decay mode is dominant for resonances in many partial waves already at masses around 1400 MeV.

In the scalar isoscalar sector, the decay mode 4π is already dominant for the $f_0(1370)$ state. Moreover, in the elastic scattering data $\pi\pi$ (extracted from the reaction πN), this state can only appear due to rescattering with other scalar states in the 4π channel and cannot be clearly observed. This property stimulated a number of discussions about the existence of this state. However, this state was observed in antiproton-proton annihilation in the analysis of data with the final states $3\pi^0$ [6] and 5π [13]. Note that the scalar isoscalar sector is rather difficult to analyze. In this sector, one expects a strong mixing between non-strange and strange quark-antiquark components and the production of exotic states. For example, many authors treat lowest-scalar states as molecular-like states or four-quark bound systems. Regarding the lowest bound states of two gluons, glueballs are also expected in this sector [14]. Such a picture makes the analysis of this sector rather complicated, and information about four-meson final states could be vital for understanding the spectrum and properties of these mesons.

In the present study, we construct a covariant approach for the analysis of resonances decaying into four pseudoscalar mesons and consider the production of these resonances in J/Ψ radiative decay and in the NN central collision reactions (pomeron-pomeron scattering). J/Ψ radiative decay is one of the main sources of the search for glueball states. In fact, the partial wave decomposition of the BES III data on the J/ψ radiative decay into two pseudo-scalar mesons [4, 5] demonstrated very complicated resonant structures in the isoscalar-scalar partial wave in the mass region of 1500−2100 MeV. The combined analysis of these data with the scattering data from $\pi\pi$ and data on proton-antiproton annihilation at rest in three pseudoscalar mesons revealed the contribution of ten scalar states [15]. The distribution of resonance production intensities demonstrated a peak in the mass re-

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† E-mail: sarantsev_AV@pnpi.nrcki.ru (Corresponding author)

gion of 1850 MeV, which was explained by the presence of two-gluon components in the observed scalar states. This idea was confirmed by the calculation of the mixing angles between $n\bar{n}$, $s\bar{s}$, and the glueball components [16]. In the alternative analysis of these data [17], it was found that a resonance with mass at approximately 1750 MeV is produced dominantly and is the main candidate for the scalar glueball. The analysis of the data with a four-pion channel should resolve this issue.

Another issue is the search for the tensor glueball. If the J/Ψ radiative decay into two pseudoscalar mesons reveals the production of a scalar glueball in the mass region 1700-1900 MeV, then one should expect the production of a tensor glueball (for example, from lattice calculations [18, 19]) in the mass region 2200−2500 MeV. The tensor partial waves extracted from the BES III data showed a strong production of $f_2(1275)$ (decaying into 2π final state) and $f_2'(1525)$ (decaying into $K\bar{K}$ final state) and practically no structure at higher energies. In fact, no clear signal was found in the analysis [20]. The only solution to this problem is related to the fact that tensor states in this mass region decay dominantly into four pseudoscalar mesons, and signals from the tensor glueball should be searched in the data on J/Ψ radiative decay into these final states.

Another prominent source for the production of glueball states is in the meson production at nucleon-nucleon central collision reactions. In these processes, the states are predominantly produced from the pomeron-pomeron collision. Considering that the pomeron is an effective way to describe the gluon lattice, it is relevant to expect that the states with a large gluon component will be strongly produced in such a reaction. Therefore, our method should be useful for planning new experiments for newly constructed colliders, such as NICA.

The covariant approach based on the tensor formalism was proposed by Zemach [21] and developed further in [22, 23]. A mathematical framework allowing the construction of angular momentum operators using recursive expressions was provided in [24]. This method was developed for the case of fermions in [25]. We should mention that in several of the analyses, the authors used an approach based on the helicity formalism. In that formalism, particle propagators are represented as a product of polarization tensors (summed over possible polarizations). In this case, the total amplitude is equal to the sum of products of amplitudes describing two particle scattering, which can be calculated in the helicity basis. However, such two-particle amplitudes should be carefully rotated, which is not a trivial task. For the case of J/Ψ-radiative decay into two pseudoscalar mesons, the exact correspondence between the helicity and tensor approach was given in [26]. However, in the case of resonance decay into a four-particle state, the helicity approach is much more complicated than a tensor formalism.

II. COVARIANT SPIN-ORBITAL FORMALISM

A. Orbital angular momentum tensor

The most detailed description of the tensor formalism was given in [25]. We briefly recall it here. Consider the decay of a composite system with spin J and momentum P ($P^2 = s$) into two spinless particles with momenta k_1 and k_2. The only quantities measured in such a reaction are the particle momenta. The angular dependent part of the wave function of the composite state is described by tensors constructed out of these momenta and the metric tensor. Such tensors (denoted as $X^{(L)}_{\mu_1\ldots\mu_L}$, where L is the orbital momentum) are called orbital angular momentum tensors and correspond to irreducible representations of the Lorentz group. They satisfy the following properties:

- Symmetry under permutation of any two indices:

$$X^{(L)}_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_L} = X^{(L)}_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\mu_L}. \tag{1}$$

- Orthogonality to the total momentum of the system $P = k_1 + k_2$:

$$P_{\mu_i} X^{(L)}_{\mu_1\ldots\mu_i\ldots\mu_L} = 0. \tag{2}$$

- Traceless property for convolution of any two indices with metric tensor:

$$g_{\mu_i\mu_j} X^{(L)}_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_L} = 0. \tag{3}$$

The orthogonality condition (2) is automatically fulfilled if the tensors are constructed from the relative momenta $k^\perp_\mu$ and tensor $g^\perp_{\mu\nu}$ orthogonal to the total momentum of the system:

$$k^\perp_\mu = \frac{1}{2}(k_1 - k_2)_\nu g^\perp_{\mu\nu}, \qquad g^\perp_{\mu\nu} = g_{\mu\nu} - \frac{P_\mu P_\nu}{s}. \tag{4}$$

In the center-of-mass system (cms), where $P = (P_0, \vec{P}) = (\sqrt{s}, 0)$, the vector $k^\perp$ is space-like: $k^\perp = (0, \vec{k})$.

The orbital tensor for $L = 0$ is a scalar value (for example, a unit), and the tensor for the orbital momentum $L = 1$ is a vector that can only be constructed from $k^\perp_\mu$. The orbital angular momentum tensors for L up to three have the following forms:

$$X^{(0)}=1\,,\qquad X^{(1)}_{\mu}=k^{\perp}_{\mu}\,,$$
$$X^{(2)}_{\mu_1\mu_2}=\frac{3}{2}\left(k^{\perp}_{\mu_1}k^{\perp}_{\mu_2}-\frac{1}{3}k^2_{\perp}g^{\perp}_{\mu_1\mu_2}\right),$$
$$X^{(3)}_{\mu_1\mu_2\mu_3}=\frac{5}{2}\left[k^{\perp}_{\mu_1}k^{\perp}_{\mu_2}k^{\perp}_{\mu_3}-\frac{k^2_{\perp}}{5}\left(g^{\perp}_{\mu_1\mu_2}k^{\perp}_{\mu_3}+g^{\perp}_{\mu_1\mu_3}k^{\perp}_{\mu_2}+g^{\perp}_{\mu_2\mu_3}k^{\perp}_{\mu_1}\right)\right]. \tag{5}$$

The tensors $X^{(L)}_{\mu_1\ldots\mu_L}$ for $L\geq 1$ can be constructed from the tensors with lower orbital momenta in the form of a recurrent expression:

$$X^{(L)}_{\mu_1\ldots\mu_L}=k^{\perp}_{\alpha}Z^{\alpha}_{\mu_1\ldots\mu_L}\,,$$
$$Z^{\alpha}_{\mu_1\ldots\mu_L}=\frac{2L-1}{L^2}\Big(\sum_{i=1}^{L}X^{(L-1)}_{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L}g^{\perp}_{\mu_i\alpha}-\frac{2}{2L-1}\sum_{\substack{i,j=1\\ i<j}}^{L}g^{\perp}_{\mu_i\mu_j}X^{(L-1)}_{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_L\alpha}\Big). \tag{6}$$

The normalization of the tensors is fixed by the convolution equality:

$$X^{(L)}_{\mu_1\ldots\mu_L}k^{\perp}_{\mu_L}=k^2_{\perp}X^{(L-1)}_{\mu_1\ldots\mu_{L-1}}. \tag{7}$$

Iterating Eq. (6), one obtains the following expression for the tensor $X^{(L)}_{\mu_1\ldots\mu_L}$:

$$X^{(L)}_{\mu_1\ldots\mu_L}(k^{\perp})=\alpha(L)\Big[k^{\perp}_{\mu_1}k^{\perp}_{\mu_2}k^{\perp}_{\mu_3}k^{\perp}_{\mu_4}\ldots k^{\perp}_{\mu_L}-\frac{k^2_{\perp}}{2L-1}\Big(g^{\perp}_{\mu_1\mu_2}k^{\perp}_{\mu_3}k^{\perp}_{\mu_4}\ldots k^{\perp}_{\mu_L}+g^{\perp}_{\mu_1\mu_3}k^{\perp}_{\mu_2}k^{\perp}_{\mu_4}\ldots k^{\perp}_{\mu_L}+\ldots\Big)+\frac{k^4_{\perp}}{(2L-1)(2L-3)}\Big(g^{\perp}_{\mu_1\mu_2}g^{\perp}_{\mu_3\mu_4}k^{\perp}_{\mu_5}k^{\perp}_{\mu_6}\ldots k_{\mu_L}+g^{\perp}_{\mu_1\mu_2}g^{\perp}_{\mu_3\mu_5}k^{\perp}_{\mu_4}k^{\perp}_{\mu_6}\ldots k_{\mu_L}+\ldots\Big)+\ldots\Big], \tag{8}$$

where

$$\alpha(L)=\prod_{l=1}^{L}\frac{2l-1}{l}=\frac{(2L-1)!!}{L!}\,. \tag{9}$$

Using normalization condition, we obtain

$$X^{(L)}_{\mu_1\ldots\mu_L}X^{(L)}_{\mu_1\ldots\mu_L}=\alpha(L)(k^2_{\perp})^L. \tag{10}$$

B. Boson projection operator

The projection operator $O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}(P)$ for the partial wave with angular momentum L is defined as

$$\int\frac{d\Omega}{4\pi}\quad X^{(L)}_{\mu_1\ldots\mu_L}(k^{\perp})X^{(L)}_{\nu_1\ldots\nu_L}(k^{\perp})=\frac{\alpha(L)}{2L+1}(k^2_{\perp})^LO^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}(P). \tag{11}$$

This tensor satisfies the following relations:

$$X^{(L)}_{\mu_1\ldots\mu_L}(k^{\perp})O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}(P)=X^{(L)}_{\nu_1\ldots\nu_L}(k^{\perp})\,,$$
$$O^{\mu_1\ldots\mu_L}_{\alpha_1\ldots\alpha_L}(P)\,O^{\alpha_1\ldots\alpha_L}_{\nu_1\ldots\nu_L}(P)=O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}(P)\,. \tag{12}$$

This tensor has the same symmetry, orthogonality, and traceless properties as X-tensors (for the same set of up and down indices), but the O-operator does not depend on the relative momentum of the constituents and does not describe decay processes. It represents the structure of the propagator of the composite system. Taking into account the definition of the projection operators (12) and properties of the X-tensors (8), we obtain

$$k_{\mu_1}\ldots k_{\mu_L}O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}=\frac{1}{\alpha(L)}X^{(L)}_{\nu_1\ldots\nu_L}(k^{\perp}). \tag{13}$$

This equation represents the basic properties of the projection operator: it projects any tensor with L indices onto the tensor that satisfies all properties of the partial wave considered.

