Preliminary analyses of the dynamics and thermodynamics of rotating regular black holes*

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Abstract: We investigate the dynamic and thermodynamic laws governing rotating regular black holes. By analyzing dynamic properties, *i.e.*, the interaction between scalar particles and rotating regular black holes, we establish the criteria that determine whether such black holes satisfy the laws of thermodynamics. In addition, we provide the general form of conserved quantities related to rotating regular black holes, including the relevant flows associated with neutral scalar particles. Meanwhile, we reexamine the relationship between the third law of thermodynamics and weak cosmic censorship conjecture for rotating regular black holes. Based on the abovementioned criteria, we discuss the laws of thermodynamics for three models of rotating regular black holes: Rotating Hayward black holes, Kerr black-bounce solutions, and loop quantum gravity black holes. Our findings indicate that none of the three models satisfies the first law of thermodynamics. In particular, the first and third models fail to comply with the three laws of thermodynamics, whereas the second model satisfies only the second and third laws of thermodynamics. Finally, we attempt to rescue the laws of thermodynamics by modifying entropy or extending the phase space. However, the two scenarios cannot ensure the three laws of thermodynamics in the three models, which reveals an unusual property of rotating regular black holes.

Keywords: rotating regular black holes, thermodynamics laws of black holes, weak cosmic censorship conjecture

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I. INTRODUCTION

As the most successful gravitational theory, general relativity has been confirmed by various recent astronomical observations [1−3]. Currently, the primary challenge that general relativity confronts remains the issue of spacetime singularity. The singularity theorem proposed by Hawking and Penrose [4−6] indicates that singularities exist inevitably in the spacetime under certain conditions. The presence of singularities disrupts the coherence and self-consistency of spacetime, but singularities are always hidden by event horizons and remain unobservable to external observers. This is known as the weak cosmic censorship conjecture [7]. As far back as the previous century, gedanken experiments [8] were employed to assess the rationality of the weak cosmic censorship conjecture. The primary objective of gedanken experiments is to examine the possibility of the destruction of event horizons by particle injection [9−12], revealing that the event horizons of Kerr and Kerr-Newman black holes with singularities are not destroyed [9, 13−15].

The weak cosmic censorship conjecture implies that an observational boundary of black holes is confined on the black hole's event horizons. Therefore, the mechanical properties of black holes are commonly described through various quantities that assume values on an event horizon, including but not limited to area, surface gravity, and angular velocity. In particular, the mechanical laws governing black holes are derived from interrelationships among these quantities [16]. Moreover, the Hawking temperature establishes the correspondence between mechanical and thermodynamic quantities of black holes, resulting in the construction of thermodynamics laws for black holes. As a particle incidence to a black hole can induce changes in mechanical quantities, such as area, mass, and angular momentum, the abovementioned gedanken experiments can be used to verify the thermodynamic laws of black holes [14, 15]. As thermodynamic quantities, such as entropy and temperature, are defined solely on horizons, black hole thermodynamics cannot be estab-

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lished if horizons are absent. In other words, the weak cosmic censorship conjecture ensures the establishment of thermodynamic laws for black holes. Conversely, the thermodynamic laws substantiate the rationality of the weak cosmic censorship conjecture. Therefore, the cosmic censorship conjecture and thermodynamic laws of black holes are complementary.

Recently, diverse techniques have been employed to construct static and spherically symmetric regular black holes that have no essential singularities [17−32], and they can be summarized into three categories:

• To solve the Einstein field equations under a special symmetry or matter source [19, 20, 24–26]. Among the black holes constructed in this category, the Hayward black hole is a typical example in which the vacuum energy density distribution is considered in Einstein's gravity [31];

• To modify a metric directly such that the corresponding spacetime has no singularity, and then to deduce a possible matter source inversely [17, 18, 21, 27–29, 33]. The key step is to determine a novel metric function that removes singularities in spacetime, which was previously achieved by experience and formulated by specific functions. Recently, a systematic modification of metrics has been proposed to construct a regular black hole [21, 28]. Applying such a modification to the Schwarzschild metric, we can derive the so-called black-bounce metric, and verify that this metric is the solution of Einstein's gravity coupled with the phantom scalar field and the electromagnetic field, where the matter source is deduced inversely from the metric [34];

• To solve a metric for regular black holes in the framework of modified theories of gravity [22, 23, 30–32, 35] . Among them, the loop quantum gravity [35] is a nonperturbative theory and exceeds general relativity, and it can resolve singularities in cosmological and black hole spacetimes. Therefore, regular black holes can be constructed naturally under this theory [30, 36].

As an astronomical black hole naturally rotate, the Newman-Janis algorithm (NJA) has been applied and improved to extend the investigation of regular black holes from a static and spherically symmetric case to a rotating and axially symmetric case [37−43]. Owing to the absence of singularities, the primary significance of the weak cosmic censorship conjecture does not lie in avoiding naked singularities but rather in upholding thermodynamic laws. However, a recent paper on rotating loop quantum gravity black holes has claimed that the incidence of scalar particles into the near-extreme configuration of rotating loop quantum gravity black holes can disrupt event horizons [44]. Although it is a special case, this result implies that the thermodynamics of regular black holes may differ from that of singular black holes [44]. In the early literature, only metric singularities were avoided in the construction of regular black holes, whereas the compatibility with thermodynamic laws was neglected. Therefore, whether regular black holes adhere to the thermodynamic laws that are valid to singular black holes remains unsolved .

In this paper, we investigate the behavior of rotating regular black holes coupled with scalar particles and propose criteria for establishing the relevant thermodynamic laws, where these criteria match the self-consistency in constructing rotating regular black holes. Moreover, we examine the correlation between the third law of thermodynamics and weak cosmic censorship conjecture, yielding different conclusions from those for Kerr and Kerr-Newman black holes. Finally, we analyze whether rotating regular black holes satisfy the thermodynamic laws deduced from singular black holes through illustrative examples and attempt to rescue these laws by modifying entropy or extending phase spaces.

The remainder of this paper is structured as follows. In Sec. II, we briefly introduce to rotating regular black holes constructed using the revised NJA and present the mass and angular momentum in their general forms. In Sec. III, we discuss the behavior of a scalar field near a rotating regular black hole and derive the scalar field flux and the Hawking temperature of rotating regular black holes. In Sec. IV, we establish the criteria to determine the laws of thermodynamics for rotating regular black holes, and we discuss the relationship between the third law of thermodynamics and weak cosmic censorship conjecture. In Sec. V, we apply these criteria to three models of rotating regular black holes and explore the possibility of their fulfillment by modifying entropy or extending phase spaces. Finally, we present our summary and outlook in Sec. VI.

II. ROTATING REGULAR BLACK HOLES

A. General form of metrics

The construction of rotating regular black holes [38, 45−47] involves the use of the NJA [42, 43, 48−50], an algebraic method for transforming a static and spherically symmetric black hole solution into a rotating and axially symmetric one. Now, we briefly introduce this method. Initially, we consider a general static and spherically symmetric metric,

$$\mathrm{d}s^2_{\text{static}} = -F(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)} + H(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right). \tag{1}$$

Because the spacetime under consideration is asymptotically flat at infinity, the functions $F(r)$ and $H(r)$ should satisfy the following conditions:

$$\lim_{r\to\infty} F(r) = 1, \qquad \lim_{r\to\infty} H(r) = r^2. \tag{2}$$

Therefore, we expand $F(r)$ and $H(r)$ as series of $1/r$ near infinity, $r\to\infty$, as follows:

$$\lim_{r\to\infty} F(r) = \sum_{n=0}^{\infty} a_n r^{-n}, \quad a_n = \frac{1}{n!}\frac{\partial^n}{\partial(1/r)^n}F(r)\Big|_{r\to\infty}, \quad a_0 = 1, \tag{3}$$

$$\lim_{r\to\infty} H(r) = \sum_{n=0}^{\infty} b_n r^{2-n}, \quad b_n = \frac{1}{n!}\frac{\partial^n}{\partial(1/r)^n}\frac{H}{r^2}\Big|_{r\to\infty}, \quad b_0 = 1. \tag{4}$$

