

Axial gravitational quasinormal modes of magnetically charged black holes*

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Abstract: In this study, we consider axial perturbations on the magnetically charged string-inspired Euler-Heisenberg black hole. As axial metric perturbation decouples from axial electromagnetic perturbation, we mainly focus on axial gravitational perturbation. By using the Wentzel–Kramers–Brillouin (WKB) approximation and asymptotic iteration method (AIM), we perform a detailed analysis of the gravitational quasinormal frequencies by varying the characteristic parameters of gravitational perturbation and black holes. The results obtained through the AIM are consistent with those obtained using the WKB method, including the results extracted from the time-domain profiles. The greybody factor is calculated using the WKB method. The effects of Q_m , ϵ , and multipole number l on the greybody factor are also studied.

Keywords: magnetically charged black holes, axial gravitational perturbation, quasinormal modes

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I. INTRODUCTION

The Euler–Heisenberg (EH) Lagrangian, first derived in 1936 [1], serves as a nonlinear extension of quantum electrodynamics (QED) and provides a classical description that goes beyond Maxwell’s theory in strong-field regimes where vacuum polarization effects become significant. By modeling vacuum as a polarizable medium, the EH framework incorporates effective polarization and magnetization responses arising from the virtual charge fluctuations associated with real charges and currents [2]. This theoretical formalism not only refines classical electrodynamics in extreme electromagnetic environments but also forms a cornerstone for studying nonlinear phenomena in astrophysics and cosmology.

Considering these unique features, the first black hole solution in Einstein–Hilbert (EH) gravity—an anisotropic, magnetically charged configuration analogous to the Reissner–Nordström metric but generalized to include dyon-type degrees of freedom—was obtained in 1956 [3]. This framework has also been extended to encompass

electrically charged solutions [3, 4], rotating black hole configurations [5, 6], and formulations within various modified theories of gravity [7, 8]. More recently, motivated by developments in string theory and Lovelock gravity, Ref. [9] introduced a novel coupling between the dilaton field and EH electrodynamics, extending the Einstein–Maxwell–dilaton framework. Further studies have investigated various physical phenomena in such spacetimes, including particle dynamics, gravitational lensing effects [10, 11], and the shadow characteristics of magnetically charged black holes [12]. Additionally, Jiang *et al.* [13] examined the properties of geometrically thin, optically thick accretion disks in these backgrounds, offering insights into the observational signatures of such compact objects.

The recent detection of gravitational waves has opened up new avenues for probing strong gravitational fields in the vicinity of black holes. In particular, during the merger of binary compact objects, the ringdown phase of the emitted gravitational waves can be interpreted as the response of the remnant black hole to perturbations

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imposed during the coalescence process. At this stage, the final Kerr black hole can be effectively described as being in a perturbed state, with the emitted gravitational radiation characterized by distinct decay timescales. These features are well modeled by the quasinormal modes (QNMs) of the black hole [14, 15], which are unique oscillation frequencies determined solely by the mass and spin of the black hole. Furthermore, QNMs provide a powerful tool for testing the validity of the no-hair conjecture, as deviations from the expected frequency spectrum could indicate violations of the Kerr metric or the presence of exotic compact objects [16–18]. In the context of modified theories of gravity [19–23], QNMs serve as critical probes for constraining alternative gravitational models. The stability of QNMs under the parametric perturbations and deformations of the background spacetime plays a key role in such investigations. However, recent studies have also shown that certain QNM spectra may exhibit instability under small perturbations to the effective potential [24, 25]. Moreover, the behavior of QNMs under various types of perturbations offers valuable insights into the stability properties of the underlying spacetime geometry [26–31].

Motivated by these considerations, this study focuses on the QNMs of the magnetically charged black hole. Note that Cho *et al.* [32, 33] introduced the asymptotic iteration method (AIM), a systematic iterative technique for deriving the accurate approximations of QNM frequencies. In this study, we employ AIM to numerically solve the perturbation equation. Additionally, we apply the Wentzel–Kramers–Brillouin (WKB) approximation [34–36], a well-established method, to estimate the QNM frequencies of the black holes. By comparing the results from both approaches, we aim to cross-validate and reinforce our findings. Another crucial aspect of black hole perturbation theory is the greybody factor [37–40], which characterizes how the gravitational potential of a black hole modifies the spectrum of emitted radiation. In gravitational wave astronomy, greybody factors and QNMs form a complementary framework for analyzing compact binary mergers: QNMs determine the ringdown phase through their characteristic frequencies and damping times, whereas greybody factors quantify the angular-dependent transmission probability of gravitational waves throughout the inspiral–merger–ringdown process [41]. Although the background geometry remains spherically symmetric, axial perturbations are particularly suitable for analyzing the dynamics of magnetically charged black holes in the string-inspired Euler–Heisenberg framework. Owing to the monopolar structure of the gauge potential, the axial sector exhibits a decoupled evolution governed by a single master equation. This simplification facilitates a clearer computation of QNMs and greybody factors and has been widely adopted in related studies of nonlinear electrodynamics.

Accordingly, we plan to study the QNMs and greybody factor of axial perturbations on the magnetically charged black hole in the string-inspired Euler–Heisenberg theory. The remainder of this paper is organized as follows. In Section II, we review the black hole solution briefly and derive the master equation for axial perturbations. In Section III, we solve the quasinormal frequencies with the AIM and WKB methods and study the effects of the black hole parameters on the QNMs. In Section IV, we investigate the time evolution profiles of the perturbation in Section III. In Section V, we calculate the greybody factor with the WKB method. The conclusions and discussions are given in Section VI.

II. MASTER EQUATION FOR BLACK HOLE PERTURBATIONS

A. Background and black hole solution

Motivated by string theory and Lovelock gravity, Bakopoulos *et al.* [9] have recently introduced an extension of the Einstein–Maxwell–dilaton theory that incorporates a nonlinear Euler–Heisenberg term coupled to the dilaton field. The action describing this model is given by [9]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\nabla^\mu \phi \nabla_\mu \phi - \mathcal{L}(\phi, \mathcal{F})), \quad (1)$$

where R denotes the scalar curvature, ϕ is the scalar field, and $\mathcal{L}(\phi, F)$ represents the Lagrangian density that governs the interaction between the dilaton and the nonlinear electromagnetic field. This Lagrangian is explicitly defined as

$$\mathcal{L}(\phi, F) = e^{-2\phi} F^2 + f(\phi) (2\alpha F_\nu^\mu F_\rho^\nu F_\delta^\rho F_\mu^\delta - \beta F^4). \quad (2)$$

Here, $f(\phi)$ is a coupling function, $F^2 = F_{\mu\nu} F^{\mu\nu}$, and $F^4 = F_{\mu\nu} F^{\mu\nu} F_{\rho\delta} F^{\rho\delta}$, where $F_{\mu\nu}$ represents the usual field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the case where $\alpha = \beta = 0$, the theory reduces to the standard Einstein–Maxwell–dilaton model.