The projection operator can also be calculated from tensors with lower rank using the recurrent expression

$$O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}=\frac{1}{L^2}\Big(\sum_{i,j=1}^{L}g^{\perp}_{\mu_i\nu_j}O^{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L}_{\nu_1\ldots\nu_{j-1}\nu_{j+1}\ldots\nu_L}-\frac{4}{(2L-1)(2L-3)}\sum_{\substack{i<j\\ k<m}}^{L}g^{\perp}_{\mu_i\mu_j}g^{\perp}_{\nu_k\nu_m}O^{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_L}_{\nu_1\ldots\nu_{k-1}\nu_{k+1}\ldots\nu_{m-1}\nu_{m+1}\ldots\nu_L}\Big). \tag{14}$$

The low order projection operators are

$$O=1,\qquad O^{\mu}_{\nu}=g^{\perp}_{\mu\nu},$$
$$O^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\Big(g^{\perp}_{\mu\alpha}g^{\perp}_{\nu\beta}+g^{\perp}_{\mu\beta}g^{\perp}_{\nu\alpha}-\frac{2}{3}g^{\perp}_{\mu\nu}g^{\perp}_{\alpha\beta}\Big). \tag{15}$$

The scattering of the two spinless particles in the partial wave with total spin $J=L$ (for example a $\pi\pi\to\pi\pi$ transition) is described as a convolution of the operators $X^{(L)}(k)$ and $X^{(L)}(q)$, where k and q are relative momenta before and after the interaction.

$$X^{(L)}_{\mu_1\ldots\mu_L}(k^{\perp})\quad O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}X^{(L)}_{\mu_1\ldots\mu_L}(q^{\perp})=\alpha(L)\left(\sqrt{k^2_{\perp}}\sqrt{q^2_{\perp}}\right)^LP_L(z)\,. \tag{16}$$

Here, $P_L(z)$ are Legendre polynomials and $z=(k^{\perp}q^{\perp})/(\sqrt{k_{\perp}^2}\sqrt{q_{\perp}^2})$ which are, in c.m.s., functions of the cosine of the angle between initial and final particles.

C. Decay of bound system into two pseudoscalar meson states

The system of two pseudoscalar particles can form a partial wave with spin and parity J^P, where $J=L$ and $P=(-1)^L$. The decay vertex of such a system is described by the orbital momentum tensor only:

$$V^J_{\mu_1\ldots\mu_L}=X^{(L)}_{\mu_1\ldots\mu_L}(k^{\perp}). \qquad (17)$$

In the two-meson decay the G-parity corresponds to the product of the G-parities of the final particles, and the isospin can have values from $|I_1-I_2|$ to I_1+I_2. If the final particles are neutral, the C-parity of the two meson system is equal to the product of the C-parities of the final particles. In the case where the final particles are the particle and its own antiparticle $C=(-1)^L$, identical particles cannot form bound states with the odd partial waves; such amplitude would be anti-symmetrical and disappear when particles are permuted. Then, for the scattering of two pseudoscalar particles, the combined system can have the quantum numbers listed in Table 1. For the two-pion system, the isospin and G parity can be $I^G=0^+$, 1^+, 2^+ for the $\pi\eta$ system $I^G=1^-$ and the $\eta\eta(\eta\eta')$ system $I^G=0^+$. In the quark model, where mesons are considered to be bound states of the quark and antiquark, the isospin can be $I=0$, 1, G-parity is connected with C-parity as $G=C(-1)^I$, and states with $CP=-1$ and $P=(-1)^J$ are forbidden.

D. Decay of the resonance into three spinless particles

Let us consider the composite system decaying into the final three pseudo-scalar mesons via an intermediate two-body system with spin $I^{G_{12}}_{12}J^{P_{12}C_{12}}_{12}$. The tensor that describes the intrinsic spin is constructed by the convolution of the orbital momentum tensor, which describes the two-particle intermediate state and projection operator of the three particle system:

$$S^{(3)}_{\mu_1\ldots\mu_n}(J_{12})=X^{(L_{12})}_{\nu_1\ldots\nu_n}(k^{\perp}_{12})O^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}(P_3). \qquad (18)$$

Here, k_1 and k_2 are the momenta of the particles from decay of the intermediate system and

$$k^{\perp}_{12\mu}=\frac{(k_1+k_2)_\nu}{2}g^{\perp P_{12}}_{\mu\nu},\quad g^{\perp P_{12}}_{\mu\nu}=g_{\mu\nu}-\frac{P_{12\mu}P_{12\nu}}{P^2_{12}}, \qquad (19)$$

where $P_{12}=k_1+k_2$, $P_3=k_1+k_2+k_3$, and $n=J_{12}=L_{12}$. If the total spin of the three-particle system is equal to J_3, the orbital momentum between the intermediate state and spectator particle can be $L_3=|J_3-J_{12}|,\ldots,J_3+J_{12}$. If the combination $(L_3+J_{12}-J_3)$ is an even number, then the decay vertex can be constructed as follows:

Table 1. Partial waves in the channel of two pseudoscalar particles.

L	0	1	2	3	4	
$\pi^0\pi^0$	0^{++}		2^{++}		4^{++}	$\cdots$
$\pi^+\pi^-$	0^{++}	1^-	2^{++}	3^-	4^{++}	$\cdots$
$\pi^\pm\pi^0$	0^+	1^-,	2^+	3^-	4^+	$\cdots$
$\eta\eta$	0^{++}		2^{++}		4^{++}	$\cdots$
$\pi^0\eta$	0^{++}	1^{-+}	2^{++}	3^{-+}	4^{++}	$\cdots$
$\pi^\pm\eta$	0^+	1^-	2^+	3^-	4^+	$\cdots$
$\eta\eta'$	0^{++}	1^{-+}	2^{++}	3^{-+}	4^{++}	$\cdots$

$$V^{(+1)}_{\mu_1\ldots\mu_{J_3}}(Q_3)=S^{(3)}_{\mu_1\ldots\mu_i\nu_1\ldots\nu_m}(J_{12})X^{(L_3)}_{\nu_1\ldots\nu_m\mu_{i+1}\ldots\mu_{J_3}}(k^{\perp}_3), \qquad (20)$$

where $k^{\perp}_3$ is the momentum of the spectator particle orthogonal to the total momentum of the three particle system. The number of convoluted indices $m=(J_{12}+L_3-J_3)/2$ and $i=J_{12}-m$. The multiindex Q_3 matches the principal quantum numbers $Q_3\equiv J_3$, L_3, J_{12}. If the combination $(J_{12}+L_3-J_3)$ is an odd number, then the amplitude is formed by means of the antisymmetric tensor:

$$V^{(-1)}_{\mu_1\ldots\mu_{J_3}}(Q_3)=\varepsilon_{\mu_1\alpha\beta P_3}\times S^{(3)}_{\alpha\mu_2\ldots\mu_i\nu_1\ldots\nu_m}(J_{12})X^{(L_3)}_{\beta\nu_1\ldots\nu_m\mu_{i+1}\ldots\mu_{J_3}}(k^{\perp}_3), \qquad (21)$$

$m=(J_{12}+L_3-J_3-1)/2$, $i=J_{12}-m$ and

$$\varepsilon_{\mu\alpha\beta P_3}\equiv\varepsilon_{\mu\alpha\beta\nu}P_{3\nu}. \qquad (22)$$

The final tensor should be symmetrical, traceless, and orthogonal to the total momentum of the three particle system, which can be performed by convolution with the projection operator:

$$A^{(3,\alpha)}_{\mu_1\ldots\mu_n}(Q_3)=V^{(\alpha)}_{\nu_1\ldots\nu_n}(Q_3)O^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}(P_3). \qquad (23)$$

Therefore, there are two classes of the vertices with $\alpha=+1,-1$, which we will refer to below as natural (tensor) and unnatural (pseudo-tensor) structures. The parity of these states is defined as

$$P=(-1)^{L_3+J_{12}}\prod_{i=1}^{3}P_i, \qquad (24)$$

where P_i denote the parities of the final states. Table 2 enumerates the potential states for the final state $3\pi^0$, considering the intermediate states of the scalar and tensor.

Table 2. List of partial wave states produced in $3\pi^0$ channel. The partial waves are denoted as $J^{PC}_{m\alpha}$ (see Eqs. (20), (21), (23)). The standard $q\bar{q}$ states in $3\pi^0$ channel have iospin $I=1$.

