# Complexity growth for AdS black holes in the presence of backreaction*

Long Guo (郭龙)† Zi-qiang Zhang (张自强)‡

School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China

**Abstract:** We investigate the holographic complexity in backreacted gravity backgrounds according to the complexity-action conjecture. The backreaction considered here originates from the presence of static strings evenly distributed over the system. We exploit a probe string in the bulk and evaluate the Nambu-Goto action and its dependence on backreaction. The results suggest that, for slower strings, the complexity increases with increasing backreaction, in accordance with the findings of holographic entanglement entropy. However, for faster strings, the situation is different. Furthermore, we analyze the relationship between complexity and space dimension as well as string velocity.

**Keywords:** AdS/CFT correspondence, complexity, backreaction

**DOI:** 10.1088/1674-1137/ada125 **CSTR:** 32044.14.ChinesePhysicsC.49035104

## I. INTRODUCTION

The concept of complexity originates from quantum information theory or computer science [1−4]. In the field of quantum information, complexity quantifies how hard it is to create a target state from a reference (initial) state through a path of elementary gate operation [5]. Moreover, complexity has been defined within geometrical approaches, which suggest introducing Finsler geometry to quantum space and then determining the complexity via the geodesic in the quantum space [6−11]. In addition, defining complexity in quantum field theory has recently become significant [12−24].

However, complexity is thought to have a holographic dual in gravity theory according to the AdS/CFT correspondence [25−27]. In this case, the complexity is described as "holographic complexity" [28, 29]. A reliable candidate is complexity-action (CA) conjecture [30, 31], where the complexity of the black hole equals the total action evaluated for the Wheeler-DeWitt patch (WdW) in the bulk (another candidate is the complexity equals volume (CV) conjecture, but we do not discuss this here). The CA conjecture has been investigated for various cases [32−39], and some features of complexity have been found; for example, complexity satisfies the second law of thermodynamics [40], it is nonlocal [41], and its variation only depends on the end point of the optimal trajectory [5]. Moreover, using the CA conjecture, some scholars have suggested that complexity can be studied using a probe [42−45]. For instance, Nagasaki proposed [44] that the complexity of AdS black holes can be determined by inserting a fundamental string in the corresponding bulk spacetime. Specifically, one may consider a Wilson line operator (a nonlocal operator) located in $AdS_5$ spacetime by inserting a fundamental string, and the Wilson line will move in a great circle in the $S^3$ part in $AdS_5$. This will then reveal how complexity is deformed when adding a time-dependent operator, which describes a test particle moving in boundary gauge theory. The studies in Refs. [44, 45] have already revealed several interesting results. For example, complexity increases with black hole mass but simultaneously exhibits a specific behavior near the speed of light. Furthermore, complexity becomes smaller as the string moves faster. Other interesting results can be found in [46−51].

The purpose of this study is to investigate the effect of backreaction on the complexity according to the CA conjecture. The motivations for this are as follows. First, in [44, 45], the author considers a Wilson line operator located in AdS spacetime by inserting a fundamental string in the bulk, and such an operator describes a test particle moving in boundary gauge theory. However, in practice, backreaction may exist owing to the presence of other strings in the system. In other words, when analyzing the complexity of AdS black holes using a probe string, the effect of backreaction due to the presence of other strings may need to be considered. Incidentally, the backreaction effect also arose in a study of quark-gluon

Received 6 October 2024; Accepted 18 December 2024; Published online 19 December 2024

* Supported by the National Natural Science Foundation of China (12375140) and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan)

† E-mail: guolong@cug.edu.cn

‡ E-mail: zhangzq@cug.edu.cn(Corresponding author)

plasma (QGP) produced in heavy ion collisions [52–55]. Second, previous studies have shown that complexity is expected to behave in a similar manner to entropy by the second law of thermodynamics [40]. It is known that entanglement entropy and black hole entropy are two typical types of entropy with potential similarities, such as their leading term divergences being proportional to the area of the subsystem. Recently, the backreaction effect on holographic entanglement entropy was studied in [56], and the results showed that entanglement entropy increased with respect to backreaction. Given the close analogy between entanglement entropy and complexity, it would be interesting to observe whether backreaction has the same effect on complexity as that on entanglement entropy.

This paper is organized as follows. In Section II, we review the backreacted gravity geometry given in [52]. In Section III, we study the complexity for the backreacted gravity background and analyze its dependence on backreaction. Finally, Section IV presents the conclusion and discussion.