In the advanced null coordinates, (u,r,θ,ϕ), defined by

$$\mathrm{d}u = \mathrm{d}t - \frac{\mathrm{d}r}{F(r)}, \tag{5}$$

the contravariant form of the metric, can be expressed in terms of a null tetrad as

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu m^{*\nu} + m^\nu m^{*\mu}, \tag{6}$$

where

$$l^\mu = \delta^\mu_r, \tag{7a}$$

$$n^\mu = \delta^\mu_u - \frac{F}{2}\delta^\mu_r, \tag{7b}$$

$$m^\mu = \frac{1}{\sqrt{2H(r)}}\left(\delta^\mu_\theta + \frac{\mathrm{i}}{\sin\theta}\delta^\mu_\phi\right), \tag{7c}$$

$$l_\mu l^\mu = m_\mu m^\mu = n^\nu n_\nu = l_\mu m^\mu = n_\mu m^\mu = 0, \tag{7d}$$

$$l_\mu n^\mu = -m_\mu m^{*\mu} = 1, \tag{7e}$$

and "$*$" denotes complex conjugate. The rotation is introduced via the complex transformation

$$r \to r + \mathrm{i}a\cos\theta, \qquad u \to u - \mathrm{i}a\cos\theta, \tag{8}$$

where a is rotation parameter, and δ^μ_ν is required to transform as a vector under the above complex transformation,

$$\delta^\mu_r \to \delta^\mu_r, \qquad \delta^\mu_u \to \delta^\mu_u,$$
$$\delta^\mu_\theta \to \delta^\mu_\theta + \mathrm{i}a\sin\theta(\delta^\mu_u - \delta^\mu_r), \qquad \delta^\mu_\phi \to \delta^\mu_\phi. \tag{9}$$

After the above fulfillment, $\{F,H\}$ are generalized to $\{B,\Psi\}$ with rotation:

$$\{F(r),H(r)\} \to \{B(r,\theta,a),\Psi(r,\theta,a)\}, \tag{10}$$

where $\{B,\Psi\}$ are real functions to be determined and should recover their static counterparts in the limit of $a\to 0$, i.e.,

$$\lim_{a\to0} B(r,\theta,a) = F(r), \qquad \lim_{a\to0}\Psi(r,\theta,a) = r^2. \tag{11}$$

In particular, the line element without rotation (see Eq. (1)), is now transformed to the one with rotation when Eqs. (9) and (10) are considered:

$$\mathrm{d}s^2 = -B\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r - 2a\sin^2\theta(1-B)\,\mathrm{d}u\mathrm{d}\phi + 2a\sin^2\theta\mathrm{d}r\mathrm{d}\phi + \Psi\mathrm{d}\theta^2 + \sin^2\theta\left[\Psi + a^2\sin^2\theta(2-B)\right]\mathrm{d}\phi^2. \tag{12}$$

Next, we rewrite the above line element with the Boyer-Lindquist coordinates and let the metric have only one non-vanishing off-diagonal term, $g_{t\varphi}$. To reach the objective, we require the following coordinate transformation:

$$\mathrm{d}u = \mathrm{d}t + \lambda(r)\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\varphi + \chi(r)\mathrm{d}r, \tag{13}$$

where $\{\lambda(r),\chi(r)\}$ depend only on r to ensure integrability. If the transformation Eq. (10) is given a priori, $\{\lambda(r),\chi(r)\}$ may not exist. Considering these constraints, we determine the formulations of $\{B(r,\theta,a),\Psi(r,\theta),\lambda(r),\chi(r)\}$:

$$B(r,\theta) = \frac{FH + a^2\cos^2\theta}{\Psi}, \tag{14a}$$

$$\Psi(r,\theta) = H + a^2\cos^2\theta, \tag{14b}$$

$$\lambda(r) = -\frac{H + a^2}{Fr^2 + a^2}, \tag{14c}$$

$$\chi(r) = -\frac{a}{FH + a^2}. \tag{14d}$$

Thus, we obtain the line element for rotating regular black holes with the Kerr-like form:

$$\mathrm{d}s^2 = \frac{\Psi}{\Sigma}\Big[-\left(1-\frac{fH}{\Sigma}\right)\mathrm{d}t^2+\frac{\Sigma}{\Delta}\mathrm{d}r^2 - \frac{2fHa}{\Sigma}\sin^2\theta\mathrm{d}t\mathrm{d}\phi+\Sigma\mathrm{d}\theta^2+\frac{A}{\Sigma}\sin^2\theta\mathrm{d}\phi^2\Big], \tag{15}$$

where

$$\Sigma = H + a^2\cos^2\theta, \tag{16}$$

$$f = 1 - F, \tag{17}$$

$$\Delta = FH + a^2, \tag{18}$$

$$A = (H+a^2)^2 - a^2\Delta\sin^2\theta. \tag{19}$$

Although the choice of Ψ has certain degrees of freedom, $\Psi=\Sigma$ is typically selected for a rotating regular black hole [29, 38, 41, 43, 45]. In this scenario, we rewrite Eq. (15) in another form:

$$\mathrm{d}s^2 = -\frac{\Delta\Sigma}{A}\mathrm{d}t^2+\frac{A\sin^2\theta}{\Sigma}(\mathrm{d}\phi-\Omega\mathrm{d}t)^2+\frac{\Sigma}{\Delta}\mathrm{d}r^2+\Sigma\mathrm{d}\theta^2, \tag{20}$$

where

$$\Omega = \frac{fHa}{A}. \tag{21}$$

In this spacetime, the locations of horizons are determined by $\Delta(r_{\rm H})=0$, and the angular velocity at the outer horizon $r_{\rm H}^+$ assumes the form

$$\Omega_{\rm H} = \Omega\big|_{r=r_{\rm H}^+} = \frac{a}{fH}\bigg|_{r=r_{\rm H}^+}. \tag{22}$$

In the next subsection, we compute the mass and angular momentum of rotating regular black holes.

B. Komar's conserved quantity

Here, we use Komar's conserved quantity for the calculation of mass and angular momentum. For asymptotically flat spacetime, the Komar conserved quantity equals the ADM one, thus enabling us to obtain the ADM mass and angular momentum. Komar's conserved quantity assumes the form [51, 52]

$$16\pi I = \int_{\partial V} {}^*\mathrm{d}\xi, \tag{23}$$

where ${}^*\mathrm{d}\xi$ denotes the dual to a two-form $\mathrm{d}\xi$, ξ is the Killing one-form, and the integration is extended over a spacelike surface ∂V of the background spacetime depicted by Eq. (20). The Killing vector field related to mass is time-like, $\xi_{(t)}$, whereas that related to angular momentum is space-like, $\xi_{(\phi)}$. We shall perform the calculations for mass and angular momentum. We note that the metric contains coupling terms of $\mathrm{d}t$ and $\mathrm{d}\phi$ (see Eq. (20)), which makes computing derivatives in the (t,r,θ,ϕ) coordinates difficult. Therefore, in the following subsections, we introduce an orthonormal frame of one-forms for the derivations:

$$X_{(0)} = -\left(\frac{\Delta\Sigma}{A}\right)^{1/2}\mathrm{d}t, \tag{24a}$$

$$X_{(1)} = \left(\frac{\Sigma}{\Delta}\right)^{1/2}\mathrm{d}r, \tag{24b}$$

$$X_{(2)} = \Sigma^{1/2}\mathrm{d}\theta, \tag{24c}$$

$$X_{(3)} = \left(\frac{A\sin^2\theta}{\Sigma}\right)^{1/2}(\mathrm{d}\phi-\Omega\mathrm{d}t). \tag{24d}$$

1. *Mass*

The mass can be precisely defined by the Komar conserved quantity $I_{(t)}$ of the time-like Killing vector field $\xi_{(t)}$. Specifically, it is determined as follows:

$$M_{\rm K} = \lim_{r\to\infty}2I_{(t)} = \lim_{r\to\infty}\frac{1}{8\pi}\int_{\partial V}{}^*\mathrm{d}\xi_{(t)}, \tag{25}$$

where the time-like Killing vector field is obtained with the aid of the orthonormal frame Eq. (24):

$$\xi_{(t)} = g_{\mu t}\mathrm{d}x^\mu = \left(\frac{\Delta\Sigma}{A}\right)^{1/2}X_{(0)} - \left(\frac{A\sin^2\theta}{\Sigma}\right)^{1/2}\Omega X_{(3)}. \tag{26}$$