	$J_{12}=0$	$J_{12}=2$
$L_3=0$	0^{-+}_{0+}	2^{-+}_{0+}
$L_3=1$	1^{++}_{0+}	1^{++}_{1+} 2^{++}_{0-} 3^{++}_{0+}
$L_3=2$	2^{-+}_{0+}	0^{-+}_{2+} 1^{-+}_{1-} 2^{-+}_{1+} 3^{-+}_{0-} 4^{-+}_{0+}
$L_3=3$	3^{++}_{0+}	1^{++}_{2+} 2^{++}_{1-} 3^{++}_{1+} 4^{++}_{0-} 5^{++}_{0+}

E. Decay of the resonance into four spinless particles through three body intermediate state

This case is very similar to the one described above. If the 3-body bound state has quantum numbers $I_3^{G_3}J_3^{P_3C_3}$, then the intrinsic spin can be described as projection of the three body amplitude (Eq. (23)) into a 4-particle system:

$$S^{(4,\alpha)}_{\mu_1\ldots\mu_{J_3}}(Q_3)=A^{(3,\alpha)}_{\nu_1\ldots\nu_{J_3}}(Q_3)O^{\nu_1\ldots\nu_{J_3}}_{\mu_1\ldots\mu_{J_3}}(P), \tag{25}$$

where $P=k_1+k_2+k_3+k_4$. The orbital momentum between the 3-body state and a spectator particle can be given by $L_4=|J_4-J_3|,\ldots,J_4+J_3$. If the combination $(L_4+J_3-J_4)$ is an even number, then the decay vertex can be written as convolution of the intrinsic spin tensor with orbital momentum tensor

$$V^{(+1,\alpha)}_{\mu_1\ldots\mu_{J_4}}(Q_4)=S^{(4,\alpha)}_{\mu_1\ldots\mu_i\nu_1\ldots\nu_m}(Q_3)X^{(L_4)}_{\nu_1\ldots\nu_m\mu_{i+1}\ldots\mu_{J_4}}(k_4^\perp), \tag{26}$$

where $k_4^\perp$ is the momentum of the spectator particle orthogonal to the momentum of the four particle system. The number of convoluted indices is equal to $m=(J_3+L_4-J_4)/2$ and $i=J_3-m$. The multiindex Q_4 lists all relevant quantum numbers $Q_4=J_4,\ L_4,\ J_3,\ L_3,\ J_{12}$. If the combination $(J_3+L_4-J_4)$ is an odd number,

$$V^{(-1,\alpha)}_{\mu_1\ldots\mu_{J_4}}(Q_4)=\varepsilon_{\mu_1\eta\beta P}\times S^{(4,\alpha)}_{\eta\mu_2\ldots\mu_i\nu_1\ldots\nu_m}(Q_3)X^{(L_4)}_{\beta\nu_1\ldots\nu_m\mu_{i+1}\ldots\mu_{J_4}}(k_4^\perp), \tag{27}$$

where $m=(J_3+L_4-J_4-1)/2$ and $i=J_3-m$. The final tensor should be symmetrical, traceless, and orthogonal to the total momentum of the four particle system, which can be conducted by means of the projection operator

$$A^{(4,\beta,\alpha)}_{\mu_1\ldots\mu_{J_4}}(Q_4)=V^{(\beta,\alpha)}_{\nu_1\ldots\nu_{J_4}}(Q_4)O^{\nu_1\ldots\nu_{J_4}}_{\mu_1\ldots\mu_{J_4}}(P), \tag{28}$$

where $\beta,\alpha=-1,+1$. Thus, we have 4 classes of amplitudes:

$$+1,+1:J_3+L_4-J_4=2n,\qquad J_{12}+L_3-J_3=2m; \tag{29}$$

$$+1,-1:J_3+L_4-J_4=2n,\qquad J_{12}+L_3-J_3=2m+1; \tag{30}$$

$$-1,+1:J_3+L_4-J_4=2n+1,\quad J_{12}+L_3-J_3=2m; \tag{31}$$

$$-1,-1:J_3+L_4-J_4=2n+1,\quad J_{12}+L_3-J_3=2m+1. \tag{32}$$

The parity of the 4-particle state is defined as

$$P=(-1)^{L_4+L_3+J_{12}}\prod_{i=1}^{4}P_i. \tag{33}$$

The partial wave amplitudes in the $4\pi^0$ channel in the case of cascade decay are listed in Table 3 for the intermediate three body states up to $J_3=2$. The exotic states, which can be produced in the $4\pi^0$ and $3\pi^0$ channels, are also included in this table.

Table 3. List of partial wave amplitude $J^{PC}_{n\beta}$ states for cascade decay into $4\pi^0$.

J_3^{PC}	0^{-+}	1^{++}	1^{-+}
$L_4=0$	0^{++}_{0+}	1^{-+}_{0+}	1^{++}_{0+}
$L_4=1$	1^{-+}_{0+}	0^{++}_{1+} 1^{++}_{0-} 2^{++}_{0+}	0^{++}_{0+} 1^{++}_{0-} 2^{++}_{0+}
$L_4=2$	2^{++}_{0+}	1^{-+}_{1+} 2^{-+}_{0-} 3^{-+}_{0+}	1^{-+}_{1+} 2^{-+}_{0-} 3^{-+}_{0+}

J_3^{PC}	2^{++}	2^{-+}
$L_4=0$	2^{-+}_{0+}	2^{++}_{0+}
$L_4=1$	1^{++}_{0+} 2^{++}_{0-} 3^{++}_{0+}	1^{-+}_{1+} 2^{-+}_{0-} 3^{-+}_{0+}
$L_4=2$	0^{-+}_{2+} 1^{-+}_{1-} 2^{-+}_{1+} 3^{-+}_{0-} 4^{-+}_{0+}	0^{++}_{2+} 1^{++}_{1-} 2^{++}_{1+} 3^{++}_{0-} 4^{++}_{0+}

F. Decay of resonance into two resonances decaying into two spinless particles

Consider the decay of the resonance into two resonances with spins J_{12} and J_{34} that decay into two spinless particles with momenta k_1,k_2 and k_3,k_4, respectively. The decay of these two states is described by $X^{(J_{12})}_{\mu_1\ldots\mu_{J_{12}}}(k_{12}^\perp)$ and $X^{(J_{34})}_{\mu_1\ldots\mu_{J_{34}}}(k_{34}^\perp)$ tensors, where

$$k^\perp_{ij\mu}=\frac{k_{i\nu}-k_{j\nu}}{2}g^{\perp P_{ij}}_{\mu\nu},\quad g^{\perp P_{ij}}_{\mu\nu}=g_{\mu\nu}-\frac{P_{ij\mu}P_{ij\nu}}{P_{ij}^2}, \tag{34}$$

where $P_{ij} = k_i + k_j$. The intrinsic spin can have values $S = |J_{12} - J_{34}|, \ldots, J_{12} + J_{34}$. If $J_{12} + J_{34} - S = 2m$, the tensor that describes this spin state can be formed by convolution of the m indices:

$$V^{(+1)}_{\mu_1\ldots\mu_S}(Q_S) = X^{(J_{12})}_{\mu_1\ldots\mu_{J_{12}-m}\nu_1\ldots\nu_m}(k^\perp_{12}) X^{(J_{34})}_{\nu_1\ldots\nu_m\mu_{J_{12}-m+1}\ldots\mu_S}(k^\perp_{34}), \quad (35)$$

where $Q_S = S,\ J_{12},\ J_{34}$. If the difference between intrinsic spin and the sum of spin resonances is an odd number $J_{12} + J_{34} - S = 2m+1$, the vertex is formed by means of the antisymmetric tensor:

$$V^{(-1)}_{\mu_1\ldots\mu_S}(Q_S) = \varepsilon_{\mu_1\eta\beta P} X^{(J_{12})}_{\eta\mu_2\ldots\mu_{J_{12}-m}\nu_1\ldots\nu_m}(k^\perp_{12}) \times X^{(J_{34})}_{\beta\nu_1\ldots\nu_m\mu_{J_{12}-m+1}\ldots\mu_S}(k^\perp_{34}), \quad (36)$$

where $P = k_1 + k_2 + k_3 + k_4$. The symmetrization of indices and traceless property can be satisfied in the standard way by convolution with the projection operator:

$$S^{(22,\alpha)}_{\mu_1\ldots\mu_S}(Q_S) = V^{(\alpha)}_{\nu_1\ldots\nu_S}(Q_S) O^{\nu_1\ldots\nu_S}_{\mu_1\ldots\mu_S}(P). \quad (37)$$

If the spin of the 4-particle partial wave is equal to J_4, the orbital momentum between the two resonances becomes $L_4 = |J_4 - S|, \ldots, J_4 + S$. If the combination $(L_4 + S - J_4) = 2n$ is an even number, then the decay vertex can be written as

$$V^{(+1,\alpha)}_{\mu_1\ldots\mu_{J_4}}(Q_{22}) = S^{(22,\alpha)}_{\mu_1\ldots\mu_{S-n}\nu_1\ldots\nu_n}(Q_S) X^{(L_4)}_{\nu_1\ldots\nu_n\mu_{S-n+1}\ldots\mu_{J_4}}(k^\perp), \quad (38)$$

where $k^\perp$ is the relative momentum between two intermediate resonances:

$$k^\perp_\mu = \frac{1}{2}(k_1 + k_2 - k_3 - k_4)_\nu g^\perp_{\mu\nu}, \quad (39)$$

and multi-index $Q_{22} = J_4,\ L_4,\ S,\ J_{12},\ J_{34}$. If the combination $(J_3 + L_4 - J_4)$ is an odd number, then

$$V^{(-1,\alpha)}_{\mu_1\ldots\mu_{J_4}}(Q_{22}) = \varepsilon_{\mu_1\eta\beta P} \times S^{(22,\alpha)}_{\eta\mu_2\ldots\mu_{S-n}\nu_1\ldots\nu_n}(Q_S) X^{(L_4)}_{\beta\nu_1\ldots\nu_n\mu_{S-n+1}\ldots\mu_{J_4}}(k^\perp), \quad (40)$$

$n = (S + L_4 - J_4 - 1)/2$. The final tensor should be symmetrical, traceless, and orthogonal to the total momentum of the four particle system, which can be achieved with the projection operator

$$A^{(22,\beta,\alpha)}_{\mu_1\ldots\mu_{J_4}}(Q_{22}) = V^{(\beta,\alpha)}_{\nu_1\ldots\nu_{J_4}}(Q_{22}) O^{\nu_1\ldots\nu_{J_4}}_{\mu_1\ldots\mu_{J_4}}(P), \quad (41)$$

Thus, we have 4 classes of amplitudes:

$$+1,+1: S + L_4 - J_4 = 2n, \qquad J_{12} + J_{34} - S = 2m; \quad (42)$$

$$+1,-1: S + L_4 - J_4 = 2n, \qquad J_{12} + J_{34} - S = 2m+1; \quad (43)$$

$$-1,+1: S + L_4 - J_4 = 2n+1, \qquad J_{12} + J_{34} - S = 2m; \quad (44)$$

$$-1,-1: S + L_4 - J_4 = 2n+1, \qquad J_{12} + J_{34} - S = 2m+1. \quad (45)$$

The parity of the 4-particle state is defined as

$$P = (-1)^{L_4 + J_{12} + J_{34}} \prod_{i=1}^{4} P_i. \quad (46)$$

G. Construction of tensors for the decay of the resonance into four pseudoscalar mesons

The formulas given in previous sections allowed one to directly construct code for the decay of a resonance into any four pseudoscalar meson states. Let us consider the decay of the tensor resonance into two tensor final states as an example.