## II. BACKGROUND GEOMETRY

Let us consider an ($n$+1)-dimensional gravity theory with a negative cosmological constant $\Lambda$ [52]:

$$I=\frac{1}{4\pi G_{n+1}}\int \mathrm{d}x^{d+1}\sqrt{g}(\mathcal{R}-2\Lambda)+S_m, \tag{1}$$

where $G_{n+1}$ denotes the ($n$+1)-dimensional Newton constant, $\mathcal{R}$ is the Ricci scalar, and $S_m$ represents the matter part,

$$S_m=-\frac{1}{2}\sum_i \mathcal{T}_i\int \mathrm{d}^2\xi\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}, \tag{2}$$

with $g_{\mu\nu}$ being the space-time metric and $h^{\alpha\beta}$ the world-sheet metric. Here, $\mu$ and $\nu$ are the space-time directions, and $\alpha$ and $\beta$ are the world-sheet coordinates.

Einstein's equations based on (1) are

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi G_{\mu\nu}T_{\mu\nu}, \tag{3}$$

with

$$T^{\mu\nu}=-\sum_i \mathcal{T}_i\int \mathrm{d}^2\xi\frac{1}{\sqrt{|g_{\mu\nu}|}}\sqrt{|h_{\alpha\beta}|}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\delta_i{}^{n-1}(x-X), \tag{4}$$

where $\delta_i{}^{n-1}(x-X)$ refers to the source divergences owing to the presence of strings.

The string cloud density is given by

$$a(x)=T\sum_i\delta_i^{(n-1)}(x-X_i),\quad \text{with}\quad a>0. \tag{5}$$

Averaging over the $(n-1)$ spatial dimensions, the constant string density can be written as

$$\tilde{a}=\frac{1}{V_{n-1}}\int a(x)d^{n-1}x=\frac{\mathcal{T}N}{V_{n-1}}, \tag{6}$$

where $V_{n-1}$ is the volume of the $(n-1)$ dimensional space, and $N$ is the total number of strings.

Choosing the static gauge $t=\xi^0$ and $r=\xi^1$, we obtain the nonzero components of $T^{\mu\nu}$

$$T^{00}=-\frac{\tilde{a}g^{tt}}{r^3},\qquad T^{rr}=-\frac{\tilde{a}g^{rr}}{r^3}, \tag{7}$$

where we assume that the strings are evenly distributed over $n-1$ directions.

The solution to Eq. (3) is

$$\mathrm{d}s^2=-V(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{V(r)}+\frac{r^2}{R^2}\delta_{ab}\mathrm{d}x^a\mathrm{d}x^b, \tag{8}$$

with

$$V(r)=K+\frac{r^2}{R^2}-\frac{2m}{r^{n-2}}-\frac{2aR^{n-3}}{(n-1)r^{n-3}}, \tag{9}$$

where $a=\tilde{a}R$ , with $R$ being the AdS radius. $K=0, -1, 1$ correspond to a flat, spherical, or hyperbolic boundary, respectively. In this study, we are primarily interested in the case of $K=0$. Correspondingly, the metric becomes

$$\mathrm{d}s^2=-\frac{r^2}{R^2}f(r)\mathrm{d}t^2+\frac{r^2}{R^2}\mathrm{d}\vec{x}^2+\frac{1}{f(r)}\frac{R^2}{r^2}\mathrm{d}r^2, \tag{10}$$

where

$$f(r)=1-\frac{2mR^2}{r^n}-\frac{2a}{n-1}\frac{R^{n-1}}{r^{n-1}}, \tag{11}$$

$$m=\left(1-\frac{2a}{n-1}\frac{R^{n-1}}{r_h^{n-1}}\right)\frac{r_h^n}{2R^2}, \tag{12}$$

with $r=\infty$ as the boundary, and $r=r_h$ as the horizon, where $r_h$ satisfies $f(r_h)=0$. Note that $m$ is proportional to the ADM (M) mass of the black hole via

$$M=\frac{m(n-1)V_{n-1}}{8\pi G_{n+1}}, \tag{13}$$

so if $V_{n-1}$ is considered a constant, $m$ may be considered the black hole mass for fixed $n$.