Differentiating $\xi_{(t)}$ and taking its dual, we obtain

$${}^*d\xi_{(t)} = \alpha X_{(2)}\wedge X_{(3)} - \beta X_{(1)}\wedge X_{(3)} + \gamma X_{(2)}\wedge X_{(0)} - \delta X_{(1)}\wedge X_{(0)}, \tag{27}$$

where

$$\alpha = \frac{A^{1/2}}{\Sigma}\frac{\partial}{\partial r}\left(\frac{\Delta\Sigma}{A}\right) - \frac{A^{3/2}\Omega\sin^2\theta}{\Sigma^2}\frac{\partial}{\partial r}\Omega, \tag{28a}$$

$$\beta=\frac{1}{\Sigma}\left(\frac{A}{\Delta}\right)^{1/2}\frac{\partial}{\partial\theta}\left(\frac{\Delta\Sigma}{A}\right)-\frac{A\Omega\sin^2\theta}{\Sigma^2}\left(\frac{A}{\Delta}\right)^{1/2}\frac{\partial}{\partial\theta}\Omega, \tag{28b}$$

$$\gamma=-\left(\frac{\Delta\sin^2\theta}{A}\right)^{1/2}\frac{\partial}{\partial r}\left(\frac{A\Omega}{\Sigma}\right), \tag{28c}$$

$$\delta=-\left(\frac{1}{A\sin^2\theta}\right)^{1/2}\frac{\partial}{\partial\theta}\left(\frac{A\Omega\sin^2\theta}{\Sigma}\right). \tag{28d}$$

To perform integration in the original coordinates of (t,r,θ,ϕ), we must reconvert ${}^*\mathrm{d}\xi_{(t)}$ to the formulation as follows:

$${}^*d\xi_{(t)}=\tilde{\alpha}\mathrm{d}r\wedge\mathrm{d}t+\tilde{\delta}\mathrm{d}\theta\wedge\mathrm{d}\phi+\tilde{\gamma}\mathrm{d}r\wedge\mathrm{d}\phi+\tilde{\beta}\mathrm{d}\theta\wedge\mathrm{d}t, \tag{29}$$

where

$$\tilde{\alpha}=\beta\Omega\left(\frac{A\sin^2\theta}{\Delta}\right)^{1/2}+\delta\Sigma\left(\frac{1}{A}\right)^{1/2}, \tag{30a}$$

$$\tilde{\delta}=\alpha\left(A\sin^2\theta\right)^{1/2}, \tag{30b}$$

$$\tilde{\gamma}=-\beta\left(\frac{A\sin^2\theta}{\Delta}\right)^{1/2}, \tag{30c}$$

$$\tilde{\beta}=-\gamma\left(\frac{\Delta}{A}\right)^{1/2}\Sigma-\alpha(A\sin^2\theta)^{1/2}\Omega. \tag{30d}$$

Here, t and r are constants since we are calculating the mass in a two-dimensional sphere over simultaneous events. Therefore, the Komar conserved quantity associated with $\xi_{(t)}$ takes the form

$$I_{(t)}=\frac{1}{16\pi}\int\int\alpha\left(A\sin^2\theta\right)^{1/2}\mathrm{d}\theta\mathrm{d}\phi. \tag{31}$$

After a tedious integral calculation, we obtain

$$I_{(t)}=\frac{1}{8}\left(H+a^2\right)\left[\frac{2HF'+(F-1)H'}{a\sqrt{H}}\tan^{-1}\left(\frac{a}{\sqrt{H}}\right)-\frac{(F-1)H'}{H+a^2}\right], \tag{32}$$

where a prime represents a derivative with respect to r. According to the definition of mass (Eq. (25)) and the asymptotic behaviors (Eqs. (3) and (4)), we finally derive the mass:

$$\begin{aligned}M_{\mathrm{K}}&=\lim_{r\to\infty}\frac{1}{4}(1-F)H'\\&=-\lim_{r\to\infty}\frac{1}{4}\sum_{n=1,m=0}^{\infty}a_nb_m(2-m)r^{1-m-n}=-\frac{a_1}{2}.\end{aligned} \tag{33}$$

The Komar mass corresponds to the mass parameter M selected in the metrics for the majority of rotating regular black holes [17−20, 45]. For convenience, in subsequent discussions, we shall omit the subscript "K" and express it simply as M.

2. *Angular momentum*

Similar to the definition of mass, the angular momentum is defined by

$$J_{\mathrm{K}}=-\lim_{r\to\infty}I_{(\phi)}=-\lim_{r\to\infty}\frac{1}{16\pi}\int_{\partial V}{}^*\mathrm{d}\xi_{(\phi)}, \tag{34}$$

where the space-like Killing vector field $\xi_{(\phi)}$ assumes the form in the orthonormal frame of Eq. (24):

$$\xi_{(\phi)}=g_{\mu\phi}dx^\mu=\left(\frac{A\sin^2\theta}{\Sigma}\right)^{1/2}X_{(3)}. \tag{35}$$

We then express the dual of $\xi_{(\phi)}$ as

$${}^*\mathrm{d}\xi_{(\phi)}=\hat{\alpha}X_{(2)}\wedge X_{(3)}-\hat{\beta}X_{(1)}\wedge X_{(3)}+\hat{\gamma}X_{(2)}\wedge X_{(0)}-\hat{\delta}X_{(1)}\wedge X_{(0)}, \tag{36}$$

where

$$\hat{\alpha}=\frac{A^{3/2}\sin^2\theta}{\Sigma^2}\frac{\partial}{\partial r}\Omega, \tag{37a}$$

$$\hat{\beta}=\frac{A\sin^2\theta}{\Sigma^2}\left(\frac{A}{\Delta}\right)^{1/2}\frac{\partial}{\partial\theta}\Omega, \tag{37b}$$

$$\hat{\gamma}=\left(\frac{\Delta}{A\sin^2\theta}\right)^{1/2}\frac{\partial}{\partial r}\left(\frac{A\sin^2\theta}{\Sigma}\right), \tag{37c}$$

$$\hat{\delta}=\left(\frac{1}{A\sin^2\theta}\right)^{1/2}\frac{\partial}{\partial\theta}\left(\frac{A\sin^2\theta}{\Sigma}\right). \tag{37d}$$

Here, t and r are also constants because we are calculating the angular momentum in a two-dimensional sphere over simultaneous events. Therefore, the Komar conserved quantity associated with $\xi_{(\phi)}$ is

$$I_\phi=\frac{1}{16\pi}\int_{\partial V}{}^*\mathrm{d}\xi_\phi=\frac{1}{16\pi}\int\int\hat{\alpha}\left(A\sin^2\theta\right)^{1/2}\mathrm{d}\theta\mathrm{d}\phi, \tag{38}$$

and the integration yields

$$I_\phi = \frac{1}{8a^2\sqrt{H(r)}}\left\{-2F'(r)H(r)\left(a^2+H(r)\right)\left[\left(a^2+H(r)\right)\tan^{-1}\left(\frac{a}{\sqrt{H(r)}}\right)-a\sqrt{H(r)}\right]\right.$$
$$\left.-[F(r)-1]H'(r)\left[\left(a^4+2a^2H(r)+[H(r)]^2\right)\tan^{-1}\left(\frac{a}{\sqrt{H(r)}}\right)-3a^3\sqrt{H(r)}-a[H(r)]^{3/2}\right]\right\}. \tag{39}$$

According to the asymptotic behaviors (Eqs. (3) and (4)), we derive the angular momentum:

$$J=-\lim_{r\to\infty}I_\phi=-\lim_{r\to\infty}\frac{a}{8}\left[(F-1)H'-2F'(H+a^2)\right]$$
$$=-\frac{aa_1}{2}=Ma, \tag{40}$$

where the subscript "K" is omitted as explained for the case of mass.