Varying the action (1) with respect to $g_{\mu\nu}$, ϕ , and A_μ , we can obtain the three field equations

$$E_{\mu\nu}^{(g)} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 2\partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \partial^\mu \phi \partial_\mu \phi - T_{\mu\nu} = 0, \quad (3)$$

$$E^{(\phi)} = \square \phi + \frac{1}{2} e^{-2\phi} F^2 - \frac{df(\phi)}{d\phi} \left(\frac{\alpha}{2} F_\nu^\mu F_\gamma^\nu F_\delta^\gamma F_\mu^\delta - \frac{\beta}{4} F^4 \right) = 0, \quad (4)$$

$$E_\nu^{(A)} = \partial^\mu \left[\sqrt{-g} (4F_{\mu\nu}(2\beta f(\phi)F^2 - e^{-2\phi}) - 16\alpha F^{\mu\kappa} F_\lambda^\kappa F_\nu^\lambda) \right] = 0, \quad (5)$$

where $T_{\mu\nu}$ is an energy-momentum tensor with

$$T_{\mu\nu} = 2e^{-2\phi} (F_\mu^\alpha F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F^2) + f(\phi) \left(8\alpha F_\mu^\alpha F_\nu^\beta F_\alpha^\eta F_{\beta\eta} - \alpha g_{\mu\nu} F_\beta^\alpha F_\gamma^\beta F_\delta^\gamma F_\alpha^\gamma - 4\beta F_\mu^\xi F_{\nu\xi} F^2 + \frac{1}{2} g_{\mu\nu} \beta F^4 \right). \quad (6)$$

With regard to the coupling function $f(\phi)$, Bakopoulos *et al.* [9] adopted

$$f(\phi) = -[3\cosh(2\phi) + 2] \equiv -\frac{1}{2} (3e^{-2\phi} + 3e^{2\phi} + 4), \quad (7)$$

and then obtained an exact analytic solution in the presence of magnetic charge and scalar hair. This hyperbolic structure reflects a symmetric dilaton coupling often encountered in string-theoretic effective actions, ensuring that both $\phi \rightarrow \infty$ and $\phi \rightarrow -\infty$ regimes are covered in a regular manner. Moreover, this form guarantees a non-trivial contribution from higher-order nonlinear electromagnetic terms and stabilizes the potential structure needed for horizon formation. Defining $\epsilon = \alpha - \beta$, we obtain the magnetically charged black hole solution [9]

$$ds^2 = -H(r)dt^2 + \frac{1}{H(r)}dr^2 + R(r)^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (8)$$

$$H(r) = 1 - \frac{2M}{r} - \frac{2\epsilon Q_m^4}{r^3(r - Q_m^2/M)^3}, \quad R(r) = r \left(r - \frac{Q_m^2}{M} \right),$$

$$\phi(r) = -\frac{1}{2} \ln \left(1 - \frac{Q_m^2}{Mr} \right), \quad A_\mu = (0, 0, 0, Q_m \cos\theta), \quad (9)$$

where M and Q_m are the mass and magnetic charge of this black hole, respectively.

In the limit $\epsilon = 0$, the solution (9) reduces to the GMGHS or GHS black holes [42, 43], which have been extensively studied [44–47]. Moreover, this solution (9) describes a black hole with a single horizon when $\epsilon = 1$, whereas for $\epsilon = -1$, the black hole horizons can range from two to none. This indicates that the magnetically charged black hole possesses different horizon structures.

To consider the QNMs and greybody factor of axial perturbation on the magnetically charged black hole in the string-inspired Euler–Heisenberg theory, we rewrite the metric (8) and solution (9) into new forms

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (10)$$

$$A(r) = 1 - \frac{4M^2}{Q_m^2 + \sqrt{Q_m^4 + 4M^2r^2}} - \frac{2\epsilon Q_m^4}{r^6},$$

$$B(r) = 1 - \frac{Q_m^4 + 4M^2r^2}{r^2(Q_m^2 + \sqrt{Q_m^4 + 4M^2r^2})} + \frac{Q_m^4}{4M^2r^2} - \frac{\epsilon Q_m^4(Q_m^4 + 4M^2r^2)}{2M^2r^8},$$

$$\phi(r) = -\frac{1}{2} \ln \left(\frac{\sqrt{Q_m^4 + 4M^2r^2} - Q_m^2}{\sqrt{Q_m^4 + 4M^2r^2} + Q_m^2} \right). \quad (11)$$

B. Axial perturbations

Here, we focus on the axial perturbation for the magnetically charged black holes. To do so, we assume

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + \delta A_\mu, \quad (12)$$

where $\bar{g}_{\mu\nu}$ and $\bar{A}_\mu$ represent the background metric and electromagnetic field, respectively, and $\delta g_{\mu\nu}$ and δA_μ denote the corresponding perturbations.

Under the Regge–Wheeler gauge [48], we expand the metric perturbation using tensor spherical harmonics. The axial gravitational field perturbation involves two modes $h_0(r)$ and $h_1(r)$, and the perturbed metric is expressed as

$$\delta g_{\mu\nu} = \sum_{l,m} e^{-i\omega t} \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} \sin\theta \partial_\theta Y_{lm}, \quad (13)$$

where the spherical harmonics can be replaced by Legendre polynomials by setting the azimuthal number $m = 0$ without loss of generality, *i.e.*, $Y_{lm}|_{m=0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$, because the background metric is spherically symmetric.

Following Refs. [49, 50], axial vector perturbation is given by

$$\delta A_\mu = \sum_{l,m} e^{-i\omega t} \left[0, 0, -u_3(r) \frac{\partial_\varphi Y_{lm}}{\sin\theta}, u_3(r) \sin\theta \partial_\theta Y_{lm} \right]. \quad (14)$$

Substituting the perturbed metric and vector potential ((12), (13), and (14)) into the gravitational field equation (3), the non-zero components of the first-order perturbed gravitational field equation are obtained as

$$E_{tt}^{(g)} = E_{rr} = E_{t\theta} = \left[r^4 - 2\epsilon Q_m^2 (4e^{2\phi} + 3e^{4\phi} + 3) \right] u_3, \quad (15)$$

$$E_{r\theta}^{(g)} = \left[r^4 - 2\epsilon Q_m^2 (4e^{2\phi} + 3e^{4\phi} + 3) \right] u_3', \quad (16)$$

$$E_{\theta\theta}^{(g)} = E_{\phi\phi} = \left[r^4 - 6\epsilon Q_m^2 (4e^{2\phi} + 3e^{4\phi} + 3) \right] u_3, \quad (17)$$

$$E_{t\phi}^{(g)} = r^2 B h_0 A'^2 - r^2 A \left[h_0 A' B' + B(i\omega h_1 A' + A' h_0' + 2h_0 A'') \right] - A^2 \left[r^2 \tilde{\mathcal{L}}(\phi, F) + 2r B' - 2 + l(l+1) + 2B(1 + r^2 \phi'^2) \right] - r^2 B' (i\omega h_1 + h_0') - 2r B (2i\omega h_1 + i\omega r h_1' + r h_0''), \quad (18)$$

$$E_{r\phi}^{(g)} = -r^2 B h_1 A'^2 + 2A^2 h_1 (r^2 \tilde{\mathcal{L}}(\phi, F) - 2 + l + l^2 + r B' + 2r^2 B \phi'^2) - r A \left[4i\omega h_0 - 2ir\omega h_0' + h_1 (r(2\omega^2 - A' B') - 2B(A' + r A'')) \right], \quad (19)$$

$$E_{\theta\phi}^{(g)} = 2i\omega h_0 + A h_1 B' + B(h_1 A' + 2A h_1'), \quad (20)$$

where $\tilde{\mathcal{L}}(\phi, F)$ denotes the Lagrangian density for black hole background solutions (11) and is expressed as

$$\tilde{\mathcal{L}}(\phi, F) = \frac{2Q_m^2 e^{-2\phi(r)}}{r^4} - \frac{4\epsilon Q_m^4 [3 \cosh(2\phi(r)) + 2]}{r^8}. \quad (21)$$

From Eqs. (15)–(17), one can easily obtain the perturbation function $u_3(r) = 0$. Moreover, the electromagnetic perturbation function $u_3(r)$ does not appear in gravitational perturbation equations ((18), (19), and (20)). This indicates that the axial metric perturbation decouples from axial electromagnetic perturbation. In fact, the phenomenon has been recovered in Ref. [51] for magnetic black holes. In addition, we substitute the perturbed metric and vector potential ((12), (13), and (14)) into electromagnetic field equation (5) and then obtain the corresponding perturbed equations. The details are shown in the Appendix, along with the recovery of this decoupling phenomenon. Then, we obtain a single Schrödinger like equation for the perturbed electromagnetic field $u_3(r)$.