The Hawking temperature of the black hole is

$$T = \frac{\sqrt{g^{rr}}\partial_r\sqrt{g_{tt}}}{4\pi}\Big|_{r=r_h} = \left(n - \frac{2aR^{n-1}}{(n-1)r_h^{n-1}}\right)\frac{r_h}{4\pi R^2}. \quad (14)$$

It has previously been demonstrated [52] that the geometry (10) is thermodynamically stable under tensor and vector perturbations. For more information on backreacted gravity backgrounds, refer to [52].

## III. COMPLEXITY GROWTH IN THE BACKREACTED GRAVITY BACKGROUND

From the holographic perspective, when a particle moves on the boundary of $AdS_5$ space, the particle will lose energy owing to the effect of shear viscosity in QGP. Such a system is dissipative, and the energy loss can be depicted by the drag force [57–59] related to the Nambu-Goto (NG) action outside of the horizon. Inspired by the AdS/CFT calculation of the drag force, Nagasaki suggested [44] that the complexity of a time dependent system inside the horizon may be obtained in a similar manner. Specifically, if one considers a Wilson line operator located in $AdS_5$ spacetime by inserting a fundamental string, the Wilson line will move in a great circle in the $S^3$ part in $AdS_5$. This process will then reveal how complexity is deformed when adding a nonlocal operator. As a general rule, inserting a Wilson loop is described by adding an NG term, and the action is expected to consist of the NG, Einstein Hilbert, and boundary terms; however, it was argued [44] that when treating NG action that includes up to the first derivative of the fields, the boundary terms can be neglected.

We now follow the prescription in [44] to study the effect of backreaction on the Wilson line operator, focusing (only) on the NG term for the background metric (10). The string is governed by the NG action,

$$S_{\rm NG} = -T_s\int {\rm d}\tau{\rm d}\sigma\,\sqrt{-g}, \quad (15)$$

where $T_s$ is the string tension, and $g$ denotes the determinant of the induced metric with

$$g_{\alpha\beta} = g_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}, \quad (16)$$

where $g_{\mu\nu}$ and $X^\mu$ are the brane metric and target space coordinates, respectively.

Supposing that the string moves in a great circle in the $S^{n-1}$ subspace, the induced metric of this part becomes ${\rm d}\vec{x}^2 = {\rm d}\phi^2$. Parametrizing the world sheet via

$$t=\tau, \qquad r=\sigma, \qquad \phi = vt+\xi(r), \quad (17)$$

we obtain nonzero components of the induced metric,

$$g_{tt} = -\frac{r^2 f(r)}{R^2}, \qquad g_{xx} = \frac{r^2}{R^2}, \qquad g_{rr} = \frac{R^2}{r^2 f(r)}. \quad (18)$$

Therefore, the Lagrangian density becomes

$$\mathcal{L} = \sqrt{-g_{rr}g_{tt} - g_{rr}g_{xx}v^2 - g_{xx}g_{tt}\xi'^2} = \sqrt{1 - \frac{v^2}{f(r)} + \frac{r^4 f(r)}{R^4}\xi'^2}, \quad (19)$$

with $\xi' = {\rm d}\xi/{\rm d}\sigma$.

Substituting (19) into (15), we find the time derivative of the NG action,

$$\frac{{\rm d}S_{\rm NG}}{{\rm d}t} = T_s\int_0^{r_h}{\rm d}\sigma\mathcal{L} = T_s\int_0^{r_h}{\rm d}r\sqrt{1 - \frac{v^2}{f(r)} + \frac{r^4 f(r)}{R^4}\xi'^2}. \quad (20)$$

Because the action in (20) does not depend on $\xi$ explicitly, we can define a constant as

$$\Pi_\xi = \frac{\partial\mathcal{L}}{\partial\xi'} = \xi'\frac{r^4 f(r)/R^4}{\sqrt{1 - \dfrac{v^2}{f(r)} + \dfrac{r^4 f(r)}{R^4}\xi'^2}}, \quad (21)$$

yielding

$$\xi'^2 = \frac{\Pi_\xi^{\,2}\left[1 - \dfrac{v^2}{f(r)}\right]}{\dfrac{r^4 f(r)}{R^4}\left[\dfrac{r^4 f(r)}{R^4} - \Pi_\xi^{\,2}\right]}. \quad (22)$$