We derive two physical quantities, mass and angular momentum, associated with a general rotating regular black hole. Further, we describe into the characteristic quantities involved in the process of the incidence of a neutral scalar particle in the background spacetime of the rotating regular black holes described by Eq. (20) and examine the impact of the particle on the aforementioned physical quantities.

III. NEUTRAL MASSIVE SCALAR FIELDS

A. Scalar field equation and flux

The action of a complex scalar field $\Phi(t,r,\theta,\phi)$ in a general spacetime is

$$S_\Phi=-\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\partial_\nu\Phi\partial^\nu\Phi^*+(\mu^2+\Xi R)\Phi\Phi^*\right], \tag{41}$$

where μ denotes the mass of scalar fields, R is the curvature, and Ξ is the non-minimal coupling constant, and $\sqrt{-g}$ assumes the form

$$\sqrt{-g}=\sqrt{-\det g_{\mu\nu}}=\Sigma\sin\theta, \tag{42}$$

where Eq. (20) has been used. By using the principle of least action, we derive the equation of motion as

$$\nabla_\nu\nabla^\nu\Phi=(\mu^2+\Xi R)\Phi. \tag{43}$$

To separate variables in the spacetime of rotating regular black holes, we make the assumption

$$\Phi(t,r,\theta,\phi)=\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}m\phi}\mathscr{S}(\theta)\mathscr{R}(r), \tag{44}$$

and then obtain the equations that govern $\mathscr{S}(\theta)$ and $\mathscr{R}(r)$, respectively:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\mathscr{S}(\theta)\right]$$
$$+\left[a^2(\omega^2-\mu^2-\Xi R)\cos^2\theta-\frac{m^2}{\sin^2\theta}+\lambda\right]\mathscr{S}(\theta)=0, \tag{45}$$

and

$$\Delta\frac{\mathrm{d}}{\mathrm{d}r}\left[\Delta\frac{\mathrm{d}}{\mathrm{d}r}\mathscr{R}(r)\right]+[\omega^2(H+a^2)^2-2afHm\omega+a^2m^2$$
$$-\Delta(\mu^2H+\Xi RH+\lambda+a^2\omega^2)]\mathscr{R}(r)=0, \tag{46}$$

where ω is the frequency of massive scalar fields, m is the azimuthal number with respect to the rotation axis, and λ is the separation parameter that can be fixed approximately as an eigenvalue of Eq. (45).

Introducing a tortoise coordinate in an outer horizon limit,

$$\frac{\mathrm{d}r_*}{\mathrm{d}r}=\frac{H+a^2}{\Delta}, \tag{47}$$

we rewrite the radial equation as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}\mathscr{R}+(\omega-m\Omega_{\mathrm{H}})^2\mathscr{R}=0. \tag{48}$$

Therefore, we obtain the solutions in the close proximity to outer horizons,

$$\mathscr{R}\sim\mathrm{e}^{\pm\mathrm{i}(\omega-m\Omega_{\mathrm{H}})r_*}, \tag{49}$$

and take the negative sign as an ingoing wave.

The alterations in energy and angular momentum induced by complex scalar particles can be derived from the fluxes of energy and angular momentum related to the particles. When a complex scalar particle crosses an outer event horizon of a black hole, it becomes indistinguishable from the black hole. Hence, the alterations in mass and angular momentum related to black holes are closely linked to the scalar field fluxes at an outer horizon of black holes. The fluxes can be derived from the energy-momentum tensor, which is obtained through the Lagrangian Eq. (41) of scalar fields. Specifically, the energy-momentum tensor is

$$\begin{aligned} T^{\mu}_{\nu} &= \sum_i \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^i)}\partial_\nu \Phi^i - \delta^{\mu}_{\nu}\mathscr{L} \\ &= \frac{1}{2}\partial^\mu\Phi\partial_\nu\Phi^* + \frac{1}{2}\partial^\mu\Phi^*\partial_\nu\Phi \\ &\quad - \delta^{\mu}_{\nu}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi^* - \frac{1}{2}(\mu^2+\Xi R)\Phi\Phi^*\right], \end{aligned} \tag{50}$$

and the correlated energy flux of scalar fields at an outer horizon takes the form

$$j_{t(\mathrm{H})} = \frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to r_{\mathrm{H}}^+}\int T^r_t\sqrt{-g}\mathrm{d}\theta\mathrm{d}\phi = \omega(\omega - m\Omega_{\mathrm{H}})(H+a^2), \tag{51}$$

where the normalization condition of $\mathscr{S}(\theta)$,

$$\int \mathscr{S}(\theta)\mathscr{S}^*(\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\phi = 1, \tag{52}$$

is used. Similarly, the angular momentum flux of scalar fields at an outer horizon is

$$j_{\phi(\mathrm{H})} = \frac{\mathrm{d}L}{\mathrm{d}t} = -\lim_{r\to r_{\mathrm{H}}^+}\int T^r_\phi\sqrt{-g}\mathrm{d}\theta\mathrm{d}\phi = m(\omega - m\Omega_{\mathrm{H}})(H+a^2). \tag{53}$$

The energy and angular momentum fluxes are crucial in verifying the second and third laws of black hole thermodynamics. This is because the fluxes represent the flow of the energy and angular momentum into black holes change with time. In addition, an important quantity closely related to black hole thermodynamics is the Hawking temperature.

B. Hawking temperature

We employ the methodology proposed in Ref. [53] to compute the Hawking temperature of rotating regular black holes. The key step lies in reducing a four-dimensional metric to a two-dimensional one near an outer horizon, which is primarily accomplished by the reformulation of action Eq. (41). We rewrite the action using Eq. (42),

$$\begin{aligned} S_\Phi = -\frac{1}{2}\int \mathrm{d}^4x\,\Sigma\sin\theta\Big[&-\frac{(H+a^2)^2-\Delta a^2\sin^2\theta}{\Delta\Sigma}\partial_t\Phi\partial_t\Phi^* - \frac{a(H+a^2-\Delta)}{\Delta\Sigma}\left(\partial_t\Phi\partial_\phi\Phi^* + \partial_\phi\Phi\partial_t\Phi^*\right) \\ &+\frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}\partial_\phi\Phi\partial_\phi\Phi^* + \frac{\Delta}{\Sigma}\partial_r\Phi\partial_r\Phi^* + \frac{1}{\Sigma}\partial_\theta\Phi\partial_\theta\Phi^* + (\mu^2+\Xi R)\Phi\Phi^*\Big], \end{aligned} \tag{54}$$

and then obtain its form in the vicinity of outer horizons by maintaining dominant terms:

$$\begin{aligned} S_\Phi\left[r\to r_{\mathrm{H}}^+\right] = \frac{1}{2}\int \mathrm{d}^4x\,\sin\theta\frac{a^2}{\Omega_{\mathrm{H}}^2\Delta}\Big[&\partial_t\Phi\partial_t\Phi^* \\ &+\Omega_{\mathrm{H}}(\partial_t\Phi\partial_\phi\Phi^* + \partial_t\Phi^*\partial_\phi\Phi) \\ &+\Omega_{\mathrm{H}}^2\partial_\phi\Phi\partial_\phi\Phi^* - \frac{\Delta^2\Omega_{\mathrm{H}}^2}{a^2}\partial_r\Phi\partial_r\Phi^*\Big]. \end{aligned} \tag{55}$$

Furthermore, using the locally non-rotating coordinate,

$$\psi = \phi - \Omega_{\mathrm{H}}t, \qquad \tilde{t} = t, \tag{56}$$

we transform the above form to

$$\begin{aligned} S_\Phi\left[r\to r_{\mathrm{H}}^+\right] = &-\frac{1}{2}\int \mathrm{d}^4x\sin\theta\frac{a}{\Omega_{\mathrm{H}}} \\ &\times\Big[-\frac{1}{f_e(r)}\partial_{\tilde{t}}\Phi\partial_{\tilde{t}}\Phi^* + f_e(r)\partial_r\Phi\partial_r\Phi^*\Big], \end{aligned} \tag{57}$$

where $f_e(r)$ is defined by

$$f_e(r) \equiv \frac{\Omega_{\mathrm{H}}\Delta}{a}. \tag{58}$$

As a result, we reduce the four-dimensional action to a two-dimensional one near an outer horizon, where the effective two-dimensional metric is given by

$$\mathrm{d}s^2 = -f_e(r)\mathrm{d}\tilde{t}^2 + \frac{1}{f_e(r)}\mathrm{d}r^2, \tag{59}$$

and we calculate the Hawking temperature according to Ref. [53]:

$$T = \frac{1}{4\pi}\partial_r f_e\Big|_{r=r_{\mathrm{H}}^+} = \frac{\Omega_{\mathrm{H}}}{4\pi a}\partial_r\Delta\Big|_{r=r_{\mathrm{H}}^+}. \tag{60}$$

We have provided all the necessary quantities, except entropy, for constructing the laws of thermodynamics. Next, we analyze the conditions required for the first, second, and third laws of thermodynamics, respectively.