Gravitational perturbations are central to black hole physics because they directly probe spacetime geometry, align with observational priorities in gravitational wave astronomy, and reflect the intrinsic properties of the black hole. In the subsequent section, we only focus on the axial gravitational perturbation. From Eqs. (18), (19), and

(20), we can observe that only two of the above three equations are independent. Considering the component $E_{\theta\phi}$ (20), we have

$$h_0 = \frac{i}{2\omega} [B(h_1 A' + 2A h_1') + A h_1 B']. \quad (22)$$

Substituting Eq. (22) into the component $E_{r\phi}$ (19), we can eliminate $h_0(r)$ and $h_0'(r)$, and then obtain a single second order differential equation for $h_1(r)$. To modify this master equation into the standard Schrödinger form, we further define the function $\Psi(r)$ with

$$h_1(r) = C_0(r) * \Psi(r). \quad (23)$$

We assume that the function $C_0(r)$ takes the following form:

$$C_0(r) = \frac{r}{\sqrt{A(r)B(r)}}, \quad (24)$$

and then obtain the final perturbed equation

$$\frac{d^2 \Psi(r_*)}{dr_*^2} + [\omega^2 - V(r)] \Psi(r_*) = 0, \quad (25)$$

where r_* represents the tortoise coordinate with

$$dr_* = \frac{1}{\sqrt{AB}} dr \quad (26)$$

and the effective potential $V(r)$ is

$$V(r) = B(r)A''(r) + A'(r) \left(\frac{B'(r)}{2} + \frac{B(r)}{2r} \right) - \frac{B(r)A'(r)^2}{2A(r)} + A(r) \left[\frac{B'(r)}{2r} + \frac{2B(r)}{r^2} + 2B(r)\phi'(r)^2 + \frac{l^2 + l - 2}{r^2} + \tilde{\mathcal{L}}(\phi, F) \right] \quad (27)$$

with the background Lagrangian density $\tilde{\mathcal{L}}(\phi, F)$; see Eq. (21). Note that the perturbation of the energy-momentum tensor of the electromagnetic part cannot be ignored. A similar phenomenon also appears in Ref. [30].

Note that the solution (11) and potential (27) are invariant under the following rescaling: $r/M \rightarrow r$, $Q_m/M \rightarrow Q_m$, and $\epsilon/M^2 \rightarrow \epsilon$. For a better analysis of the behavior of the potential function, we set $M = 1$ throughout the paper and leave ϵ and Q_m free without loss of generality. The effective potentials $V(r)$ are plotted in Figs. 1, 2, and 3, respectively. The height of the effective potential increases as Q_m increases in Fig. 1. Moreover, the potential increases with an increase in the multipole moment l in

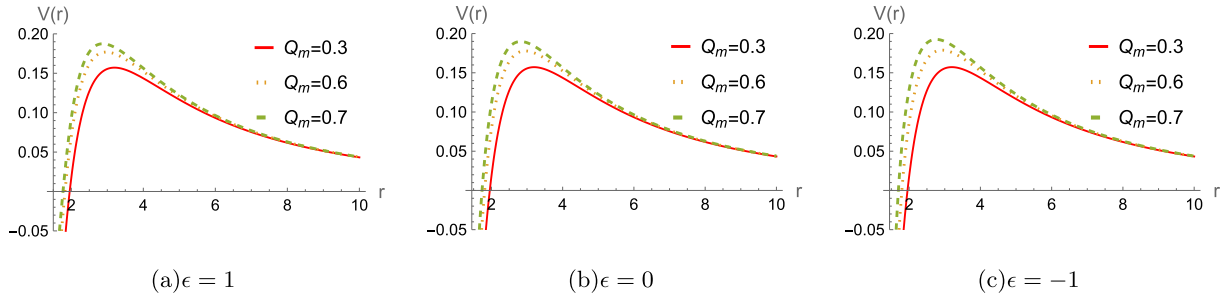


Fig. 1. (color online) Effective potential $V(r)$ for the gravitational perturbation with $M = 1$ and $l = 2$.

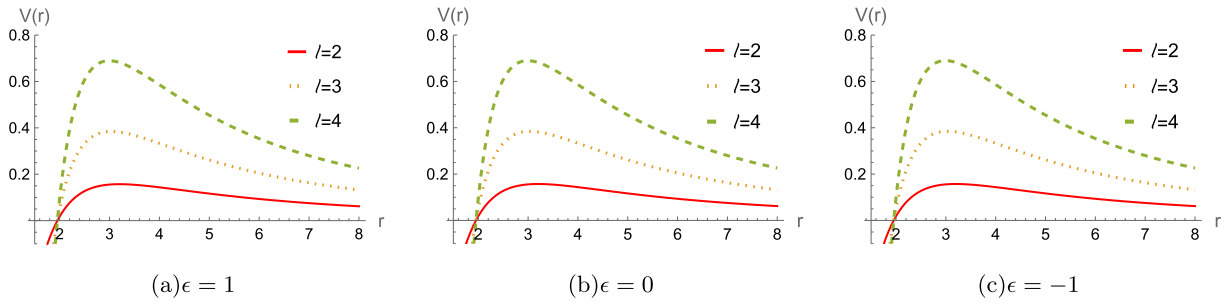


Fig. 2. (color online) Effective potential $V(r)$ for the gravitational perturbation with $M = 1$ and $Q_m = 0.3$.

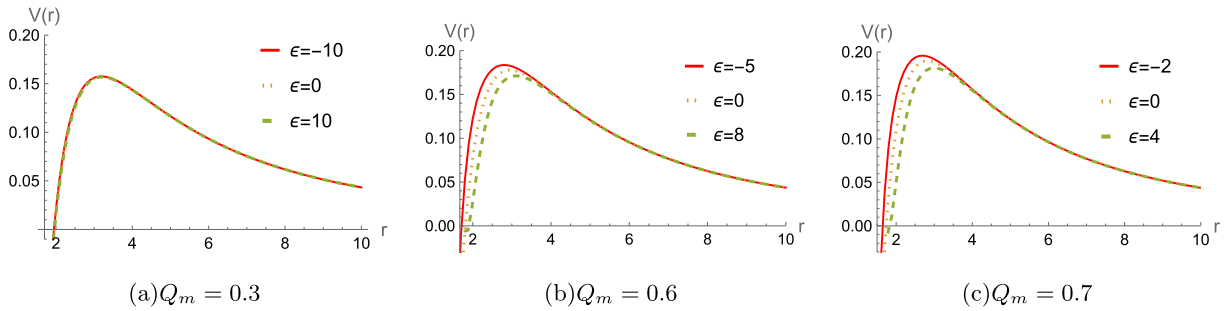


Fig. 3. (color online) Effective potential $V(r)$ for the gravitational perturbation with $M = 1$ and $l = 2$.

Fig. 2. For different Q_m values, the effective potential exhibits distinct behaviors as the parameter ϵ varies, as shown in Fig. 3. Note that the effective potentials are always positive, indicating that the system is stable under the axial gravitational field perturbation.