It can be seen that the two final tensor states can form intrinsic spin $S^P = 0^+,\ 1^+,\ 2^+,\ 3^+,\ 4^+$ (see Table 4). If the intrinsic spin is equal to 2^+, the tensor for this state is constructed with $S^P_{m\alpha} = 2^+_{1+}$, which means that one index is convoluted and no antisymmetric tensors are involved. Then:

$$S^{(22,+)}_{\mu\nu}(Q_S) = O^{\mu\nu}_{\eta\chi}(P) X^{(2)}_{\eta\xi}(k^\perp_{12}) X^{(2)}_{\xi\chi}(k^\perp_{34}). \quad (47)$$

If the initial state is the tensor one, it can be constructed with $L_4 = 0$ or $L_4 = 2$ (see Table 5). In the first case, the $n\beta = 0+$ and tensor for the decay into four mesons has a simple form:

$$V^{(2+,1)}_{\mu\nu} = S^{(22,+)}_{\mu\nu}(Q_S). \quad (48)$$

In the second case, $n\beta = 1+$, and one convolution with orbital momentum tensor

$$V^{(2+,2)}_{\mu\nu} = X^{(2)}_{\mu\chi}(k^\perp) S^{(22,+)}_{\chi\nu}(Q_S). \quad (49)$$

Here, for convenience, we have changed the upper in-

Table 4. List of possible spin combinations $S^P_{m\alpha}$ states ($C=+1$) in four particle decay via two resonance decays.

J^{PC}_{12}	0^{++}	0^{++}	2^{++}	2^{++}
J^{PC}_{34}	0^{++}	2^{++}	2^{++}	4^{++}
$S^P_{m\alpha}$	0^+_{0+}	2^+_{0+}	0^+_{2+} 1^+_{1-} 2^+_{1+} 3^+_{0-} 4^+_{0+}	2^+_{2+} 3^+_{1-} 4^+_{1+} 5^+_{0-} 6^+_{0+}

Table 5. List of states $J^{PC}_{n\beta}$ in $4\pi^0$ channel decaying into two resonances forming intrinsic spin S^{PC}.

S^{PC}	0^{++}	1^{++}	2^{++}
$L_4=0$	0^{++}_{0+}	1^{++}_{0+}	2^{++}_{0+}
$L_4=1$	1^{-+}_{0+}	0^{-+}_{1+} 1^{-+}_{0-} 2^{-+}_{0+}	1^{-+}_{1+} 2^{-+}_{0-} 3^{-+}_{0+}
$L_4=2$	2^{++}_{0+}	1^{++}_{1+} 2^{++}_{0-} 3^{++}_{0+}	0^{++}_{2+} 1^{++}_{1-} 2^{++}_{1+} 3^{++}_{0-} 4^{++}_{0+}
S^{PC}		3^{++}	4^{++}
$L_4=0$		3^{++}_{0+}	4^{++}_{0+}
$L_4=1$		2^{-+}_{1+} 3^{-+}_{0-} 4^{-+}_{0+}	3^{-+}_{1+} 4^{-+}_{0-} 5^{-+}_{0+}
$L_4=2$		1^{++}_{2+} 2^{++}_{1-} 3^{++}_{1+} 4^{++}_{0-} 5^{++}_{0+}	2^{++}_{2+} 3^{++}_{1-} 4^{++}_{1+} 5^{++}_{0-} 6^{++}_{0+}

dices for the tensor $V^{J^P,m}_{\mu_1...\mu_J}$ given the spin and parity of the 4-particle resonance and just the tensor number.

The Appendix presents the tensor list associated with the decay of a resonance into a final state of $4\pi^0$. An extra symmetry arises from the permutations of pions. When a resonance decays into two other resonances, amplitudes in which the sum of intrinsic spin and orbital momentum is odd are nullified. The scenario is more intricate in cascade decays, and all possible configurations are enumerated.

H. Production of states decaying into 4 pseudoscalar mesons

A partial wave amplitude is a scalar value. The convolution of the tensors with an antisymmetric tensor creates a pseudo-tensor structure (unnatural). Thus, the scalar value should be either a convolution of any number of tensors (natural structures) or a convolution of an even number of unnatural structures. Therefore, if the produced state is a natural state, the decay amplitude can have only (++) or (−) structures. If the produced state is an unnatural state, the decay should be described by the (+−) or (−+) combination.

1. *Production of the state from pomeron-pomeron, f_0f_0 or $\pi^0\pi^0$ collision*

Consider the production of resonances in central collision reactions. In this case, the states can be produced, for example, from the pomeron-pomeron collision. A similar mechanism is responsible for the production of resonances in two-scalar or two neutral pseudo-scalar mesons. In such processes, only states with $G=+1$ and $C=+1$ are produced. In the standard quark model, only resonances with isospin 0 and even spin can be produced in such a reaction: $I^GJ^{PC}=0^+J^{++}$ and $J=2n$. The produced vertex is described by the orbital momentum tensor $X^{(J_4)}_{\mu_1...\mu_{J_4}}(q^\perp)$, where q represents the relative momenta of the collided particles. Then, the partial wave amplitude for the partial wave with spin J_4 has the form

$$A^{(t\alpha\beta)}_{J_4}=X^{(J_4)}_{\mu_1...\mu_{J_4}}(q^\perp)A^{(t\alpha\beta)}_{\mu_1...\mu_{J_4}}(Q_t), \tag{50}$$

where index t describes topologies $t=4$ (cascade) and $t=22$ (decay into two resonances). Consider the case with the final state $4\pi^0$ (or 4η). Then, only resonances with $I^GJ^{PC}=0^+J^{++}_{ij}$, where $J_{ij}=2n$, can be produced in the two-particle channel. If we have two scalar resonances $J_{12}=J_{34}=0$ in the final states, only ++ amplitudes are possible:

$$\begin{aligned}A^{(22++)}_{J_4}&=X^{(J_4)}_{\mu_1...\mu_{J_4}}(q^\perp)O^{\mu_1...\mu_{J_4}}_{\nu_1...\nu_{J_4}}(P)X^{(J_4)}_{\nu_1...\nu_{J_4}}(k^\perp)\\&=X^{(J_4)}_{\mu_1...\mu_{J_4}}(q^\perp)X^{(J_4)}_{\mu_1...\mu_{J_4}}(k^\perp).\end{aligned} \tag{51}$$

We obtain the standard Legendre polynomial dependence in the rest system of the 4-particle state (see Eq. 16). In the cascade topology, the amplitudes have ++ or − signatures. Examples of other amplitudes are given in the Appendix.

2. *Production of the states decaying into 4 pseudoscalar particles in J/Ψ radiative decays*

The states produced in radiative decay J/Ψ can be natural or unnatural states. All states have isospin 0, $G=+1$ and $C=+1$. This process can be considered as the production of resonances in collisions $J/\Psi\gamma$. In this case, the intrinsic spin can have values $S=0$, 1, 2, and the corresponding spin tensors are

$$\begin{aligned}S^0&=g_{\mu\nu}\epsilon^\Psi_\mu\epsilon^{\gamma*}_\nu, && S=0;\\S^1_\eta&=\varepsilon_{\eta\mu\nu P}\epsilon^\Psi_\mu\epsilon^{\gamma*}_\nu, && S=1;\\S^2_{\eta\xi}&=\epsilon^\Psi_\mu\epsilon^{\gamma*}_\nu O^{\mu\nu}_{\eta\xi}(P), && S=2.\end{aligned} \tag{52}$$

In the spin-orbital basis (SL), the partial waves can be described as ${}^{2S+1}L_J$. In this basis, the partial waves with $S=0$ have $L=J$ and parity $P=(-1)^J$. These production vertices are described as

$$V^0_{\mu_1...\mu_J}=S^0X^{(J)}_{\mu_1...\mu_J}(q^\perp)\qquad {}^1J_J, \tag{53}$$

and all amplitudes are the natural ones. The natural amplitudes can be constructed with intrinsic spin 1 with only $L=J$ (${}^3L_J$):

$$V^{1,0}_{\mu_1\ldots\mu_J} = \varepsilon_{\mu_1\eta\xi P} S^1_\eta X^{(J)}_{\xi\mu_2\ldots\mu_J}(q^\perp) \qquad {}^3J_J. \tag{54}$$

This vertex can be symmetrized and traceless by convolution with the projection operator. However, we already applied this operator to the decay vertices, and it is not necessary to apply it to the production vertices. For the amplitudes with $S=2$, the natural amplitudes are produced with $L=J-2$, J, $J+2$:

$$\begin{aligned} V^{5,-2}_{\mu_1\ldots\mu_J} &= S^2_{\mu_1\mu_2} X^{(J-2)}_{\mu_3\ldots\mu_J}(q^\perp) \qquad {}^5(J-2)_J, \\ V^{5,0}_{\mu_1\ldots\mu_J} &= S^2_{\mu_1\eta} X^{(J)}_{\eta\mu_2\ldots\mu_J}(q^\perp) \qquad {}^5J_J, \\ V^{5,+2}_{\mu_1\ldots\mu_J} &= S^2_{\xi\eta} X^{(J+2)}_{\xi\eta\mu_1\ldots\mu_J}(q^\perp) \qquad {}^5(J+2)_J. \end{aligned} \tag{55}$$

Recall that the decay of the natural states can be described with − or ++ amplitudes in the decay into 4 pseudoscalar mesons. In the gauge-invariant limit for the partial waves with $J\geq 2$, only three amplitudes are linearly independent. For $J=0$, only one amplitude is linearly independent, and for $J=1$, we have two linearly independent amplitudes (see Appendix).