IV. THERMODYNAMICS OF ROTATING REGULAR BLACK HOLES UNDER INCIDENCE OF NEUTRAL SCALAR FIELDS

With the incidence of a neutral scalar particle, the parameters of black holes undergo a transformation from $(M, J, r_{\rm H})$ to $(M+{\rm d}M, J+{\rm d}J, r_{\rm H}+{\rm d}r_{\rm H})$. In this process, we assume the perpetual existence of black hole event horizons, thereby ensuring that the condition $\Delta=0$ is always satisfied. Specifically, the necessary conditions for the existence of event horizons is

$$\Delta(M, J, r_{\rm H}) = F(M, r_{\rm H})H(M, r_{\rm H}) + \frac{J^2}{M^2} = 0, \tag{61}$$

where we have rewritten $F(r)$ and $H(r)$ as $F(M,r)$ and $H(M,r)$, respectively, because M is a parameter of these functions, as shown in Eq. (33). However, the angular momentum J is not a parameter of F and H because they are defined in a static and spherically symmetric spacetime and independent of any rotation introduced later. After the incidence of the neutral scalar particle, the condition in Eq. (61) changes to

$$\begin{aligned}&\Delta(M+{\rm d}M, J+{\rm d}J, r_{\rm H}+{\rm d}r_{\rm H})\\ &= \frac{\partial\Delta}{\partial M}\Big|_{r=r_{\rm H}}{\rm d}M + \frac{\partial\Delta}{\partial J}\Big|_{r=r_{\rm H}}{\rm d}J + \frac{\partial\Delta}{\partial r}\Big|_{r=r_{\rm H}}{\rm d}r_{\rm H} = 0,\end{aligned} \tag{62}$$

where

$$\frac{\partial\Delta}{\partial M}\Big|_{r=r_{\rm H}} = -\frac{1}{H}\left(a^2\frac{\partial H}{\partial M} - H^2\frac{\partial F}{\partial M} + 2H\frac{J^2}{M^3}\right)\Big|_{r=r_{\rm H}}, \tag{63}$$

$$\frac{\partial\Delta}{\partial J}\Big|_{r=r_{\rm H}} = \frac{2J}{M^2}, \tag{64}$$

$$\frac{\partial\Delta}{\partial r}\Big|_{r=r_{\rm H}} = -\frac{1}{H}\left(a^2\frac{\partial H}{\partial r} - H^2\frac{\partial F}{\partial r}\right)\Big|_{r=r_{\rm H}}. \tag{65}$$

Thus, we establish the correlation among ${\rm d}r_{\rm H}$, ${\rm d}M$, and ${\rm d}J$ from the above four equations,

$${\rm d}r_{\rm H} = -\frac{\partial\Delta}{\partial M}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}\Big|_{r=r_{\rm H}}{\rm d}M - \frac{\partial\Delta}{\partial J}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}\Big|_{r=r_{\rm H}}{\rm d}J, \tag{66}$$

with which we further construct the laws of thermodynamics.

A. First law of thermodynamics

The differential form of the first law of thermodynamics for black holes establishes a relationship among the first-order differentials of physical quantities. As neutral scalar particles induce changes in mass and angular momentum, the first law of thermodynamics should describe the correlation among first-order differentials of entropy, mass, and angular momentum. A significant thermodynamic quantity is the entropy of a black hole. As adopted in the thermodynamics of singular black holes, we define the entropy of a regular black hole by using the Bekenstein-Hawking entropy,

$$S_{\rm BH} = \frac{1}{4}\mathscr{A}_{\rm H}, \tag{67}$$

and compute the area of out horizons using Eq. (20):

$$\mathscr{A}_{\rm H} = \int\sqrt{g_{\theta\theta}g_{\phi\phi}}\,{\rm d}\theta{\rm d}\phi\Big|_{r=r_{\rm H}} = 4\pi(H+a^2)\Big|_{r=r_{\rm H}}. \tag{68}$$

When the black hole parameters change, the associated entropy undergoes a corresponding change:

$${\rm d}S_{\rm BH} = \frac{\partial S_{\rm BH}}{\partial M}{\rm d}M + \frac{\partial S_{\rm BH}}{\partial J}{\rm d}J + \frac{\partial S_{\rm BH}}{\partial r_{\rm H}}{\rm d}r_{\rm H}. \tag{69}$$

After utilizing the relationship in Eq. (66), we derive

$${\rm d}S_{\rm BH} = \hat{A}{\rm d}M + \hat{B}{\rm d}J, \tag{70}$$

where

$$\begin{aligned}\hat{A} = &-\frac{1}{4TH(H+a^2)}\left\{H^2\left(\frac{\partial H}{\partial r_{\rm H}}\frac{\partial F}{\partial M} - \frac{\partial H}{\partial M}\frac{\partial F}{\partial r_{\rm H}}\right)\right.\\ &\left.+\frac{2a^2}{M}\left[H^2\frac{\partial F}{\partial r_{\rm H}} - (H+a^2)\frac{\partial H}{\partial r_{\rm H}}\right]\right\},\end{aligned} \tag{71a}$$

$$\hat{B} = \frac{a}{2TMH(H+a^2)}\left[H^2\frac{\partial F}{\partial r_{\rm H}} - (H+a^2)\frac{\partial H}{\partial r_{\rm H}}\right], \tag{71b}$$

and the following relationship is used:

$$4\pi TS_{\rm BH} = \frac{\partial\Delta}{\partial r_{\rm H}} = -\frac{1}{H}\left(a^2\frac{\partial H}{\partial r_{\rm H}} - H^2\frac{\partial F}{\partial r_{\rm H}}\right). \tag{72}$$

In the thermodynamics of singular black holes, the first law takes the form

$${\rm d}M = T{\rm d}S_{\rm BH} + \Omega_{\rm H}{\rm d}J. \tag{73}$$

When comparing Eq. (73) with Eq. (70), we give $\hat{A}$ and $\hat{B}$ if the first law remains unchanged for regular black holes:

$$\frac{1}{\hat{A}} = T, \tag{74a}$$

$$-\frac{\hat{B}}{\hat{A}} = \Omega_{\rm H}. \tag{74b}$$

Therefore, we determine the necessary conditions using Eqs. (71a) and (67) for a rotating regular black hole to satisfy the first law of thermodynamics:

$$\frac{\partial H}{\partial r_{\rm H}}\frac{\partial F}{\partial M} - \frac{\partial H}{\partial M}\frac{\partial F}{\partial r_{\rm H}} = -4, \tag{75a}$$

$$H\frac{\partial F}{\partial r_{\rm H}} - (1-F)\frac{\partial H}{\partial r_{\rm H}} = -2M. \tag{75b}$$

B. Second law of thermodynamics

The second law of black hole thermodynamics dictates that the entropy of a system can only increase or remain constant during the system's evolution without external interaction. This evolution naturally includes the incidence of a scalar particle into a black hole. Therefore, by utilizing Eqs. (51) and (53), we derive the temporal variations of both energy and angular momentum of black holes:

$$\mathrm{d}M = \frac{\mathrm{d}E}{\mathrm{d}t}\bigg|_{r=r_{\rm H}}\mathrm{d}t = \omega(\omega - m\Omega_{\rm H})(H+a^2)\mathrm{d}t, \tag{76a}$$

$$\mathrm{d}J = \frac{\mathrm{d}L}{\mathrm{d}t}\bigg|_{r=r_{\rm H}}\mathrm{d}t = m(\omega - m\Omega_{\rm H})(H+a^2)\mathrm{d}t. \tag{76b}$$