III. QUASINORMAL MODE FREQUENCIES

To accurately determine the QNM frequencies of magnetically charged black holes, we employ two distinct computational approaches: the AIM and WKB approximation method. This dual-methodology approach enables us to perform the cross-validation of our results.

A. Asymptotic iteration method

In Refs. [32, 33], the authors applied the AIM to the computation of QNMs. This method has been widely employed in the analysis of black hole perturbations across various spacetime geometries. In the present study, we utilized the AIM to numerically solve the axial gravitational perturbation equation (25), enabling us to determine the QNM frequencies associated with the considered black hole background.

We can rewrite the gravitational perturbation equation (25) in terms of $u = 1 - r_+/r$ as

$$\Psi''(u) + \frac{1}{2} \left(\frac{A'(u)}{A(u)} + \frac{B'(u)}{B(u)} + \frac{4}{u-1} \right) \Psi'(u) + \frac{r_+^2}{(u-1)^4 A(u) B(u)} \left[\omega^2 + \frac{(u-1)^2}{2r_h^8} \left(-4e^{-2\phi(u)} r_+^4 (u-1)^2 A(u) Q_m^2 \right. \right.$$

$$+ \frac{r_+^6(u-1)^2 B(u) A'(u)^2}{A(u)} - A(u) \left(-8(\alpha - \beta)(u-1)^6 (2 + 3 \cosh(2\phi(u))) Q_m^4 + r_+^6 (2(l^2 + l - 2) - (u-1)B'(u)) \right. \\ \left. + 4r_+^6 B(u) (1 + (u-1)^2 \phi'(u)^2) \right) - r_+^6(u-1) \left(A'(u) (3B(u) + (u-1)B'(u)) + 2(u-1)B(u)A'(u) \right) \Big] \Psi(u) = 0. \quad (28)$$

Here, the range of u satisfies $0 \leq u < 1$. The boundary conditions are pure ingoing waves ($\Psi \sim e^{-i\omega r_*}$, $r_* \rightarrow -\infty$) at the black hole horizon and pure outgoing waves ($\Psi \sim e^{i\omega r_*}$, $r_* \rightarrow +\infty$) at spatial infinity.

To propose an ansatz for Eq. (28), we examine the behavior of the function $\Psi(u)$ at the horizon ($u=0$) and at the boundary $u=1$. Near the horizon ($u=0$), we have $A(0) \approx uA'(0)$ and $B(0) \approx uB'(0)$. Thus, Eq. (28) reduces to

$$\Psi''(u) + \frac{1}{u} \Psi'(u) + \frac{r_+^2 \omega^2}{u^2 A'(0) B'(0)} \Psi(u) = 0. \quad (29)$$

Then, we can obtain the solution

$$\Psi(u \rightarrow 0) \sim C_1 u^{-\xi} + C_2 u^\xi, \quad \xi = \frac{i r_+ \omega}{\sqrt{A'(0) B'(0)}}, \quad (30)$$

where we must set $C_2 = 0$ to respect the ingoing condition at the black hole horizon.

At infinity ($u=1$), the asymptotic form of Eq. (28) can be written as

$$\Psi''(u) - \frac{2}{1-u} \Psi'(u) + \frac{r_+^2 \omega^2}{(1-u)^4} \Psi(u) = 0, \quad (31)$$

which has the solution

$$\Psi(u \rightarrow 1) \sim D_1 e^{-\zeta} + D_2 e^\zeta, \quad \zeta = \frac{i r_+ \omega}{1-u}. \quad (32)$$

To impose the outgoing boundary condition, we should set $D_1 = 0$.

From the above solutions at horizon and infinity, we can define the general ansatz for Eq. (28) as

$$\Psi(u) = u^{-\xi} e^\zeta \chi(u). \quad (33)$$

Substituting Eq. (33) into Eq. (28), we have

$$\chi'' = \lambda_0(u) \chi' + s_0(u) \chi, \quad (34)$$

where

$$\lambda_0(u) = \frac{1}{2} \left(\frac{4i r_+ \omega}{u \sqrt{A'(0)} \sqrt{B'(0)}} - \frac{A'(u)}{A(u)} - \frac{B'(u)}{B(u)} - \frac{4(i r_+ \omega + u - 1)}{(u-1)^2} \right), \quad (35)$$

and

$$s_0(u) = \frac{1}{2} \left[\frac{2r_+ \omega (r_+ (u-1)^2 \omega + i(u^2 - 1 + 2i r_+ \omega u) \sqrt{A'(0)} \sqrt{B'(0)})}{(u-1)^2 u^2 A'(0) B'(0)} - \frac{A'(u)^2}{A(u)^2} + \frac{2A''(u)}{A(u)} \right. \\ + \frac{4(u-1)^2 + 2r_+^2 \omega^2}{(u-1)^4} + 4\phi'(u)^2 + \frac{A'(u)}{B(u) - uB(u)} + \frac{1}{B(u)} \left(\frac{2(l^2 + l - 2)}{(u-1)^2} + \frac{4e^{-2\phi(u)} Q_m^2}{r_+^2} \right. \\ \left. - \frac{8(\alpha - \beta)(u-1)^4 (2 + 3 \cosh(2\phi(u))) Q_m^4}{r_+^6} - i r_+ \omega B'(u) \left(\frac{1}{(u-1)^2} - \frac{1}{u \sqrt{A'(0)} \sqrt{B'(0)}} \right) \right. \\ \left. + \frac{1}{A(u)} \left(A'(u) B'(u) - \frac{2r_+^2 \omega^2}{(u-1)^4} \right) \right) + \frac{A'(0)}{A(u)} \left(\frac{3u - 3 - i r_+ \omega}{(u-1)^2} + \frac{i r_+ \omega}{u \sqrt{A'(0)} \sqrt{B'(0)}} \right) \Big]. \quad (36)$$

Finally, we obtain the functions λ_0 and s_0 for the gravitational perturbation equation (34).

B. Padé averaged WKB method

The WKB method is a well-established approach to solve the black hole perturbation equation in the frequency domain [34–38]. However, the accuracy of the estimated frequencies degrades for $n \geq l$. To ameliorate

this issue, the Padé approximation [52] can be used to evaluate QNMs with higher precision. Within this method, the oscillation frequency ω can be determined using the following expression:

$$\omega = \sqrt{-i \left[(n + 1/2) + \sum_{k=2}^6 \bar{\Lambda}_k \right] \sqrt{-2V_0''} + V_0}, \quad (37)$$

where $n = 0, 1, 2, \dots$ represents the overtone number, $V_0 = V|_{r=r_{\max}}$, and $V_0'' = \frac{d^2 V}{dr^2}|_{r=r_{\max}}$. The position $r_{\max}$ corresponds to the location where the potential function $V(r)$ attains its highest value. To achieve higher precision in the calculations, correction terms $\bar{\Lambda}_k$ are introduced. The analytical expressions for these terms, together with the methodology for Padé averaging, are comprehensively presented in Refs. [39, 53].

Table 1 shows the fundamental quasinormal frequencies ($n = 0$) of black holes with $\epsilon = 0$ and $\epsilon = \pm 1$ obtained through the application of the AIM and sixth-order Padé averaged WKB approximation methods for different values of Q_m with $M = 1$ and $l = 2$. Evidently, these fundamental quasinormal frequencies obtained by numerical methods agree well with each other for each branch of these black holes.