Before continuing, we comment on (22). First, the denominator and numerator are both positive for large $r$ and negative for small $r$ (near the horizon). Second, $\xi'^2$ should be non-negative. Taken together, we infer that the denominator and numerator should change sign at the same point, *i.e*, the critical point. For the numerator, the critical point $r_c$ satisfies

$$f(r_c) = v^2, \quad (23)$$

with $f(r_c) \equiv f(r)|_{r=r_c}$, yielding

$$1 - \frac{2mR^2}{r_c^n} - \frac{2a}{n-1}\frac{R^{n-1}}{r_c^{n-1}} - v^2 = 0. \quad (24)$$

For the denominator, $r_c$ should satisfy

$$\Pi_{\xi}^{2} = \frac{r_c^4 f(r_c)}{R^4}. \tag{25}$$

Substituting (22), (23), and (24) into (20), we end up with the growth of the NG action as

$$\frac{1}{T_s}\frac{\mathrm{d}S_{\rm NG}}{\mathrm{d}t} = \int_0^{r_h} \mathrm{d}r \sqrt{\frac{r^4(f(r)-v^2)}{r^4 f(r) - r_c^4 f(r_c)}}. \tag{26}$$

As described, the growth of the NG action is closely connected to the increase in complexity. Therefore, the dependence of complexity on backreaction can be obtained by analyzing (26). However, it is difficult to solve (26) analytically, and we must resort to numerical methods (for convenience of calculations, we take the AdS radius $R$ as the unit). The numerical procedure is summarized as follows:

1) Choose a dimension $n$ and obtain $r_h$ for different values of $a$ at fixed $m$ using (12).

2) Then, obtain $r_c$ with $r_h$ for different values of $v$ and $a$ using (24).

3) Next, substitute $r_h$ and $r_c$ into (26) to obtain the numerical result of the complexity increase.

4) Repeat the same procedure for different values of $n$ to obtain the relationship of the increase in complexity with $a$ and $v$.

5) Similarly, we can analyze other cases by varying the value of $m$.

In the left panel of Fig. 1, we plot $\frac{\mathrm{d}S_{\rm NG}}{\mathrm{d}t}\frac{1}{T_s}$ versus $v$ with fixed $m$ for $n=4$ (where $r_h=1$ and $a=0$ give $m=0.5$; therefore, for comparative purposes, we take $m=0.5$ for all other situations). In all the plots, the action growth is a maximum when the string is stationary, *i.e*, $v=0$, and it decreases as $v$ increases, implying that complexity decreases as the probe string moves faster, in accordance with [44, 45]. Furthermore, for slower strings, the action grows as $a$ increases, but for faster strings, this is reversed. This is suggested by the left panel of Fig. 1 , which shows that the curves intersect as $v$ increases. To more clearly show the difference, we plot $\frac{\mathrm{d}S_{\rm NG}}{\mathrm{d}t}\frac{1}{T_s}$ versus $a$ with fixed $v$ in the right panel of Fig. 1. For slower strings, *e.g*, $v=0.6$, the complexity increase is an increasing function, whereas for faster strings, *e.g*, $v=0.9$, it is a decreasing function. This indicates that complexity has a different velocity dependence with respect to backreaction in the relativistic region. Interestingly, similar phenomena appear in the mass dependence at different velocities: for slower strings, the mass dependence increases with mass, whereas for faster strings, it changes to a decreasing function of mass [45].

Figures 2 and 3 show the results of $n=6$ and $n=8$, respectively. The results are similar to those in Fig. 1: complexity decreases as the probe string moves faster, and for slower strings, the inclusion of backreaction tends to enhance the complexity. However, there are some differences. First, the slopes of the plots in Figs. 2 and 3 are smaller than their counterparts in Fig. 1. Second, for $n=6$ , the critical velocity is approximately $v=0.97$, whereas for $n=8$, there is no critical velocity until $v=0.99$, i.e., the increase in complexity is still an increasing function for $v=0.99$(but with a very small slope). Therefore, we conclude that the higher the dimension, the smaller the change in complexity owing to backreaction. Furthermore, for probe strings, the effect of backreaction becomes insensitive in higher dimensions.