The unnatural amplitudes are formed with intrinsic spin $S=1$, 2 and orbital momentum $L=J-1$, and $L=J+1$:

$$\begin{aligned} V^{3,-1}_{\mu_1\ldots\mu_J} &= S^1_{\mu_1} X^{(J-1)}_{\mu_2\ldots\mu_J}(q^\perp) \qquad {}^3(J-1)_J, \\ V^{3,+1}_{\mu_1\ldots\mu_J} &= S^1_\eta X^{(J+1)}_{\eta\mu_1\ldots\mu_J}(q^\perp) \qquad {}^3(J+1)_J, \\ V^{5,-1}_{\mu_1\ldots\mu_J} &= \varepsilon_{\mu_1\eta\xi P} S^2_{\eta\mu_2} X^{(J-1)}_{\xi\mu_3\ldots\mu_J}(q^\perp) \qquad {}^5(J-1)_J, \\ V^{5,+1}_{\mu_1\ldots\mu_J} &= \varepsilon_{\mu_1\eta\xi P} S^2_{\eta\chi} X^{(J+1)}_{\xi\chi\mu_2\ldots\mu_J}(q^\perp) \qquad {}^5(J+1)_J. \end{aligned} \tag{56}$$

In the case of the decay a vector meson into virtual photon (or a massive vector particle like ω-meson) and four pseudoscalar mesons, all these vertices are linearly independent. However, in the case of the real photon and $J\geq 2$, only three amplitudes are linearly independent. For $J=0$, only one amplitude is linearly independent, and for $J=1$, we have two linearly independent amplitudes. The amplitudes' basis set might vary; when dealing with an energy-independent fit, the specific choice of amplitudes does not matter for the analysis. Conversely, in the context of an energy-dependent fit, it is essential to manage the amplitudes' asymptotic behavior. From this perspective, we suggested using amplitudes with the lowest orbital momentum and different intrinsic spins. The details of the amplitude linear dependence for the case of the real photon are given in the Appendix.

III. CONCLUSION

We developed a formalism for the partial-wave analysis of data with four pseudoscalar meson states. The method is covariant and can be directly applied to event-by-event analysis of the data, for example, in the framework of the maximum likelihood method. In particular, we consider $4\pi^0$ production in the radiative decay J/Ψ and the production of the four pseudo-scalar mesons in the central collision.

APPENDIX

A. Linearly dependent vertices for J/Ψ radiative decay

Consider the general structure of J/Ψ radiative decay. The production of the state with spin J can be described as a convolution projection operator with vertex

$$O^{\nu_1\ldots\nu_J}_{\mu_1\ldots\mu_J}(P) V_{\mu_1\ldots\mu_J}(q^\perp), \tag{A1}$$

where $q^\perp$ is the momentum of J/Ψ orthogonal to the momentum of the resonance. Considering the orthogonality and traceless properties of the projection operator, the general structure of the vertex for the natural states is

$$\begin{aligned} V_{\mu_1\ldots\mu_J} = {} & (\epsilon^\Psi \epsilon^{\gamma*}) q^\perp_{\mu_1}\ldots q^\perp_{\mu_J} F_1 + \epsilon^\Psi_{\mu_1}\epsilon^{\gamma*}_{\mu_2} q^\perp_{\mu_3}\ldots q^\perp_{\mu_J} F_2 \\ & + \epsilon^\Psi_{\mu_1}(\epsilon^{\gamma*} q^\perp) q^\perp_{\mu_2}\ldots q^\perp_{\mu_J} F_3 + (\epsilon^\Psi q^\perp)\epsilon^{\gamma*}_{\mu_1} q^\perp_{\mu_2}\ldots q^\perp_{\mu_J} F_4 \\ & + (\epsilon^\Psi q^\perp)(\epsilon^{\gamma*} q^\perp) q^\perp_{\mu_1}\ldots q^\perp_{\mu_J} F_5, \end{aligned} \tag{A2}$$

where F_i are scalar functions. Recall that the polarization vector of the particle is orthogonal to its momentum, and we can use $q^\perp$ as universal momentum in convolution with both polarization vectors. In the case of the real photon, its polarization vector is orthogonal to all momenta in the vertex, and therefore, $(\epsilon^{\gamma*} q^\perp)=0$. Consequently, only structures with F_1, F_2, and F_4 will contribute to the decay. In the case of the scalar state ($J=0$), the vertex has only structures F_1, F_5, and only F_1 is not zero in this limit. For the vector state ($J=1$), only structures F_1, F_3, F_4, F_5 can contribute to the vertex, and only F_1 and F_4 survive.

For the production of unnatural states, the vertex has the structure

$$\begin{aligned} V_{\mu_1\ldots\mu_J} = {} & \varepsilon_{q^\perp\alpha\beta P}\epsilon^\Psi_\alpha \epsilon^{\gamma*}_\beta q^\perp_{\mu_1}\ldots q^\perp_{\mu_J} F_1 \\ & + \varepsilon_{\mu_1\alpha\beta P}\epsilon^\Psi_\alpha \epsilon^{\gamma*}_\beta q^\perp_{\mu_2}\ldots q^\perp_{\mu_J} F_2 \\ & + \varepsilon_{\mu_1\alpha q^\perp P}(\epsilon^\Psi_\alpha \epsilon^{\gamma*}_{\mu_2} + \epsilon^\Psi_{\mu_2}\epsilon^{\gamma*}_\alpha) q^\perp_{\mu_3}\ldots q^\perp_{\mu_J} F_3 \\ & + \varepsilon_{\mu_1\alpha q^\perp P}\left(\epsilon^\Psi_\alpha(\epsilon^{\gamma*} q^\perp) + (\epsilon^\Psi q^\perp)\epsilon^{\gamma*}_\alpha\right) q^\perp_{\mu_2}\ldots q^\perp_{\mu_J} F_4. \end{aligned} \tag{A3}$$

For the $J^{PC}=0^{-+}$ state, there is only one structure (with F_1). Decomposing the polarization vector of the real photon ϵ^Ψ_α into components parallel to $q^\perp$, parallel to P and orthogonal to both $q^\perp$ and P, we have

$$\epsilon^{\Psi}_{\alpha} = Cq^{\perp}_{\alpha} + \epsilon^{\perp\perp}_{\alpha} + \delta P_{\alpha},$$
$$\varepsilon_{\mu_1\alpha\beta P}\epsilon^{\Psi}_{\alpha} = C\varepsilon_{\mu_1 q_{\perp}\beta P} + \varepsilon_{\mu_1\alpha\beta P}\epsilon^{\perp\perp}_{\alpha},$$
$$\varepsilon_{\mu_1\alpha\beta P}\epsilon^{\Psi}_{\alpha}\epsilon^{\gamma*}_{\beta} = C\varepsilon_{\mu_1 q^{\perp}\beta P}\epsilon^{\gamma*}_{\beta} + \varepsilon_{\mu_1\alpha\beta P}\epsilon^{\perp\perp}_{\alpha}\epsilon^{\gamma*}_{\beta}. \quad \text{(A4)}$$

As both $\epsilon^{\perp\perp}_{\alpha}, \epsilon^{\gamma*}_{\beta}$ are orthogonal to momentum $q^{\perp}$ and P, then

$$\varepsilon_{\mu_1\alpha\beta P}\epsilon^{\perp\perp}_{\alpha}\epsilon^{\gamma*}_{\beta} = Bq^{\perp}_{\mu_1}. \quad \text{(A5)}$$

Then, we obtain the following structures for the vertex:

$$V_{\mu_1\ldots\mu_J} = B(q^2_{\perp}F_1 + F_2)q^{\perp}_{\mu_1}\ldots q^{\perp}_{\mu_J}$$
$$-(CF_2 + (\epsilon^{\Psi}q^{\perp})F_4)\varepsilon_{\mu_1\beta q^{\perp}P}\epsilon^{\gamma*}_{\beta}q^{\perp}_{\mu_2}\ldots q^{\perp}_{\mu_J}$$
$$+F_3\varepsilon_{\mu_1\alpha q^{\perp}P}(\epsilon^{\Psi}_{\alpha}\epsilon^{\gamma*}_{\mu_2} + \epsilon^{\Psi}_{\mu_2}\epsilon^{\gamma*}_{\alpha})q^{\perp}_{\mu_3}\ldots q^{\perp}_{\mu_J}. \quad \text{(A6)}$$

The amplitudes F_1, F_2, F_4 have only two independent structures. This means that only two of them are linearly independent. This corresponds to the case $J^P = 1^+$, where the structure with F_3 cannot be produced. Thus, for $J \geq 2$, only three partial waves are linearly independent for every unnatural state. The list of the lowest partial waves is given in Table A1.

Table A1. List of partial waves produced in radiative J/Ψ decay. The partial waves that became linearly dependent from other partial waves are shown in third column. The states that are forbidden in the quark model are not listed.