The term Δ_6 in the sixth column of Table 1 serves to quantify the error between two adjacent order approximations, defined as [36]

$$\Delta_6 = \frac{|\omega_7 - \omega_5|}{2}, \quad (38)$$

where ω_7 and ω_5 represent the QNMs computed using the seventh and fifth-order Padé-averaged WKB methods, respectively. By analyzing these errors, we provide a detailed assessment of the accuracy of our QNM estimates. We further consider the percentage deviation Δ_{AW} of QNMs obtained via the AIM and Padé-averaged WKB methods. The relative error Δ_{AW} between the two methods is defined by

$$\Delta_{\text{AW}} = \frac{|\omega_{\text{AIM}} - \omega_{\text{PWKB}}|}{|\omega_{\text{PWKB}}|} \times 100\%. \quad (39)$$

Now, we discuss the influence of the magnetic charge Q_m on the QNM frequencies of the lowest modes ($n = 0$) in gravitational field perturbation. These fundamental QNM frequencies with respect to different values of Q_m

are plotted in Fig. 4. For different branch solutions, these QNM frequencies exhibit distinct characteristics with an increase in Q_m . The real QNM frequencies for $\epsilon = 0, 1$ increase as Q_m increases, and the damping rate or decay rate of the perturbed field increases significantly with an increase in Q_m ; see Figs. 4(a)–4(d). With regard to $\epsilon = -1$ in Figs. 4(e)–4(f), the real QNM frequencies also increase as Q_m increases. However, the damping rate increases with the increase in Q_m and then reaches a maximum decay rate at $Q_m = 0.6$. As Q_m surpasses this point, the damping rate begins to decrease.

The effects of the parameter ϵ on the QNM frequencies for gravitational perturbation can also be investigated. By fixing the magnetic charge $Q_m = 0.3, 0.6$, and 0.7 , we show the variation in the fundamental QNM frequencies and damping rate with different values of ϵ in Fig. 5. As the parameter ϵ increases, these QNM frequencies decrease for $Q_m = 0.6$ and 0.7 while remaining almost unchanged for $Q_m = 0.3$.

IV. TIME DOMAIN INTEGRATION

To study QNM properties via gravitational wave propagation, we analyze the time-domain behavior by converting Eq. (25) into a wave equation, replacing the second-order term with $-\frac{d^2}{dt^2}$. This yields a generalized PDE for the gravitational field

$$\left(\frac{d^2}{dr_*^2} - \frac{d^2}{dt^2} - V(r) \right) \Psi(r, t) = 0, \quad (40)$$

where the potential $V(r)$ is presented in Eq. (27).

Following the method in Refs. [40, 54–57], we can numerically solve the above time-dependent wave-like equation. Here, the finite difference method is applied for temporal integration, with a Gaussian wave serving as the initial spatial configuration. The radial coordinate is discretized based on the tortoise coordinate transformation

Table 1. Fundamental QNM frequencies for gravitational field perturbation with $M = 1$.

ϵ	l	Q_m	AIM	Padé averaged WKB	Δ_6	$\Delta_{\text{AW}}(\%)$
1	2	0.3	0.3812939 – 0.0895414i	0.381247 – 0.0895259i	0.0000272575	0.00412788
		0.6	0.4055129 – 0.0928762i	0.405312 – 0.0929958i	0.0000877586	0.0561951
		0.7	0.4174759 – 0.0960118i	0.41732 – 0.0962066i	0.000171463	0.0583149
0	2	0.3	0.3813581 – 0.0894781i	0.381338 – 0.0894572i	0.0000259841	0.00733116
		0.6	0.4071892 – 0.0912672i	0.407187 – 0.0912665i	4.24181×10^{-6}	0.0006046086
		0.7	0.4214127 – 0.0922712i	0.421409 – 0.0922722i	1.8842×10^{-6}	0.000862789
-1	2	0.3	0.3814223 – 0.0894146i	0.381425 – 0.0893987i	0.0000502269	0.00334693
		0.6	0.4091798 – 0.0892181i	0.40922 – 0.0893678i	0.000308716	0.0370056
		0.7	0.4261004 – 0.0874892i	0.426335 – 0.0878444i	0.00114384	0.0978375

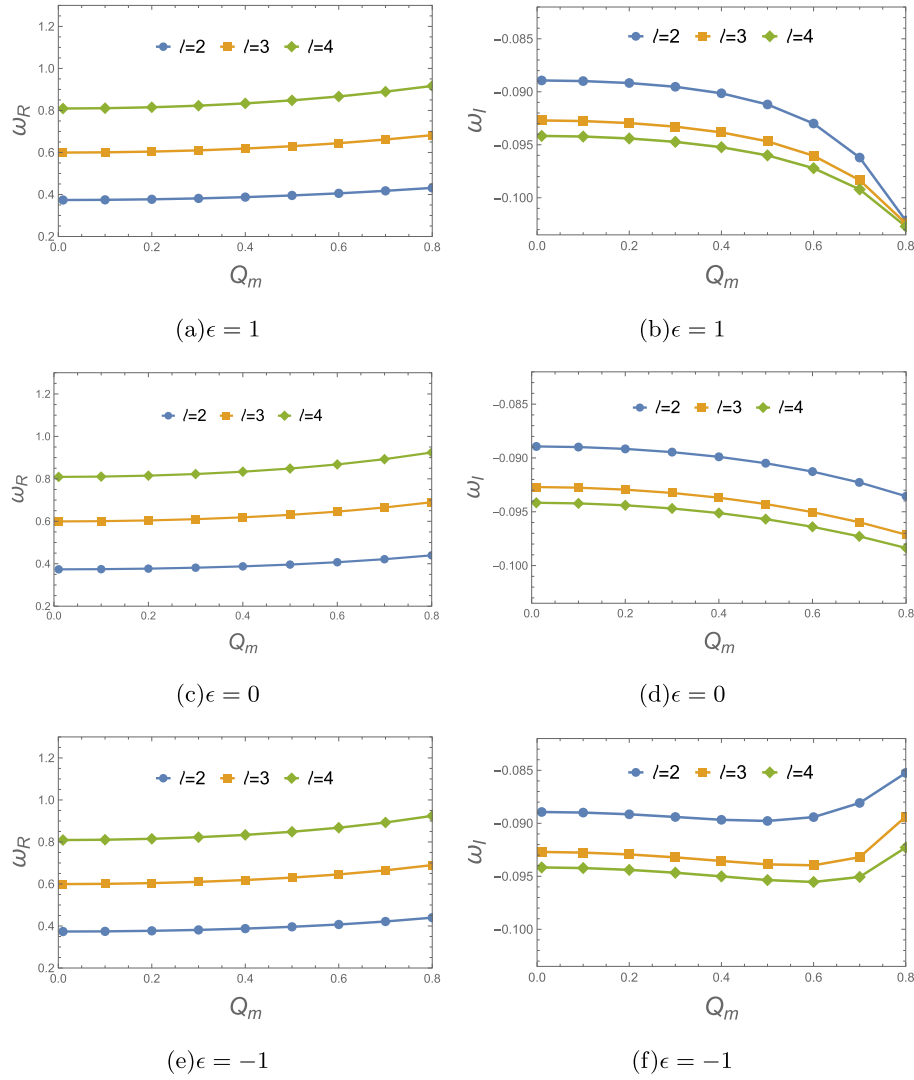


Fig. 4. (color online) Variation in fundamental QNM frequencies with respect to the magnetic charge Q_m with $M = 1$.