Next, we compare our results with (holographic) entanglement entropy. It has been argued [56] that entanglement entropy increases with respect to backreaction. Here, we find that for slower strings, complexity increases as backreaction increases (whereas for faster strings, the opposite may occur (except for $n=8$)). However, it is worth noting that in [56], entanglement entropy was determined as the area of a minimal surface ex-

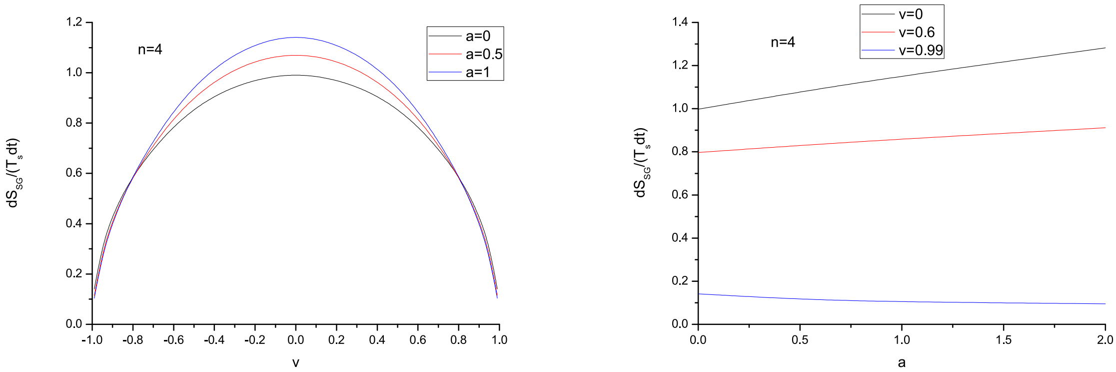


**Fig. 1.** (color online) Left: Action growth vs. string velocity for $n=4$; Right: Action growth vs. $a$.

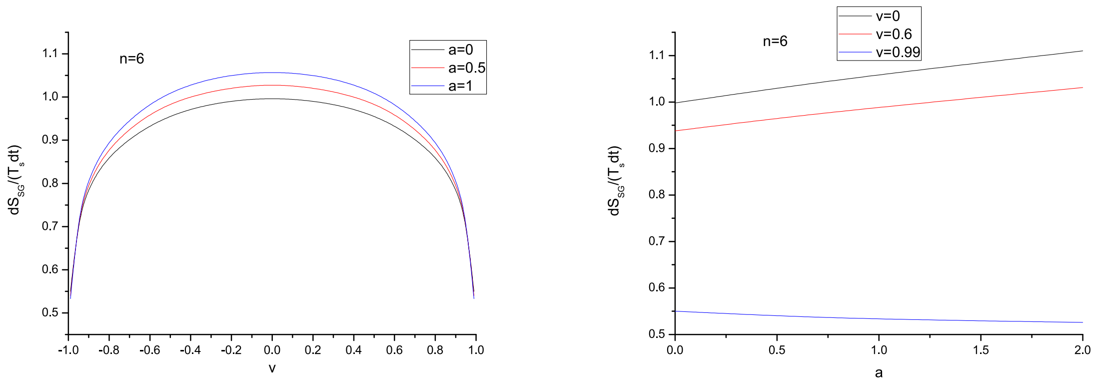


**Fig. 2.** (color online) Left: Action growth vs. string velocity for $n = 6$; Right: Action growth vs. $a$.

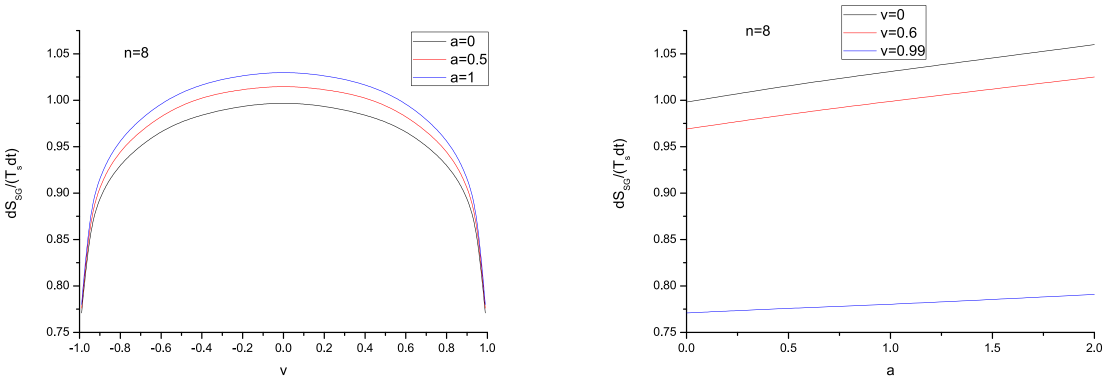


**Fig. 3.** (color online) Left: Action growth vs. string velocity for $n = 8$; Right: Action growth vs. $a$.

tending from some predefined surface on the boundary into the bulk, and the velocity effect has not yet been considered. It would be interesting to investigate whether the effect of backreaction on entanglement entropy also depends on velocity. Thus far, the results suggest that under the condition of low velocity, backreaction has the same effect on entanglement entropy and complexity, i.e., the inclusion of backreaction enhances entanglement entropy and complexity and is thus bounded by the second law of thermodynamics [40].