Thus, we obtain the relationship between $\mathrm{d}S_{\rm BH}$ and $\mathrm{d}t$:

$$\mathrm{d}S_{\rm BH} = \hat{A}\mathrm{d}M + \hat{B}\mathrm{d}J = (\hat{A}\omega + \hat{B}m)(\omega - m\Omega_{\rm H})(H+a^2)\mathrm{d}t. \tag{77}$$

Moreover, considering that frequency ω and integer m are arbitrary, we deduce the necessary and sufficient conditions for $\mathrm{d}S_{\rm BH} \geq 0$:

$$\hat{A} > 0 \qquad \text{and} \qquad \frac{\hat{B}}{\hat{A}} = -\Omega_{\rm H}. \tag{78}$$

Again, using Eqs. (71a) and (67), we reformulate the necessary and sufficient conditions for a rotating regular black hole to satisfy the second law of thermodynamics:

$$\frac{\partial H}{\partial r_{\rm H}}\frac{\partial F}{\partial M} - \frac{\partial H}{\partial M}\frac{\partial F}{\partial r_{\rm H}} < 0, \tag{79a}$$

$$\left(H + \frac{M}{2}\frac{\partial H}{\partial M}\right)\frac{\partial F}{\partial r_{\rm H}} - \left(1 - F + \frac{M}{2}\frac{\partial F}{\partial M}\right)\frac{\partial H}{\partial r_{\rm H}} = 0. \tag{79b}$$

By comparing Eqs. (75a) and (75b) with Eqs. (79a) and (79b), we observe that the validity of the first law indicates the validity of the second law. Alternatively, according to Eqs. (74a) and (74b), we observe that the first law yields

$$\mathrm{d}S_{\rm BH} = \frac{1}{T}(\omega - m\Omega_{\rm H})^2(H+a^2)\mathrm{d}t, \tag{80}$$

which implies that the second law is naturally satisfied. This result appears to be plausible but is actually incorrect. The reason is two fold: we must compute the entropies of black holes and scalar particles, not that only for black holes, and a (rotating) regular black hole does not satisfy the first law of a (rotating) singular black hole. In practice, the first law of mechanics for static and spherically symmetric regular black holes often contains additional terms related to regularized parameters, [54], which has a non-negligible role in constructing the first law of thermodynamics [55]. Further investigation is required to establish the first and second laws for (rotating) regular black holes.

Based on the above clarifications in this section, we note that both the first and second laws of thermodynamics are established on Bekenstein-Hawking's area theorem Eq. (67). Therefore, if Bekenstein-Hawking's area theorem is violated, the two laws are also violated. Conversely, if the second law is violated, the entropy decreases. This implies that the basis of Bekenstein-Hawking's area theorem, *i.e.*, the entropy increases during evolution, is lost, indicating the nonexistence of the theorem.

C. Third law of thermodynamics and weak cosmic censorship conjecture

The above discussions on thermodynamics are based on the general premise that an incident scalar particle does not destroy an event horizon of black holes. In spacetime with singularities, this premise is also required by the weak cosmic censorship conjecture, where no naked singularities are permitted. However, the weak cosmic censorship conjecture loses its necessity in regular black holes, where no singularities exist. Therefore, we can comprehend the weak cosmic censorship conjecture as a mechanism that safeguards the thermodynamic stability of black holes because thermodynamic quantities are only well-defined on event horizons. Next, we reexamine whether an incident scalar particle breaks an event horizon, *i.e.*, to verify whether the weak cosmic censorship conjecture holds for a rotating regular black hole. Because the energy and angular momentum of scalar particles are significantly smaller than those of black holes, incident scalar particles can only destroy an event horizon when a black hole is near its ultimate state.

Generally, the ultimate states for a rotating regular black hole exist in two primary categories. The first scenario entails that a black hole initially has multiple horizons, which subsequently coalesce into a one with a nonzero radius in the final stage of black holes; this is referred to as an extreme black hole case. The alternative scenario is that a black hole ultimately has one event horizon with a vanishing radius, which can be considered a one-way wormhole featuring a null throat [21, 45].

1. *In the first scenario*

Let us scrutinize the first scenario. Based on the existence condition Eq. (61) of horizons, we rewrite $r_{\rm H}$ as a function of M and J:

$$r_{\rm H}=r_{\rm H}(M,J). \tag{81}$$

In terms of $\lim_{r\to\infty}\Delta=r^2$, we know that the value of Δ is negative between an inner horizon and an outer one and that a minimum value of Δ, denoted by $\Delta_{\min}$, satisfies the condition in terms of Eq. (18):

$$\frac{\partial\Delta(M,J,r_{\min})}{\partial r_{\min}}=F(M,r_{\min})\frac{\partial H(M,r_{\min})}{\partial r_{\min}}+H(M,r_{\min})\frac{\partial F(M,r_{\min})}{\partial r_{\min}}=0, \tag{82}$$

from which we can deduce that $r_{\min}$ depends only on M:

$$r_{\min}=r_{\min}(M). \tag{83}$$

If multiple minimum values exist, the largest one among them will be selected. We note that the extreme horizon radius r_e satisfies both Eq. (61) and Eq. (82). Thus, we can express the angular momentum J_e of an extreme black hole as a function of M:

$$J_e=\sqrt{-M^2F(M,r_e(M))H(M,r_e(M))}\equiv j_e(M), \tag{84}$$

where

$$r_e(M)=r_{\min}(M). \tag{85}$$

Eq. (84) shows that the upper bound on the angular momentum is determined only by mass; thus, it is termed as an "extreme function." Based on Eq. (60), the Hawking temperature of an extreme black hole is zero. The third law of thermodynamics stipulates that it requires an infinite amount of time, or equivalently, an infinite number of steps to reach the absolute zero temperature. This implies that the evolution of a black hole into its extreme configuration cannot be achieved through any finite number of steps. Subsequently, we demonstrate that the weak cosmic censorship conjecture is compatible with the third law of thermodynamics in the first scenario.

From the above analyses, the condition for existence of event horizons is

$$\Delta_{\min}\le 0. \tag{86}$$

By using Eqs. (18) and (84), we rewrite it as follows:

$$J^2\le -M^2F(M,r_{\min}(M))H(M,r_{\min}(M))\equiv j_e^2(M), \tag{87}$$

where the equality is valid only for extreme black holes. Therefore, for a near-extreme black hole, we assume the relationship between angular momentum J_{ne} and mass M is

$$J_{ne}=(1-\epsilon)j_e(M), \tag{88}$$

where ϵ is an arbitrary positive infinitesimal parameter, $0<\epsilon\ll 1$. After the incidence of a neutral scalar particle into a near-extreme black hole, when we consider Eqs. (76b) and (88), the angular momentum becomes

$$J_{ne}+dJ=(1-\epsilon)j_e(M)+m(\omega-m\Omega_{ne})(H+a^2){\rm d}t, \tag{89}$$

and when we consider Eq. (76a), the extreme function then becomes

$$j_e(M+{\rm d}M)=j_e(M)+\frac{{\rm d}j_e(M)}{{\rm d}M}\omega(\omega-m\Omega_{ne})(H+a^2){\rm d}t. \tag{90}$$