$$\begin{aligned}
 \frac{dr(r_*)}{dr_*} &= \sqrt{A(r(r_*))B(r(r_*))} \Rightarrow \frac{r(r_{*j} + \Delta r_*) - r(r_{*j})}{\Delta r_*} \\
 &= \frac{r_{j+1} - r_j}{\Delta r_*} = \sqrt{A(r_j)B(r_j)} \Rightarrow r_{j+1} \\
 &= r_j + \Delta r_* \sqrt{A(r_j)B(r_j)}.
 \end{aligned} \quad (41)$$

Then, one can further discretize the effective potential into $V(r(r_*)) = V(j\Delta r_*) = V_j$ and the field into $\Psi(r, t) = \Psi(j\Delta r_*, i\Delta t) = \Psi_{ji}$. Subsequently, the wave-like equation [49] is converted to a discretized equation

$$\begin{aligned}
 &-\frac{\Psi_{ji+1} - 2\Psi_{ji} + \Psi_{ji-1}}{\Delta t^2} + \frac{\Psi_{j+1,i} - 2\Psi_{ji} + \Psi_{j-1,i}}{\Delta r_*^2} \\
 &- V_j \Psi_{ji} + O(\Delta t^2) + O(\Delta r_*^2) = 0,
 \end{aligned} \quad (42)$$

from which one can isolate Ψ_{ji+1} after algebraic operations

$$\Psi_{ji+1} = \frac{\Delta t^2}{\Delta r_*^2} \Psi_{j+1,i} + \left(2 - 2\frac{\Delta t^2}{\Delta r_*^2} - \Delta t^2 V_j \right) \Psi_{ji} + \Psi_{j-1,i} - \Psi_{ji-1}. \quad (43)$$

The above equation is an iterative equation that can be solved if one gives a Gaussian wave packet $\Psi_{j,0}$ as the initial perturbation. In our calculations, setting the seed $r_{i=1} = r_h + 10^{-12}$, after imposing the initial condition $\Psi_{ji < 0} = 0$, $\Psi_{j,0} = \exp\left[-\frac{(r_j - a)^2}{2b^2}\right]$. We choose the parameters $a = 40$ and $b = 20$ in the Gaussian profile, and setting $\frac{\Delta t}{\Delta r_*} = \frac{0.05}{0.1} = \frac{1}{2}$, this iterative equation provides us the evolution of the gravitational field perturbation in the time profile.

Figure 6 shows the temporal profiles for gravitational perturbation while varying the parameter Q_m . These profiles were obtained with an overtone number of $n = 0$ and a multipole number of $l = 2$ in both cases. For $\epsilon = 0, 1$

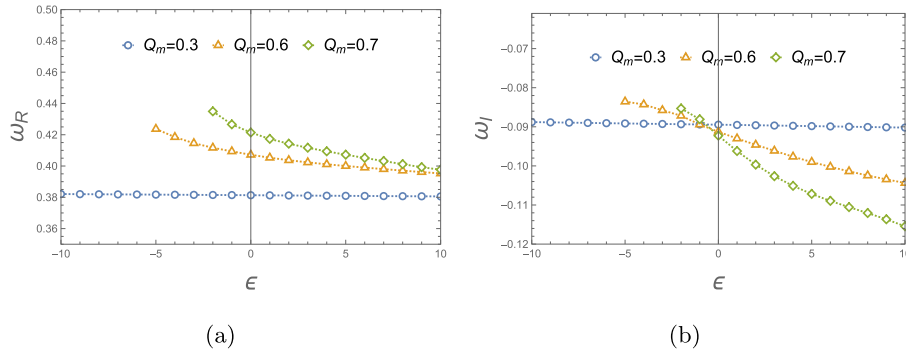


Fig. 5. (color online) Variation in fundamental QNM frequencies with respect to ϵ with $M = 1$ and $l = 2$.

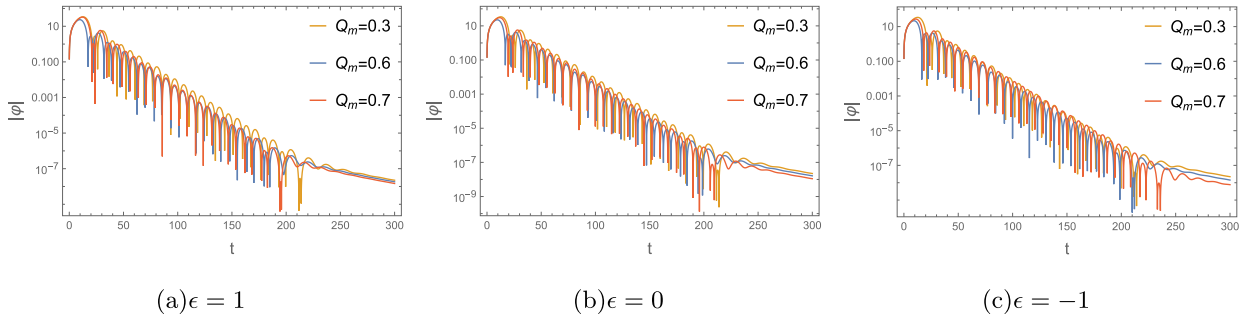


Fig. 6. (color online) Time evolution for the perturbing gravitational field with $M = 1$ and $l = 2$.

(see Figs. 6(a) and 6(b)), a large Q_m makes the ringing stage of the perturbation waveform more intensive and shorter, which corresponds to a larger value of $\text{Re}(\omega)$ and absolute value of $\text{Im}(\omega)$ (see Figs. 4(a)–4(d)). In Fig. 6(c) for $\epsilon = -1$, the perturbation wave with $Q_m = 0.6$ exhibits a shorter ringing stage, indicating a larger absolute value of $\text{Im}(\omega)$. These observations agree well with the results in the frequency domain (Figs. 4(e) and 4(f)). Additionally, as a validation check, we employ the Prony method to fit the fundamental mode, and the results are presented in Table 2.

In Fig. 7, we show the time domain profiles for gravitational perturbation with different values of ϵ . For $Q_m = 0.3$, the real and imaginary parts of these QNM frequencies remain almost unchanged as the parameter ϵ varies. This is consistent with the variation in the QNM plots studied in Fig. 5 and potential behavior in Fig. 3(a). For $Q_m = 0.6$ and 0.7 , a large ϵ makes the ringing stage of the perturbation waveform sparser and shorter, which corresponds to a smaller $\text{Re}(\omega)$ and larger absolute value of $\text{Im}(\omega)$ (see Fig. 5). Moreover, the QNM frequencies with different ϵ are obtained using the Prony method; see Table 3.

V. GREYBODY FACTORS

In this section, we outline the application of the WKB method to the analysis of the greybody factor, a quantity that provides valuable insight into the transmission properties of the effective potential governing wave propaga-

tion in the given spacetime background. We begin by considering the wave equation under boundary conditions that allow for incoming waves originating from spatial infinity. Owing to the symmetry inherent in the scattering process, this configuration is mathematically equivalent to examining the scattering of a wave emanating from the black hole horizon. The appropriate boundary conditions for this scattering scenario are given by

$$\psi = T(\omega)e^{-i\omega r_*}, \quad r_* \rightarrow -\infty, \quad (44)$$

$$\psi = e^{-i\omega r_*} + R(\omega)e^{i\omega r_*}, \quad r_* \rightarrow +\infty, \quad (45)$$

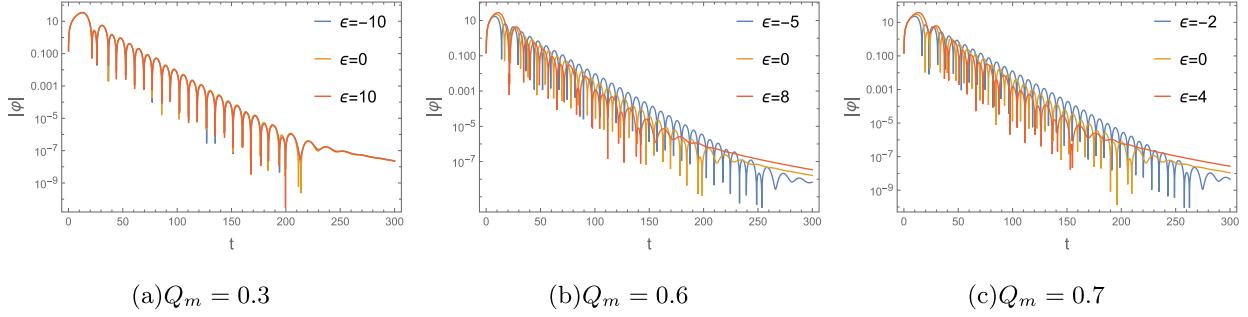
where T is the transmission coefficient, and R is the reflection coefficient.