0^{++}	$^1S_0$	$^5D_0$
2^{++}	$^1D_2$ $^3D_2$ $^5S_2$	$^5D_2$ $^5G_2$
4^{++}	$^1G_4$ $^3G_4$ $^5D_4$	$^5G_4$ $^5I_4$
0^{-+}	$^3P_0$	
1^{++}	$^3S_1$ $^3D_1$	$^5D_1$
2^{-+}	$^3P_2$ $^5P_2$ $^3F_2$	$^5F_2$
3^{++}	$^3D_3$ $^3G_3$ $^5D_3$	$^5G_3$

B. Decay of resonances into $4\pi^0$ final state

Let us list the basic momenta and tensors:

$$P_{12\mu} = (k_1+k_2)_{\mu}, \quad g^{\perp P_{12}}_{\mu\nu} = g_{\mu\nu} - \frac{P_{12\mu}P_{12\nu}}{P^2_{12}},$$
$$k^{\perp}_{12\mu} = \frac{1}{2}(k_1-k_2)_{\nu}g^{\perp P_{12}}_{\nu\mu}, \quad P_{34\mu} = (k_3+k_4)_{\mu},$$
$$g^{\perp P_{34}}_{\mu\nu} = g_{\mu\nu} - \frac{P_{34\mu}P_{34\nu}}{P^2_{34}}, \quad k^{\perp}_{34\mu} = \frac{1}{2}(k_3-k_4)_{\nu}g^{\perp P_{34}}_{\nu\mu},$$

$$P_{3\mu} = (P_{12}+k_3)_{\mu}, \quad g^{\perp P_3}_{\mu\nu} = g_{\mu\nu} - \frac{P_{3\mu}P_{3\nu}}{P^2_3},$$
$$k^{\perp}_{3\mu} = \frac{1}{2}(k_3-P_{12})_{\nu}g^{\perp P_3}_{\nu\mu},$$
$$P_{\mu} = (P_3+k_4)_{\mu}, \quad g^{\perp}_{\mu\nu} = g_{\mu\nu} - \frac{P_{\mu}P_{\nu}}{P^2},$$
$$k^{\perp}_{4\mu} = \frac{1}{2}(k_4-P_3)_{\nu}g^{\perp}_{\nu\mu},$$
$$k^{\perp}_{\mu} = \frac{1}{2}(k_1+k_2-k_3-k_4)_{\nu}g^{\perp}_{\nu\mu},$$
$$O^{\nu_1\ldots\nu_J}_{\mu_1\ldots\mu_J} \equiv O^{\nu_1\ldots\nu_J}_{\mu_1\ldots\mu_J}(P). \quad \text{(B1)}$$

Consider the decay of states into the $4\pi^0$ channel. Let us start from decay via two intermediate resonances for natural states :

$$J^{PC} \to J^{PC}_{n,\beta}(L_4(S^{PC}_{m\alpha} \to f_{J_{12}}f_{J_{34}})),$$
$$0^{++} \to 0^{++}_{0,+}(0(0^{++}_{0+} \to f_0f_0)),$$
$$V^{(0+,1)} = 1; \quad \text{(B2)}$$

$$0^{++} \to 0^{++}_{2,+}(2(2^{++}_{0+} \to f_0f_2)),$$
$$V^{(0+,2)} = X^{(2)}_{\alpha\beta}(k^{\perp})X^{(2)}_{\alpha\beta}(k^{\perp}_{34}); \quad \text{(B3)}$$

$$0^{++} \to 0^{++}_{0,+}(0(0^{++}_{2+} \to f_2f_2)),$$
$$V^{(0+,3)} = X^{(2)}_{\alpha\beta}(k^{\perp}_{12})X^{(2)}_{\alpha\beta}(k^{\perp}_{34}); \quad \text{(B4)}$$

$$0^{++} \to 0^{++}_{0,+}(2(2^{++}_{1+} \to f_2f_2)),$$
$$V^{(0+,4)} = X^{(2)}_{\alpha\beta}(k^{\perp})X^{(2)}_{\alpha\xi}(k^{\perp}_{12})X^{(2)}_{\beta\xi}(k^{\perp}_{34}); \quad \text{(B5)}$$

$$2^{++} \to 2^{++}_{0,+}(2(0^{++}_{0+} \to f_0f_0)),$$
$$V^{(2+,1)}_{\alpha\beta} = X^{(2)}_{\alpha\beta}(k^{\perp}); \quad \text{(B6)}$$

$$2^{++} \to 2^{++}_{0,+}(0(2^{++}_{0+} \to f_0f_2)),$$
$$V^{(2+,2)}_{\alpha\beta} = X^{(2)}_{\alpha\beta}(k^{\perp}_{34}); \quad \text{(B7)}$$

$$2^{++} \to 2^{++}_{1,+}(2(2^{++}_{0+} \to f_0f_2)),$$
$$V^{(2+,3)}_{\alpha\beta} = X^{(2)}_{\alpha\chi}(k^{\perp})O^{\mu\nu}_{\beta\chi}X^{(2)}_{\mu\nu}(k^{\perp}_{34}); \quad \text{(B8)}$$

$$2^{++} \to 2^{++}_{0,+}(2(0^{++}_{2+} \to f_2f_2)),$$
$$V^{(2+,4)}_{\alpha\beta} = X^{(2)}_{\alpha\beta}(k^{\perp})X^{(2)}_{\eta\zeta}(k^{\perp}_{12})X^{(2)}_{\eta\zeta}(k^{\perp}_{34}); \quad \text{(B9)}$$

$$2^{++} \to 2^{++}_{0,+}(0(2^{++}_{1+} \to f_2f_2)),$$
$$V^{(2+,5)}_{\alpha\beta} = X^{(2)}_{\alpha\zeta}(k^{\perp}_{12})X^{(2)}_{\beta\zeta}(k^{\perp}_{34}); \quad \text{(B10)}$$

$$2^{++} \to 2^{++}_{1,+}(2(2^{++}_{1+} \to f_2f_2)),$$
$$V^{(2+,6)}_{\alpha\beta} = X^{(2)}_{\alpha\chi}(k^{\perp})O^{\mu\nu}_{\chi\beta}X^{(2)}_{\mu\zeta}(k^{\perp}_{12})X^{(2)}_{\nu\zeta}(k^{\perp}_{34}); \quad \text{(B11)}$$

$$4^{++}\to 4^{++}_{0,+}(2(2^{++}_{0+}\to f_0 f_2)),$$
$$V^{(4+,1)}_{\alpha\beta\mu\nu}=X^{(2)}_{\alpha\beta}(k^{\perp})O^{\eta\zeta}_{\mu\nu}X^{(2)}_{\eta\zeta}(k^{\perp}_{34});\qquad (B12)$$

$$4^{++}\to 4^{++}_{0,+}(2(2^{++}_{1+}\to f_2 f_2)),$$
$$V^{(4+,2)}_{\alpha\beta\mu\nu}=X^{(2)}_{\alpha\beta}(k^{\perp})O^{\eta\zeta}_{\mu\nu}X^{(2)}_{\eta\chi}(k^{\perp}_{12})X^{(2)}_{\chi\zeta}(k^{\perp}_{34});\qquad (B13)$$

$$4^{++}\to 4^{++}_{0,+}(0(4^{++}_{0+}\to f_2 f_2)),$$
$$V^{(4+,3)}_{\alpha\beta\mu\nu}=X^{(2)}_{\alpha\beta}(k^{\perp}_{12})X^{(2)}_{\mu\nu}(k^{\perp}_{34});\qquad (B14)$$

$$4^{++}\to 4^{++}_{1,+}(2(4^{++}_{0+}\to f_2 f_2)),$$
$$V^{(4+,4)}_{\alpha\beta\mu\nu}=X^{(2)}_{\alpha\eta}(k^{\perp})O^{\zeta\chi\delta\gamma}_{\eta\beta\mu\nu}X^{(2)}_{\zeta\chi}(k^{\perp}_{12})X^{(2)}_{\delta\gamma}(k^{\perp}_{34}).\qquad (B15)$$

Here, we provide the amplitudes with $L_4\le 2$ and resonances decaying into two pions up to $J=2$. The unnatural states can only decay into two resonances with $J\ge 2$. In the case of two tensor states, we get the following:

$$0^{-+}\to 0^{-+}_{1,+}(1(1^{++}_{1-}\to f_2 f_2)),$$
$$V^{(0-,1)}=X^{(1)}_{\chi}(k^{\perp})\varepsilon_{\chi\mu\nu P}X^{(2)}_{\mu\zeta}(k^{\perp}_{12})X^{(2)}_{\nu\zeta}(k^{\perp}_{34});\qquad (B16)$$

$$2^{-+}\to 2^{-+}_{0,+}(1(1^{++}_{1-}\to f_2 f_2)),$$
$$V^{(2-,1)}_{\alpha\beta}=X^{(1)}_{\alpha}(k^{\perp})\varepsilon_{\beta\mu\nu P}X^{(2)}_{\mu\zeta}(k^{\perp}_{12})X^{(2)}_{\nu\zeta}(k^{\perp}_{34}).\qquad (B17)$$

For the cascade decays, we provide examples of amplitudes or cascade decays for the orbital momentum $L_4, L_3\le 2$:

$$J^{PC}\to J^{PC}_{n,\beta}(L_4(\pi J_{3\,m\alpha}^{PC}\to L_3(\pi f_{J_{12}})),$$
$$0^{++}\to 0^{++}_{0,+}\left(0\left(\pi 0^{-+}_{0+}\to 0(\pi f_0)\right)\right),$$
$$V^{(0+,5)}=1;\qquad (B18)$$

$$0^{++}\to 0^{++}_{1,+}\left(1\left(\pi 1^{++}_{0+}\to 1(\pi f_0)\right)\right),$$
$$V^{(0+,6)}=X^{(1)}_{\nu}\left(k^{\perp}_4\right)X^{(1)}_{\nu}\left(k^{\perp}_3\right);\qquad (B19)$$

$$0^{++}\to 0^{++}_{2,+}\left(2\left(\pi 2^{-+}_{0+}\to 2(\pi f_0)\right)\right),$$
$$V^{(0+,7)}=X^{(2)}_{\nu\mu}(k^{\perp}_4)X^{(2)}_{\nu\mu}\left(k^{\perp}_3\right);\qquad (B20)$$

$$0^{++}\to 0^{++}_{2,+}\left(2\left(\pi 2^{-+}_{0+}\to 0(\pi f_2)\right)\right),$$
$$V^{(0+,8)}=X^{(2)}_{\alpha\beta}(k^{\perp}_4)O^{\nu\mu}_{\alpha\beta}(P_3)X^{(2)}_{\nu\mu}\left(k^{\perp}_{12}\right);\qquad (B21)$$

$$0^{++}\to 0^{++}_{1,+}\left(1\left(\pi 1^{++}_{1+}\to 1(\pi f_2)\right)\right),$$
$$V^{(0+,9)}=X^{(1)}_{\mu}\left(k^{\perp}_4\right)X^{(1)}_{\nu}\left(k^{\perp}_3\right)O^{\xi\zeta}_{\mu\nu}(P_3)\times X^{(2)}_{\xi\zeta}\left(k^{\perp}_{12}\right);\qquad (B22)$$