Next, we prove that the weak cosmic censorship conjecture is consistent with the third law of thermodynamics in the first scenario, that is, both are valid or both are broken. According to Eq. (87), the condition for absence of horizons after the incidence of a neutral scalar particle assumes the form

$$J_{ne}+{\rm d}J>j_e(M+{\rm d}M), \tag{91}$$

which implies the invalidity of the weak cosmic censorship conjecture, and the condition for a black hole to reach its extreme configuration is

$$J_{ne} + \mathrm{d}J = j_e(M + \mathrm{d}M), \tag{92}$$

which implies the invalidity of the third law of thermodynamics. More specifically, considering Eqs. (89) and (90), we combine Eqs. (91) and (92) as

$$-\epsilon j_e(M) \geq \left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\omega - m\right)(\omega - m\Omega_{ne})(H + a^2)\mathrm{d}t, \tag{93}$$

where the equality holds for an extreme configuration. From Eqs. (76a) and (76b), we deduce $\frac{\mathrm{d}j_e(M)}{\mathrm{d}M} > 0$. Therefore, we determine that the right-hand side of Eq. (93) satisfies[1)] the following inequality:

$$\begin{aligned}&\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\omega - m\right)(\omega - m\Omega_{ne})(H + a^2)\mathrm{d}t\\ &= m^2\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\left[\frac{\omega}{m} - \left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1}\right]\left(\frac{\omega}{m} - \Omega_{ne}\right)(H + a^2)\mathrm{d}t\\ &\geq -\frac{m^2}{4}\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\left[\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1} - \Omega_{ne}\right]^2(H + a^2)\mathrm{d}t,\end{aligned} \tag{94}$$

where the equality is valid for the condition $\frac{\omega}{m} = \frac{1}{2}\left[\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1} + \Omega_{ne}\right]$. To simplify Eq. (94), we expand $\Omega_{ne}(M, J_{ne}, r_{ne})$ near the extreme configuration $\epsilon = 0$ as

$$\Omega_{ne}(M, J_{ne}, r_{ne}) = \Omega_e - j_e(M)\kappa\epsilon + O(\epsilon^2), \tag{95}$$

where $\kappa \equiv \frac{\mathrm{d}\Omega_{\mathrm{H}}(M, J, r_{\mathrm{H}})}{\mathrm{d}J}\Big|_{J = j_e(M)}$. Considering Eqs. (94) and (95), we refine Eq. (93) as

$$\epsilon \leq \frac{m^2}{4j_e(M)}\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\left[\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1} - \Omega_e + j_e(M)\kappa\epsilon\right]^2(H + a^2)\mathrm{d}t. \tag{96}$$

- If we have the condition

$$\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1} - \Omega_e = 0, \tag{97}$$

the above inequality becomes

$$\epsilon \leq \frac{m^2}{4j_e(M)}\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\left[j_e(M)\right]^2\kappa^2(H + a^2)\epsilon^2\mathrm{d}t. \tag{98}$$

After comparing the orders of infinitesimals on both sides, we conclude that this inequality never holds, indicating that both the third law of thermodynamics and weak cosmic censorship conjecture are valid in this scenario.

- If we have the condition

$$\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1} - \Omega_e \neq 0, \tag{99}$$

we maintain the finite term and ignore infinitesimals in Eq. (96):

$$\epsilon \leq \frac{m^2}{4j_e(M)}\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\left[\left(\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}\right)^{-1} - \Omega_e\right]^2(H + a^2)\mathrm{d}t. \tag{100}$$

Because both ϵ and $\mathrm{d}t$ are the same order of infinitesimals, we can always determine an appropriate time interval $\mathrm{d}t$ such that this inequality holds, which means that both the third law of thermodynamics and weak cosmic censorship conjecture are broken.

2. *In the second scenario*

Now, we examine the second scenario. Considering Eq. (84), we redefine $j_e(M)$ as follows:

$$j_e(M) \equiv \sqrt{-M^2F(M,r)H(M,r)}\Big|_{r=0}. \tag{101}$$

According to Eq. (60), the Hawking temperature of this ultimate state is no longer zero. Therefore, following the analyses for the first scenario, we conclude that the third law holds, but the weak cosmic censorship conjecture may not hold in the second scenario because event horizons may disappear or black holes may transform into a wormhole.

3. *Results in the two scenarios*

In the two scenarios, Eq. (97) is the sufficient and necessary condition for the weak cosmic censorship conjecture to be valid, and it is also the necessary condition for the third law of thermodynamics to be valid for an extreme black hole. However, if the ultimate state of a black hole is a one-way wormhole in the second scenario, the weak cosmic censorship conjecture may not hold, but the third law of thermodynamics is still valid. This results in an interesting challenge to establish the so-called wormhole thermodynamics to describe the evolutionary process from a black hole to a wormhole.

Moreover, we express Eq. (97) in a specific form, *i.e.*, in terms of $F(M,r)$ and $H(M,r)$. By using Eqs. (84) and

1) For a function, $f(x) = (x-a)(x-b)$, it takes its minimum, $-(a-b)^2/4$, when $x = (a+b)/2$.

(82), we derive

$$\frac{\mathrm{d}j_e(M)}{\mathrm{d}M}=\frac{j_e(M)}{M}-\frac{M^2}{2j_e(M)}\left(H(M,r_e)\frac{\partial F(M,r_e)}{\partial M}+F(M,r_e)\frac{\partial H(M,r_e)}{\partial M}\right), \tag{102}$$

and then substituting the above equation into Eq. (97) and considering Eqs. (22) and (85), we finally rewrite Eq. (97) as

$$H(M,r_e)\frac{\partial F(M,r_e)}{\partial M}+F(M,r_e)\frac{\partial H(M,r_e)}{\partial M}=-\frac{2H(M,r_e)}{M}, \tag{103}$$

where the definition of r_e is given by Eqs. (82), (83), and (85) in the first scenario and $r_e=0$ in the second scenario.

D. Interpretation

In the above three subsections, we have examined the conditions for the first, second, and third laws of thermodynamics to be valid, as well as for the weak cosmic censorship conjecture (see Eqs. (75a), (75b), (79a), (79b), and (103)). We note that the definition of r_e given by Eqs. (82), (83), and (85) is solely determined by metric functions of a static spacetime, thus rendering the validity condition Eq. (103) for both the third law and weak cosmic censorship conjecture completely dependent on metric functions of a static spacetime. However, in the validity conditions for the first and second laws (Eqs. (75a), (75b), (79a), and (79b)), we note that a variable $r_{\rm H}$ exists that varies with the rotation parameter a in addition to metric functions of a static spacetime.

When a static metric is given, the minimum horizon of a rotating black hole corresponds to r_e of an extreme configuration, where r_e is independent of the rotation parameter a. Moreover, the maximum horizon $r_{\max}$ depends only on a static metric. Therefore, the range of horizons for a rotating black hole is $r\in[r_e,r_{{\rm H}(s)}]$, where "(s)" denotes a static black hole considered a seed. We now rewrite the validity conditions for the first law as

$$\frac{\partial H}{\partial r}\frac{\partial F}{\partial M}-\frac{\partial H}{\partial M}\frac{\partial F}{\partial r}=-4, \tag{104a}$$

$$H\frac{\partial F}{\partial r}-(1-F)\frac{\partial H}{\partial r}=-2M. \tag{104b}$$

That for the second law is expressed as

$$\frac{\partial H}{\partial r}\frac{\partial F}{\partial M}-\frac{\partial H}{\partial M}\frac{\partial F}{\partial r}<0, \tag{105a}$$

$$\left(H+\frac{M}{2}\frac{\partial H}{\partial M}\right)\frac{\partial F}{\partial r}-\left(1-F+\frac{M}{2}\frac{\partial F}{\partial M}\right)\frac{\partial H}{\partial r}=0, \tag{105b}$$

where $r\in[r_e,r_{{\rm H}(s)}]$. Thus, we express the validity conditions only by metric functions F and H of a static seed spacetime. In other words, we determine the conditions under which a rotating black hole satisfies the laws of thermodynamics only using the metric functions of its static seed black hole. We note that such conditions are valid for any rotating black holes constructed using the revised NJA as our calculations are performed in the spacetime beyond event horizons.

V. APPLICATION TO THREE MODELS

Based on the number of shape functions that appear in a metric, we classify regular black holes into two types [56], where these regular black holes are static and spherically symmetric and considered a seed of rotating regular black holes. In the first type, $H(r)=r^2$ and only $F(r)$ is to be determined (see Eq. (1)), which is called the "single function case," such as for Bardeen black holes [57], Hayward black holes [19], noncommutative black holes [20], and the other widely studied black holes [17, 18, 47]. In the second type, both $F(r)$ and $H(r)$ are to be determined, which is called the "double function case," such as for black-bounce solutions [21], loop quantum gravity black holes [45], and the other black holes that have recently attracted interest and discussions [58]. Next, we discuss the thermodynamics of these two types of black holes, where Hayward, black-bounce, and loop quantum gravity black holes are selected as specific examples. There are two reasons for our choices: One reason is that these three black holes correspond to the three categories of construction for regular black holes, which has been mentioned in Sec. I; the other is that these three black holes correspond to different types of shape functions, from which we can gain a more comprehensive understanding of the thermodynamic laws for regular black holes.