The square of the amplitude of the wave function at a particular point of spacetime determines the probability of finding it in the given point. The wave incoming toward a regular black hole is partially transmitted and partially reflected by the potential barrier. The greybody factor is defined as the probability of an outgoing wave reaching infinity or an incoming wave being absorbed by the black hole. Therefore, $|T(\omega)|^2$ is called the greybody factor, and $R(\omega)$ and $T(\omega)$ should satisfy the following relation:

$$|R(\omega)|^2 + |T(\omega)|^2 = 1. \quad (46)$$

Table 2. Fundamental QNM frequencies for gravitational field perturbation with $M = 1$ and $l = 2$.

Q_m	$\epsilon = 1$	$\epsilon = 0$	$\epsilon = -1$
0.3	$0.381192 - 0.090218i$	$0.381057 - 0.089666i$	$0.382067 - 0.090253i$
0.6	$0.405460 - 0.092283i$	$0.407387 - 0.090434i$	$0.409135 - 0.089114i$
0.7	$0.417572 - 0.096208i$	$0.421273 - 0.092543i$	$0.425926 - 0.087750i$

**Fig. 7.** (color online) Time evolution for the perturbing gravitational field with $M = 1$ and $l = 2$.**Table 3.** Fundamental QNM frequencies for gravitational field perturbation with $M = 1$ and $l = 2$.

Q_m	ϵ	WKB method	Prony method
0.3	-10	$0.382084 - 0.0887889i$	$0.381169 - 0.0889776i$
	0	$0.381338 - 0.0894572i$	$0.381057 - 0.0896657i$
	10	$0.380574 - 0.0902011i$	$0.380790 - 0.0895344i$
0.6	-5	$0.423692 - 0.0835524i$	$0.423566 - 0.0835848i$
	0	$0.407187 - 0.0912665i$	$0.407387 - 0.0904337i$
	8	$0.397065 - 0.102459i$	$0.396758 - 0.1022570i$
0.7	-2	$0.434931 - 0.0853049i$	$0.434850 - 0.0856028i$
	0	$0.421409 - 0.0922722i$	$0.421495 - 0.0923237i$
	4	$0.409369 - 0.105113i$	$0.409109 - 0.1047280i$

Using the sixth-order WKB method, the reflection and transmission coefficients can be obtained

$$|R(\omega)|^2 = \frac{1}{1 + e^{-2\pi i K(\omega)}},$$

$$|T(\omega)|^2 = \frac{1}{1 + e^{2\pi i K(\omega)}} = 1 - |R(\omega)|^2, \quad (47)$$

where K is a parameter that can be obtained by the WKB formula

$$K = \frac{i(\omega^2 - V(r_0))}{\sqrt{-2V''(r_0)}} + \sum_{i=2}^6 \Lambda_i. \quad (48)$$

For more detailed information, refer to reviews such as [37–40] and the references therein.

In Figs. 8–10, we show the behaviors of the greybody factors for gravitational perturbation under varying magnetic charge Q_m , multipole number l , and parameter

ϵ . The greybody factors exhibit a noticeable decrease as Q_m increases (see Fig. 8). This suggests that, when the magnetic charge Q_m is reduced, the black hole becomes more effective at capturing and interacting with incoming matter or radiation. We also investigate how a change in the multipole number l affects the corresponding behavior of the greybody factors (see Fig. 9). The greybody factors also gradually decrease as l increases, which reveals that the greybody factors are larger for smaller values of l . With regard to the changes in ϵ , the greybody factors gradually increase as ϵ increases (see Fig. 10). This indicates that black holes become less interactive with the surrounding radiation and allow more of the perturbed field to escape. These results are consistent with the effective potentials in Figs. 1–3.

VI. CONCLUSION AND DISCUSSION

In this study, we examined the axial field perturbations of magnetically charged black holes within the

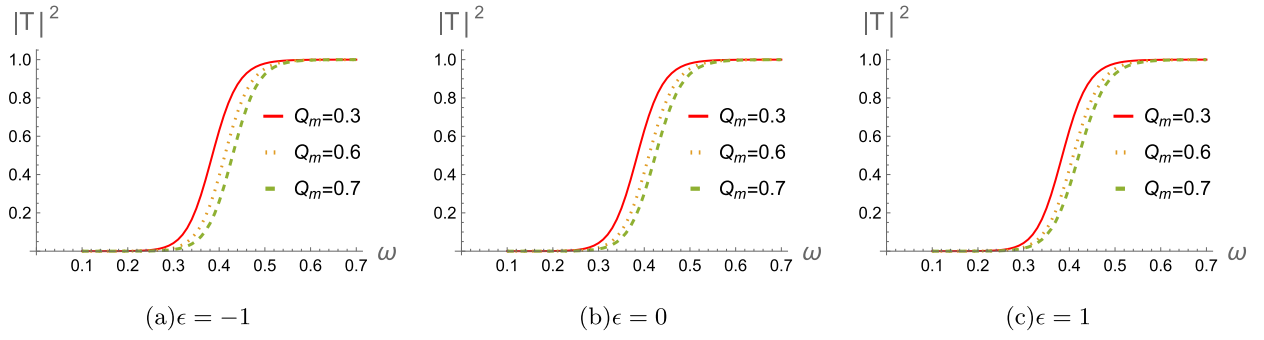


Fig. 8. (color online) Greybody factors for the gravitational field with $M = 1$ and $l = 2$.

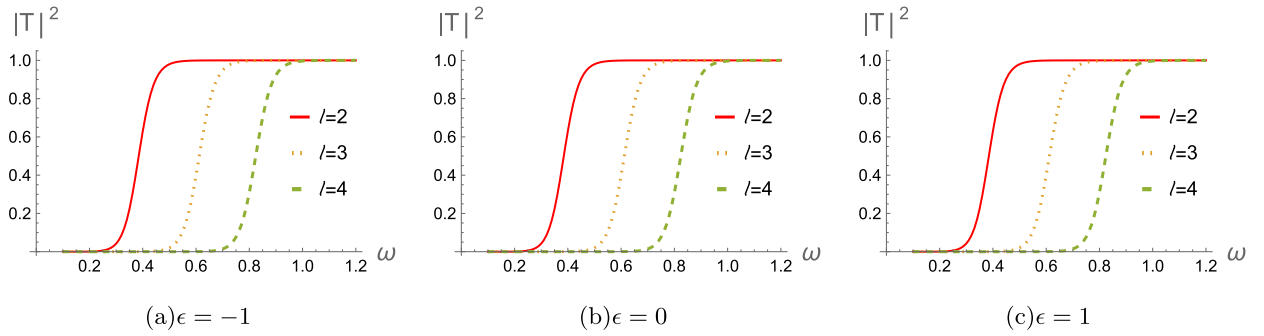


Fig. 9. (color online) Greybody factors for the gravitational field with $M = 1$ and $l = 2$.