$$0^{++}\to 0^{++}_{0,+}\left(0\left(\pi 0^{-+}_{2+}\to 2(\pi f_2)\right)\right),$$
$$V^{(0+,10)}=X^{(2)}_{\nu_1\nu_2}\left(k^{\perp}_3\right)X^{(2)}_{\nu_1\nu_2}\left(k^{\perp}_{12}\right);\qquad (B23)$$

$$0^{++}\to 0^{++}_{2,+}\left(2\left(\pi 2^{-+}_{1+}\to 2(\pi f_2)\right)\right),$$
$$V^{(0+,11)}=X^{(2)}_{\alpha\beta}(k^{\perp}_4)O^{\eta\zeta}_{\alpha\beta}(P_3)X^{(2)}_{\eta\mu}\left(k^{\perp}_3\right)O^{\mu\zeta}_{\xi\nu}(P_3)X^{(2)}_{\xi\nu}\left(k^{\perp}_{12}\right);\qquad (B24)$$

$$2^{++}\to 2^{++}_{0,+}\left(2\left(\pi 0^{-+}_{0+}\to 0(\pi f_0)\right)\right),$$
$$V^{(2+,7)}_{\alpha\beta}=X^{(2)}_{\mu_1\mu_2}\left(k^{\perp}_4\right);\qquad (B25)$$

$$2^{++}\to 2^{++}_{0,+}\left(1\left(\pi 1^{++}_{0+}\to 1(\pi f_0)\right)\right),$$
$$V^{(2+,8)}_{\alpha\beta}=X^{(1)}_{\alpha}\left(k^{\perp}_4\right)X^{(1)}_{\beta}\left(k^{\perp}_3\right);\qquad (B26)$$

$$2^{++}\to 2^{++}_{0,+}\left(0\left(\pi 2^{-+}_{0+}\to 2(\pi f_0)\right)\right),$$
$$V^{(2+,9)}_{\alpha\beta}=X^{(2)}_{\alpha\beta}\left(k^{\perp}_3\right);\qquad (B27)$$

$$2^{++}\to 2^{++}_{1,+}\left(2\left(\pi 2^{-+}_{0+}\to 2(\pi f_0)\right)\right),$$
$$V^{(2+,10)}_{\alpha\beta}=X^{(2)}_{\alpha\nu}\left(k^{\perp}_4\right)X^{(2)}_{\nu\beta}\left(k^{\perp}_3\right);\qquad (B28)$$

$$2^{++}\to 2^{++}_{0,+}\left(0\left(\pi 2^{-+}_{0+}\to 0(\pi f_2)\right)\right),$$
$$V^{(2+,11)}_{\alpha\beta}=O^{\mu\nu}_{\alpha\beta}(P_3)X^{(2)}_{\mu\nu}\left(k^{\perp}_{12}\right);\qquad (B29)$$

$$2^{++}\to 2^{++}_{1,+}\left(2\left(\pi 2^{-+}_{0+}\to 0(\pi f_2)\right)\right),$$
$$V^{(2+,12)}_{\alpha\beta}=X^{(2)}_{\alpha\nu}\left(k^{\perp}_4\right)O^{\nu\beta}_{\xi\zeta}(P_3)X^{(2)}_{\xi\zeta}\left(k^{\perp}_{12}\right);\qquad (B30)$$

$$2^{++}\to 2^{++}_{0,+}\left(1\left(\pi 1^{++}_{1+}\to 1(\pi f_2)\right)\right),$$
$$V^{(2+,13)}_{\alpha\beta}=X^{(1)}_{\alpha}\left(k^{\perp}_4\right)X^{(1)}_{\nu_1}\left(k^{\perp}_3\right)\times O^{\rho_1\rho_2}_{\beta\nu_1}(P_3)X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right);\qquad (B31)$$

$$2^{++}\to 2^{++}_{0,-}\left(1\left(\pi 2^{++}_{0-}\to 1(\pi f_2)\right)\right),$$
$$V^{(2+,14)}_{\alpha\beta}=\varepsilon_{\nu_1\rho_2\alpha P}X^{(1)}_{\nu_1}\left(k^{\perp}_4\right)O^{\rho_2\beta}_{\rho_3\rho_4}(P_3),$$
$$\times\varepsilon_{\nu_2\nu_3\rho_3 P_3}X^{(1)}_{\nu_2}\left(k^{\perp}_3\right)X^{(2)}_{\nu_3\rho_4}\left(k^{\perp}_{12}\right);\qquad (B32)$$

$$2^{++}\to 2^{++}_{0,+}\left(2\left(\pi 0^{-+}_{2+}\to 2(\pi f_2)\right)\right),$$
$$V^{(2+,15)}_{\alpha\beta}=X^{(2)}_{\alpha\beta}\left(k^{\perp}_4\right)X^{(2)}_{\nu_1\nu_2}\left(k^{\perp}_3\right)X^{(2)}_{\nu_1\nu_2}\left(k^{\perp}_{12}\right);\qquad (B33)$$

$$2^{++}\to 2^{++}_{0,+}\left(2\left(\pi 0^{-+}_{2+}\to 2(\pi f_2)\right)\right),$$
$$V^{(2+,15)}_{\alpha\beta}=X^{(2)}_{\alpha\beta}\left(k^{\perp}_4\right)X^{(2)}_{\nu_1\nu_2}\left(k^{\perp}_3\right)X^{(2)}_{\nu_1\nu_2}\left(k^{\perp}_{12}\right);\qquad (B34)$$

$$2^{++} \to 2^{++}_{0,+}\left(0\left(\pi 2^{-+}_{1+} \to 2(\pi f_2)\right)\right),$$
$$V^{(2+,16)}_{\alpha\beta} = O^{\rho_1\rho_2}_{\alpha\beta}(P_3) X^{(2)}_{\rho_1\nu_1}\left(k^{\perp}_3\right) O^{\nu_1\rho_2}_{\mu\nu}(P_3) \times X^{(2)}_{\mu\nu}\left(k^{\perp}_{12}\right); \quad \text{(B35)}$$

$$2^{++} \to 2^{++}_{1,+}\left(2\left(\pi 2^{-+}_{1+} \to 2(\pi f_2)\right)\right),$$
$$V^{(2+,17)}_{\alpha\beta} = X^{(2)}_{\rho_1\alpha}\left(k^{\perp}_4\right) O^{\beta\rho_1}_{\rho_3\rho_2}(P_3), \times X^{(2)}_{\rho_3\nu_1}\left(k^{\perp}_3\right) O^{\rho_4\rho_5}_{\nu_1\rho_2}(P_3) X^{(2)}_{\rho_4\rho_5}\left(k^{\perp}_{12}\right); \quad \text{(B36)}$$

$$2^{-+} \to 2^{-+}_{0,-}\left(2\left(\pi 1^{++}_{0+} \to 1(\pi f_0)\right)\right),$$
$$V^{(2-,2)}_{\alpha\beta} = \varepsilon_{\nu_1\nu_2\alpha P} X^{(2)}_{\nu_1\beta}\left(k^{\perp}_4\right) X^{(1)}_{\nu_2}\left(k^{\perp}_3\right); \quad \text{(B37)}$$

$$2^{-+} \to 2^{-+}_{0,-}\left(1\left(\pi 2^{-+}_{0+} \to 2(\pi f_0)\right)\right),$$
$$V^{(2-,3)}_{\alpha\beta} = \varepsilon_{\nu_1\nu_2\alpha P} X^{(1)}_{\nu_1}\left(k^{\perp}_4\right) X^{(2)}_{\nu_2\beta}\left(k^{\perp}_3\right); \quad \text{(B38)}$$

$$2^{-+} \to 2^{-+}_{0,-}\left(1\left(\pi 2^{-+}_{0+} \to 0(\pi f_2)\right)\right),$$
$$V^{(2-,4)}_{\alpha\beta} = \varepsilon_{\nu_1\nu_2\alpha P} X^{(1)}_{\nu_1}\left(k^{\perp}_4\right) O^{\rho_1\rho_2}_{\nu_2\beta}(P_3) \times X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B39)}$$

$$2^{-+} \to 2^{-+}_{0,-}\left(2\left(\pi 1^{++}_{1+} \to 1(\pi f_2)\right)\right),$$
$$V^{(2-,5)}_{\alpha\beta} = \varepsilon_{\nu_1\nu_2\alpha P} X^{(2)}_{\nu_1\beta}\left(k^{\perp}_4\right) X^{(1)}_{\nu_3}\left(k^{\perp}_3\right) \times O^{\rho_1\rho_2}_{\nu_3\nu_2}(P_3) X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B40)}$$

$$2^{-+} \to 2^{-+}_{0,+}\left(0\left(\pi 2^{++}_{0-} \to 1(\pi f_2)\right)\right),$$
$$V^{(2-,6)}_{\alpha\beta} = \varepsilon_{\nu_1\nu_2\alpha P} X^{(1)}_{\nu_1}\left(k^{\perp}_3\right) \times O^{\rho_1\rho_2}_{\nu_2\beta}(P_3) X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B41)}$$

$$2^{-+} \to 2^{-+}_{1,+}\left(2\left(\pi 2^{++}_{0-} \to 1(\pi f_2)\right)\right),$$
$$V^{(2-,7)}_{\alpha\beta} = X^{(2)}_{\rho_2\alpha}\left(k^{\perp}_4\right) O^{\rho_2\beta}_{\rho_3\rho_1}(P_3) \times \varepsilon_{\nu_1\nu_2\rho_3 P_3} X^{(1)}_{\nu_1}\left(k^{\perp}_3\right) X^{(2)}_{\nu_2\rho_1}\left(k^{\perp}_{12}\right); \quad \text{(B42)}$$

$$2^{-+} \to 2^{-+}_{0,-}\left(1\left(\pi 2^{-+}_{1+} \to 2(\pi f_2)\right)\right),$$
$$V^{(2-,8)}_{\alpha\beta} = X^{(1)}_{\nu_1}(k^{\perp}_4) \varepsilon_{\nu_1\rho_2\alpha P_3} O^{\rho_2\beta}_{\rho_3\rho_1}(P_3) \times X^{(2)}_{\nu_2\rho_3}\left(k^{\perp}_3\right) O^{\rho_4\rho_5}_{\nu_2\rho_1}(P_3) X^{(2)}_{\rho_4\rho_5}\left(k^{\perp}_{12}\right); \quad \text{(B43)}$$