## IV. CONCLUSION

Complexity is expected to be a useful tool for understanding gravitational physics. In this study, we follow the prescription in [44] to investigate holographic complexity in backreacted gravity backgrounds according to the CA conjecture. Such backreaction originates from the presence of static strings evenly distributed over the system. Specifically, we use a probe string in the bulk that contributes an NG action and determine the growth of the action and its dependence on the backreaction, time-space dimension, and string velocity. Our conclusions on the complexity in backreacted gravity backgrounds can be summarized as follows:

1) The effect of backreaction on complexity depends on velocity: for $n = 4$ and $n = 6$, the complexity increases as backreaction increases for slower strings, whereas for faster strings, this is reversed. This specific behavior is similar to that of the mass dependence [45]. However, for $n = 8$, the complexity increases as backreaction increases at all velocities.

2) For $n = 4$ and $n = 6$ , the critical velocities are approximately $v = 0.8$ and $v = 0.97$, respectively, whereas for $n = 8$, there is no critical velocity. Moreover, we find that the higher the dimension, the smaller the change in complexity owing to velocity and backreaction.

3) Under the condition of low velocity, backreaction has the same effect on entanglement entropy and complexity.

Now, we discuss the physical significance of the results. It is known that complexity defines how complex a physical system is. If a probe string in the bulk is used to investigate the complexity, the backreaction from other strings may affect it. Our results indicate that for slower strings, complexity increases as backreaction increases, implying the system may have more information, in accordance with our physical intuitions. However, for lower dimension space, backreaction seems to decrease the complexity for faster strings. This is an intriguing and unexpected result for which we cannot yet provide a clear interpretation.

There are several questions that require investigating in the future. First, the backreaction effect on entanglement entropy including the velocity effect may be studied, and the results can be compared with ours. Moreover, the relationships between complexity and temperature, electromagnetic field, and angular momentum deserves further research.