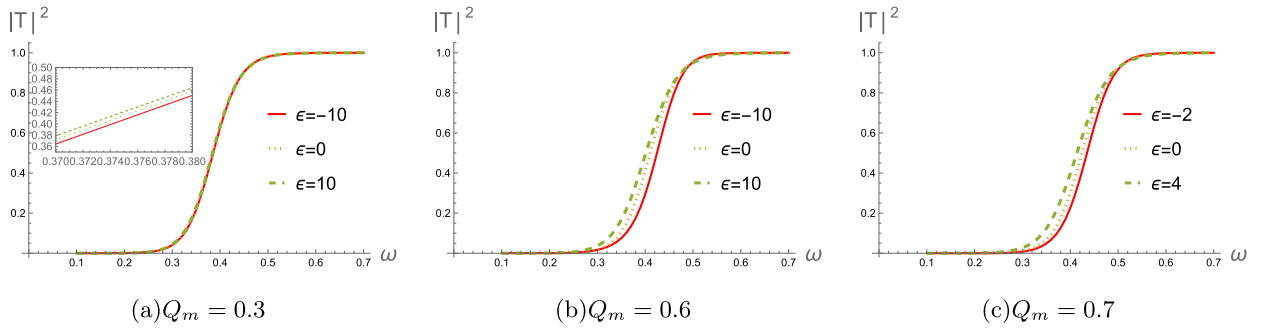


Fig. 10. (color online) Greybody factors for the gravitational field with $M = 1$ and $l = 2$.

framework of the string-inspired Euler–Heisenberg theory. Under the consideration of axial gravitational perturbations decoupled from axial electromagnetic perturbations, we systematically analyzed the effects of the magnetic charge Q_m , parameter ϵ , and angular quantum number l on the corresponding effective potentials associated with the axial gravitational perturbations.

Then, we adopted the AIM and Padé-averaged sixth order WKB methods to compute QNMs in each scenario. We considered how the QNMs change with the multipole moment l , magnetic charge Q_m , and parameter ϵ . For $\epsilon = 0, 1$, the real part of the QNM frequencies increases as Q_m increases, and the damping rate or decay rate of gravitational waves increases significantly with an increase in Q_m . With regard to $\epsilon = -1$, the real part of the QNM frequencies also increases as Q_m increases. However, the damping rate increases with the increase in Q_m and then

reaches a maximum decay rate at $Q_m = 0.6$. As Q_m surpasses this point, the damping rate slowly begins to decrease. In contrast, these QNM frequencies decrease for $Q_m = 0.6$ and 0.7 while remaining almost unchanged for $Q_m = 0.3$ as the parameter ϵ increases. These observations in frequency domains agree well with the results we obtained in the time domains.

Using the sixth-order WKB approximation, we also computed the greybody factor associated with the gravitational field perturbations. The results indicate that the magnetic charge Q_m has a suppressing effect on the tunneling probability; specifically, a smaller fraction of the perturbed field can penetrate the effective potential barrier as Q_m increases. In contrast, varying the parameter ϵ results in an opposite trend. These observations are consistent with the qualitative behavior of the corresponding effective potential profiles.

Note that Ref. [51] recovered the axial metric perturbation decoupled from axial electromagnetic perturbation for magnetic black holes. Some papers [51, 58, 59] further discussed certain combinations: axial gravitational perturbation coupled with polar electromagnetic perturbation or polar gravitational perturbation coupled with axial electromagnetic perturbation for magnetic regular black holes. We will consider these combinations for magnetically charged black holes in future studies.

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APPENDIX A: AXIAL ELECTROMAGNETIC PERTURBATION

In this appendix, we show the perturbed equation for axial electromagnetic perturbation. Substituting the perturbed metric and vector potential (Eqs. (12), (13), and (14)) into the Maxwell equation (5), we obtain

$$E_t^{(A)} = h_0(r) \left[4Q_m^2 e^{2\phi(r)} (\alpha + 2\beta + 3\beta \cosh(2\phi(r))) + r^4 \right], \quad (A1)$$

$$E_r^{(A)} = h_1(r) \left[4Q_m^2 e^{2\phi(r)} (\alpha + 2\beta + 3\beta \cosh(2\phi(r))) + r^4 \right], \quad (A2)$$

$$\begin{aligned} E_\theta^{(A)} = & 4Q_m^2 e^{2\phi(r)} \left[\alpha + 2\beta + 3\beta \cosh(2\phi(r)) \right] \left[rB(r)h_1(r)A'(r) + A(r)(rh_1(r)B'(r) + 2B(r)(rh_1'(r) - 6h_1(r))) + 2ir\omega h_0(r) \right] \\ & + r^5 B(r)A'(r)h_1(r) + 24\beta Q_m^2 rA(r)B(r)e^{4\phi(r)}\phi'(r)h_1(r) + 2ir^5\omega h_0(r) \\ & + rA(r) \left[r^4 B'(r)h_1(r) + 2B(r)(r^4 h_1'(r) - 2h_1(r)((6\beta Q_m^2 + r^4)\phi'(r) + r^3)) \right], \end{aligned} \quad (A3)$$

$$\begin{aligned} E_\varphi^{(A)} = & \left[\omega^2 - \frac{3l(l+1)A(r)}{r^2} + \frac{2(l+1)lr^2 A(r)}{4Q_m^2 e^{2\phi(r)}(\alpha + 2\beta + 3\beta \cosh(2\phi(r))) + r^4} \right] u_3(r) \\ & + \left[\frac{4A(r)B(r)(\phi'(r)(-6\beta Q_m^2 - 2Q_m^2(\alpha + 2\beta)e^{2\phi(r)} - r^4) + r^3)}{4Q_m^2 e^{2\phi(r)}(\alpha + 2\beta + 3\beta \cosh(2\phi(r))) + r^4} \right. \\ & \left. + 2A(r)B(r)\phi'(r) - \frac{4A(r)B(r)}{r} + \frac{1}{2}B(r)A'(r) + \frac{1}{2}A(r)B'(r) \right] u_3'(r) + A(r)B(r)u_3''(r). \end{aligned} \quad (A4)$$

From these equations, it is evident that the gravitational perturbation functions $h_0(r)$ and $h_1(r)$ must vanish and there only exists a single perturbation equation for the perturbed electromagnetic field.

To modify this master equation (A4) into the standard Schrödinger form, we further define the function $u_3(r)$ with

$$u_3(r) = C_1(r) * \Psi_A(r). \quad (A5)$$

We assume the function $C_1(r)$ taking the following form:

$$C_1(r) = \frac{r^2 e^{\phi(r)}}{\sqrt{2Q^2(3\beta + 2(\alpha + 2\beta)e^{2\phi(r)} + 3\beta e^{4\phi(r)}) + r^4}}, \quad (A6)$$

and then obtain the final perturbed equation

$$\frac{d^2 \Psi_A(r_*)}{dr_*^2} + [\omega^2 - V_A(r)] \Psi_A(r_*) = 0, \quad (A7)$$

where r_* represents the tortoise coordinate with $dr_* = \frac{1}{\sqrt{AB}} dr$, and $V_A(r)$ denotes the effective potential for axial electromagnetic perturbation. Owing to the cumbersome expression of $V_A(r)$, its specific form is not provided here.

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