$$0^{-+} \to 0^{-+}_{2,+}\left(2\left(\pi 2^{++}_{0-} \to 1(\pi f_2)\right)\right),$$
$$V^{(0-,2)} = X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_4\right) O^{\rho_1\rho_2}_{\rho_3\rho_4}(P_3) \varepsilon_{\nu_1\nu_2\rho_3 P_3} \times X^{(1)}_{\nu_1}\left(k^{\perp}_3\right) O^{\rho_5\rho_6}_{\nu_2\rho_4}(P_3) X^{(2)}_{\rho_5\rho_6}\left(k^{\perp}_{12}\right); \quad \text{(B44)}$$

$$1^{++} \to 1^{++}_{0,-}\left(1\left(\pi 1^{++}_{0+} \to 1(\pi f_0)\right)\right),$$
$$V^{(1+,1)}_{\alpha} = \varepsilon_{\alpha\nu_1\nu_2 P} X^{(1)}_{\nu_1}\left(k^{\perp}_4\right) X^{(1)}_{\nu_2}\left(k^{\perp}_3\right); \quad \text{(B45)}$$

$$1^{++} \to 1^{++}_{1,-}\left(2\left(\pi 2^{-+}_{0+} \to 2(\pi f_0)\right)\right),$$
$$V^{(1+,2)}_{\alpha} = \varepsilon_{\alpha\nu_1\nu_2 P} X^{(2)}_{\nu_1\nu_3}\left(k^{\perp}_4\right) X^{(2)}_{\nu_2\nu_3}\left(k^{\perp}_3\right); \quad \text{(B46)}$$

$$1^{++} \to 1^{++}_{1,-}\left(2\left(\pi 2^{-+}_{0+} \to 0(\pi f_2)\right)\right),$$
$$V^{(1+,3)}_{\alpha} = \varepsilon_{\alpha\nu_1\nu_2 P} X^{(2)}_{\nu_1\nu_3}\left(k^{\perp}_4\right) O^{\rho_1\rho_2}_{\nu_2\nu_3}(P_3) \times X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B47)}$$

$$1^{++} \to 1^{++}_{0,-}\left(1\left(\pi 1^{++}_{1+} \to 1(\pi f_2)\right)\right),$$
$$V^{(1+,4)}_{\alpha} = \varepsilon_{\alpha\nu_1\nu_2 P} X^{(1)}_{\nu_1}\left(k^{\perp}_4\right) X^{(1)}_{\nu_3}\left(k^{\perp}_3\right) \times O^{\rho_1\rho_2}_{\nu_3\nu_2}(P_3) X_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B48)}$$

$$1^{++} \to 1^{++}_{1,+}\left(1\left(\pi 2^{++}_{0-} \to 1(\pi f_2)\right)\right),$$
$$V^{(1+,5)}_{\alpha} = X^{(1)}_{\rho_1}\left(k^{\perp}_4\right) O^{\rho_3\rho_2}_{\alpha\rho_1}(P_3) \varepsilon_{\nu_1\nu_2\rho_2 P_3} \times X^{(1)}_{\nu_1}\left(k^{\perp}_3\right) O^{\rho_4\rho_5}_{\nu_2\rho_3}(P_3) X^{(2)}_{\rho_4\rho_5}\left(k^{\perp}_{12}\right); \quad \text{(B49)}$$

$$1^{++} \to 1^{++}_{1,-}\left(2\left(\pi 2^{-+}_{1+} \to 2(\pi f_2)\right)\right),$$
$$V^{(1+,6)}_{\alpha} = \varepsilon_{\alpha\nu_1\rho_1 P} X^{(2)}_{\nu_1\rho_2}\left(k^{\perp}_4\right) O^{\rho_1\rho_2}_{\rho_3\rho_4}(P_3) \times X^{(2)}_{\nu_2\rho_3}\left(k^{\perp}_3\right) O^{\rho_5\rho_6}_{\nu_2\rho_4}(P_3) X^{(2)}_{\nu_2\rho_4}\left(k^{\perp}_{12}\right); \quad \text{(B50)}$$

$$4^{++} \to 4^{++}_{0,+}\left(2\left(\pi 2^{-+}_{0+} \to 2(\pi f_0)\right)\right),$$
$$V^{(4+,5)}_{\alpha\beta\mu\nu} = X^{(2)}_{\alpha\beta}\left(k^{\perp}_4\right) X^{(2)}_{\mu\nu}\left(k^{\perp}_3\right); \quad \text{(B51)}$$

$$4^{++} \to 4^{++}_{0,+}\left(2\left(\pi 2^{-+}_{0+} \to 0(\pi f_2)\right)\right),$$
$$V^{(4+,6)}_{\alpha\beta\mu\nu} = X^{(2)}_{\alpha\beta}\left(k^{\perp}_4\right) O^{\rho_1\rho_2}_{\mu\nu}(P_3) X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B52)}$$

$$4^{++} \to 4^{++}_{0,+}\left(2\left(\pi 2^{-+}_{1+} \to 2(\pi f_2)\right)\right),$$
$$V^{(4+,7)}_{\alpha\beta\mu\nu} = X^{(2)}_{\alpha\beta}\left(k^{\perp}_4\right) O^{\rho_3\rho_4}_{\mu\nu}(P_3) \times X^{(2)}_{\nu_1\rho_3}\left(k^{\perp}_3\right) O^{\rho_1\rho_2}_{\nu_1\rho_4}(P_3) X^{(2)}_{\rho_1\rho_2}\left(k^{\perp}_{12}\right); \quad \text{(B53)}$$

The code for the generation of these tensors can be found at https://pwa.hiskp.uni-bonn.de/version/jpsi.html.

C. Transition amplitudes for the central production due to pomeron-pomeron ($f_0 f_0$) collision

In the pomeron-pomeron central collision (or in $f_0 f_0$ collision), only isoscalar states with even spin and positive P,C-parity are produced. The produced vertices are described by the orbital momentum tensors. Therefore, we obtain the following expressions for the partial-wave amplitudes:

$$A^{(0+,i)} = V^{(0+,i)}, \quad i=1-11,$$
$$A^{(2+,i)} = X^{(2)}_{\mu\nu}(q^\perp)O^{\mu\nu}_{\alpha\beta}V^{(2+,i)}_{\alpha\beta}, \quad i=1-17,$$
$$A^{(4+,i)} = X^{(4)}_{\mu\nu\chi\xi}(q^\perp)O^{\alpha\beta\eta\zeta}_{\mu\nu\chi\xi}V^{(4+,i)}_{\alpha\beta\eta\zeta}, \quad i=1-7. \qquad (C1)$$

D. Radiative J/Ψ decay into $4\pi^0$ final state

In the case of radiative J/ψ, it is usually assumed that after emission of the photon, the $c\bar{c}$ system is annihilated to a resonance that decays into hadron final states. From this perspective, all states that are allowed in the quark model can be produced. These states have isospin $I=0$ and charged parity $C=+1$. Let us give the list of the amplitudes up to $J=4$ as follows:

$$A^{(0+,i)} = S^0 V^{(0+,i)}, \quad i=1-11,$$
$$A^{(2+,i)}_1 = S^0 X^{(2)}_{\mu\nu}(q^\perp)O^{\mu\nu}_{\alpha\beta}V^{(2+,i)}_{\alpha\beta}, \quad i=1-17,$$
$$A^{(2+,i)}_2 = \varepsilon_{\mu\eta\xi P}S^1_\eta X^{(2)}_{\xi\nu}(q^\perp)O^{\mu\nu}_{\alpha\beta}V^{(2+,i)}_{\alpha\beta}, \quad i=1-17,$$
$$A^{(2+,i)}_3 = S^2_{\mu\nu}O^{\mu\nu}_{\alpha\beta}V^{(2+,i)}_{\alpha\beta}, \quad i=1-17,$$
$$A^{(4+,i)}_1 = S^0 X^{(4)}_{\mu\nu\alpha\beta}(q^\perp)O^{\mu\nu\alpha\beta}_{\eta\chi\zeta\varrho}V^{(4+,i)}_{\eta\chi\zeta\varrho}, \quad i=1-7,$$
$$A^{(4+,i)}_2 = \varepsilon_{\mu\eta\xi P}S^1_\eta X^{(4)}_{\xi\nu\alpha\beta}(q^\perp)O^{\mu\nu\alpha\beta}_{\chi\zeta\varrho\delta}V^{(4+,i)}_{\chi\zeta\varrho\delta}, \quad i=1-7,$$
$$A^{(4+,i)}_3 = S^2_{\mu\nu}X^{(2)}_{\alpha\beta}(q^\perp)O^{\mu\nu\alpha\beta}_{\eta\chi\zeta\varrho}V^{(4+,i)}_{\eta\chi\zeta\varrho}, \quad i=1-7,$$
$$A^{(0-,i)} = S^1_\eta X^{(1)}_\eta(q^\perp)V^{(0-,i)}, \quad i=1-2,$$
$$A^{(1+,i)}_1 = S^1_\mu V^{(1+,i)}_\mu, \quad i=1-6,$$
$$A^{(1+,i)}_2 = S^1_\eta X^{(2}_{\eta\mu}(q^\perp)V^{(1+,i)}_\mu, \quad i=1-6,$$
$$A^{(2-,i)}_1 = S^1_\mu X^{(1)}_\nu(q^\perp)O^{\alpha\beta}_{\mu\nu}V^{(2-,i)}_{\alpha\beta}, \quad i=1-8,$$
$$A^{(2-,i)}_2 = S^1_\eta X^{(3)}_{\eta\mu\nu}(q^\perp)O^{\alpha\beta}_{\mu\nu}V^{(2-,i)}_{\alpha\beta}, \quad i=1-8,$$
$$A^{(2-,i)}_3 = \varepsilon_{\mu\eta\xi P}S^2_{\eta\nu}X^{(1)}_\xi(q^\perp)O^{\alpha\beta}_{\mu\nu}V^{(2-,i)}_{\alpha\beta}, \quad i=1-8. \qquad (D1)$$

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