## References


[1] J. Watrous, *Quantum computational complexity*, arXiv: 0804.3401
[2] N. Bao and J. Liu, JHEP **08**, 144 (2018)
[3] S. Arora and B. Barak, *Computational Complexity: A Modern Approach*, 1st ed. (Cambridge University Press, New York, 2009)
[4] C. Moore and S. Mertens, *The Nature of Computation*, (Oxford University Press, New York, 2011)
[5] A. Bernamonti, F. Galli, J. Hernandez *et al.*, Phys. Rev. Lett. **123**, 081601 (2019)
[6] L. Susskind and Y. Zhao, *Switchbacks and the bridge to nowhere*, arXiv: 1408.2823
[7] M. A. Nielsen, M. R. Dowling, M. Gu *et al.*, Science **311**, 1133 (2006)
[8] M. R. Dowling and M. A. Nielsen, *The geometry of quantum computation*, arXiv: quant-ph/0701004
[9] R. Jefferson and R. C. Myers, JHEP **10**, 107 (2017)
[10] R. Q. Yang, Y. S. An, C. Niu *et al.*, *Axiomatic complexity in quantum field theory and its applications*, arXiv: 1803.01797
[11] L. Hackl and R. C. Myers, JHEP **07**, 139 (2018)
[12] S. Chapman, M. P. Heller, H. Marrochio *et al.*, Phys. Rev. Lett. **120**, 121602 (2018)
[13] R. Q. Yang, Y. S. An, C. Niu *et al.*, JHEP **03**, 161 (2019)
[14] A. Bhattacharyya, P. Caputa, S. R. Das *et al.*, JHEP **07**, 086 (2018)
[15] V. Vanchurin, JHEP **06**, 001 (2016)
[16] J. Jiang, J. Shan, and J. Yang, Nucl. Phys. B **954**, 114988 (2020)
[17] J. Molina-Vilaplana and A. Del Campo, JHEP **08**, 012 (2018)
[18] T. Ali, A. Bhattacharyya, S. Shajidul Haque *et al.*, JHEP **04**, 087 (2019)
[19] J. Couch, S. Eccles, W. Fischler *et al.*, JHEP **03**, 108 (2018)
[20] M. Guo, J. Hernandez, R. C. Myers *et al.*, JHEP **10**, 011 (2018)
[21] R. Q. Yang, Y. S. An, C. Niu *et al.*, Eur. Phys. J. C **79**(2), 109 (2019)
[22] R. Khan, C. Krishnan, and S. Sharma, Phys. Rev. D **98**(12), 126001 (2018)
[23] S. Chapman and H. Z. Chen, JHEP **02**, 187 (2021)
[24] M. Sinamuli and R. B. Mann, Phys. Rev. D **99**(10), 106013 (2019)
[25] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998)
[26] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998)
[27] O. Aharony, S. S. Gubser, J. Maldacena *et al*, Phys. Rept. **323**, 183 (2000)
[28] M. Alishahiha, Phys. Rev. D **92**(12), 126009 (2015)
[29] J. Tao, P. Wang, and H. Yang, Eur. Phys. J. C **77**(12), 817 (2017)
[30] A. R. Brown, D. A. Roberts, L. Susskind *et al.*, Phys. Rev. Lett. **116**, 191301 (2016)
[31] A. R. Brown, D. A. Roberts, L. Susskind *et al.*, Phys. Rev. D **93**, 086006 (2016)
[32] J. L. F. Barbon and J. Martin-Garcia, JHEP **11**, 181 (2015)
[33] D. Carmi, R. C. Myers, and P. Rath, JHEP **03**, 118 (2017)
[34] K. Y. Kim, C. Niu, and R. Q. Yang, JHEP **09**, 042 (2017)
[35] A. Reynolds and S. F. Ross, Classical Quantum Gravity **34**, 175013 (2017)
[36] S. Chapman, H. Marrochio, and R. C. Myers, JHEP **01**, 062 (2017)
[37] Z. Y. Fan and M. Guo, JHEP **08**, 031 (2018)
[38] J. Jiang and B. X. Ge, Phys. Rev. D **99**(12), 126006 (2019)
[39] J. Jiang and X.-W. Li, Phys. Rev. D **100**(6), 066026 (2019)
[40] A. R. Brown and L. Susskind, Phys. Rev. D **97**, 086015 (2018)
[41] Z. Fu, A. Maloney, D. Marolf *et al.*, JHEP **02**, 072 (2018)
[42] D. Ageev, I. Arefeva, A. Bagrov *et al.*, JHEP **08**, 071 (2018)
[43] F. J. G. Abad, M. Kulaxizi, and A. Parnachev, JHEP **01**, 127 (2018)
[44] K. Nagasaki, Phys. Rev. D **96**(12), 126018 (2017)
[45] K. Nagasaki, Phys. Rev. D **98**(12), 126014 (2018)
[46] K. Nagasaki, Int. J. Mod. Phys. A **38**(04n05), 2350027 (2023)
[47] K. Nagasaki, PTEP **4**, 043B02 (2022)
[48] Y. T. Zhou, X. M. Kuang, and J. P. Wu, Eur. Phys. J. C **81**(8), 768 (2021)
[49] F. F. Santos, Eur. Phys. J. Plus **135**(10), 810 (2020)
[50] M. B. Gaete and F. F. Santos, Eur. Phys. J. C **82**(2), 101 (2022)
[51] W. B. Chang, D. F. Hou, Chin. Phys. C **48**(3), 034106 (2024)
[52] S. Chakrabortty, Phys. Lett. B **705**, 244 (2011)
[53] S. Chakrabortty and T. K. Dey, JHEP **05**, 094 (2016)
[54] P. P. Wu, X. R. Zhu, and Z. Q. Zhang, Nucl. Phys. B **952**, 114917 (2020)
[55] Z. q. Zhang, Phys. Rev. D **101**(10), 106005 (2020)
[56] S. Chakrabortty, S. Pant, and K. Sil, JHEP **06**, 061 (2020)
[57] S. S. Gubser, Phys. Rev. D **74**, 126005 (2006)
[58] C. P. Herzog, A. Karch, P. Kovtun *et al.*, JHEP **07**, 013 (2006)
[59] J. Casalderrey-Solana and D. Teaney, Phys. Rev. D **74**, 085012 (2006)