A. Single function case

1. *Conditions*

In Eq. (1), $H(r)$ is fixed, $H(r)=r^2$; thus, the necessary conditions for a rotating regular black hole to satisfy the first law of thermodynamics (Eqs. (104a) and (104b)) become

$$\frac{\partial F(M,r)}{\partial M}=-\frac{2}{r}, \tag{106a}$$

$$r^2\frac{\partial F(M,r)}{\partial r}-2[1-F(M,r)]r+2M=0, \tag{106b}$$

where $r \in [r_e, r_{H(s)}]$. From Eq. (106a), we solve $F(M,r)$ as follows:

$$F(M,r) = 1 - \frac{2M}{r} + \sigma(r), \tag{107}$$

where $\sigma(r)$ is a function independent of M. Subsequently, substituting Eq. (107) into Eq. (106b), we obtain

$$\sigma(r) = \frac{C_1}{r^2}, \tag{108}$$

where C_1 is an integration constant. Thus, we fix $F(M,r)$ in the single function case:

$$F(M,r) = 1 - \frac{2M}{r} + \frac{C_1}{r^2}, \tag{109}$$

with which the rotating regular black hole of the single function case satisfies the same first law of thermodynamics as its seed black hole does.

From Eqs. (109) and (18), we obtain the horizon of rotating regular black holes, $r_H = M + \sqrt{M^2 - a^2 - C_1}$, whose lower limit is M, corresponding to $a = \sqrt{M^2 - C_1}$, and whose upper limit is $M + \sqrt{M^2 - C_1}$, corresponding to $a = 0$. To ensure the existence of horizons and a non-vanishing rotation parameter, C_1 should satisfy $C_1 < M^2$. Therefore, note that the range of r is no longer from zero to infinity but coincides with that of event horizons of rotating regular black holes, $[M, M + \sqrt{M^2 - C_1})$. Within this range, the rotating black hole is reduced to a Kerr black hole when $C_1 = 0$ and to a Kerr-Newman black hole when $C_1 = Q^2$. We have seen that the first law of thermodynamics imposes a highly stringent constraint on metrics, which indicates that a static seed black hole must have a metric with such an $F(M,r)$ as in Eq. (109). However, it is significantly challenging to construct a continuous metric function $F(M,r)$ in the range of $r \in [0, \infty)$, where this function returns to Eq. (109) when r is fixed to the range of $[M, M + \sqrt{M^2 - C_1})$.

Next, we examine the second law of thermodynamics. The necessary conditions, Eqs. (105a) and (105b), become

$$\frac{\partial F(M,r)}{\partial M} < 0, \tag{110a}$$

$$r\frac{\partial F(M,r)}{\partial r} - M\frac{\partial F(M,r)}{\partial M} = 2(1 - F(M,r)), \tag{110b}$$

where $r \in [r_e, r_{H(s)}]$. To verify the validity of the aforementioned conditions, we must rely on a specific metric function $F(M,r)$, which is discussed in detail for the following models.

Finally, owing to $\Delta(M,r)\big|_{r=0} = a^2 > 0$, the ultimate state of black holes is an extreme configuration in the single function case. Therefore, for the third law of thermodynamics and weak cosmic censorship conjecture, the validity condition Eq. (103) is reduced to

$$\frac{\partial F(M,r_e)}{\partial M} = -\frac{2}{M}. \tag{111}$$

Because r_e is a function of M, we cannot determine $F(M,r)$ using a direct integration from the above equation, but we leave it to specific models.

2. *Hayward model*

The specific model we select here is the static and spherically symmetric Hayward black hole [19, 41], whose shape function is

$$F_H(r) = 1 - \frac{2Mr^2}{r^3 + 2L^2M}, \tag{112}$$

where the regularization parameter L is a convenient encoding of the central energy density $3/(8\pi L^2)$. The field source of this black hole solution corresponds to the vacuum energy density distribution

$$\rho(r) = \frac{3L^2M^2}{2\pi(r^3 + 2L^2M)^2}, \tag{113}$$

and its radial pressure, p_r, and transverse pressures, p_θ and p_ϕ, are

$$p_r = -\frac{3L^2M^2}{2\pi(r^3 + 2L^2M)^2}, \tag{114}$$

$$p_\theta = p_\phi = \frac{3(r^3 - L^2M)L^2M^2}{\pi(r^3 + 2L^2M)^3}. \tag{115}$$

If the first law of thermodynamics is required, when comparing Eq. (112) with Eq. (109), we obtain the integration constant

$$C_1 = \frac{4M^2L^2r}{r^3 + 2L^2M}, \tag{116}$$

where $r \in [M, M + \sqrt{M^2 - C_1})$ and $C_1 < M^2$. We observe that the right-hand side of Eq. (116) is not a constant; therefore, the first law of thermodynamics does not hold for rotating Hayward black holes.

For the second law of thermodynamics, Eq. (110a) is valid owing to $r > 0$:

$$\frac{\partial F(M,r)}{\partial M} = -\frac{2r^5}{(r^3+2L^2M)^2} < 0. \tag{117}$$

However, Eq. (110b) assumes the form

$$-\frac{16L^2M^2r^2}{(2L^2M+r^3)^2} = 0, \tag{118}$$

which does not hold for $L \neq 0$. Therefore, the second law of thermodynamics is also invalid for rotating Hayward black holes.

To check the third law of thermodynamics and weak cosmic censorship conjecture, we define a function U and derive its form by using Eq. (112) and $r_e = M$:

$$U \equiv \frac{\partial F(M,r_e)}{\partial M} + \frac{2}{M} = -\frac{2M^3}{(M^2+2L^2)^2} + \frac{2}{M}. \tag{119}$$

In terms of this definition, Eq. (111) becomes $U = 0$.

In Fig. 1, we observe that $U = 0$ holds only for $L = 0$, suggesting that rotating Hayward black holes do not satisfy the third law of thermodynamics and weak cosmic censorship conjecture, where $M = 1$ is set without loss of generality.

B. Double function case

In this case, our discussions are based on the relationship between H and M in two categories: where H does not depend on M and where H does depend on M.

1. *Category of H independent of M*

In this category, the necessary condition Eq. (104a) of the first law of thermodynamics can be expressed as

$$\frac{\partial H}{\partial r}\frac{\partial F}{\partial M} = -4, \tag{120}$$

where $r \in [r_e, r_{\mathrm{H}(s)}]$, and the condition Eq. (104b) remains unchanged. For the second law of thermodynamics, the

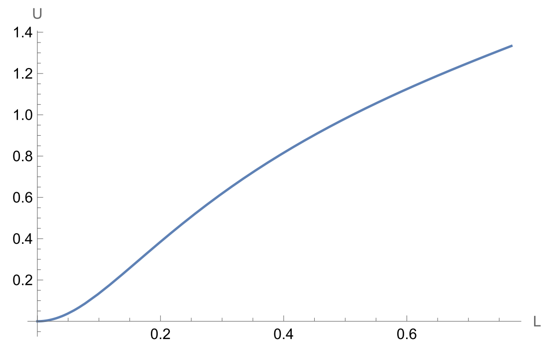


Fig. 1. (color online) Relationship between U and L in rotating Hayward black holes, where $M = 1$.

necessary conditions, Eqs. (105a) and (105b), become

$$\frac{\partial H}{\partial r}\frac{\partial F}{\partial M} < 0, \tag{121}$$

and

$$H\frac{\partial F}{\partial r}\left(\frac{\partial H}{\partial r}\right)^{-1} - \frac{M}{2}\frac{\partial F}{\partial M} = 1 - F, \tag{122}$$

where the range of r is still $r \in [r_e, r_{\mathrm{H}(s)}]$. For the third law of thermodynamics and weak cosmic censorship conjecture, the necessary condition is the same as that in Eq. (111), but r_e is taken to be zero if the ultimate state of black holes is a one-way wormhole.

Here, we use the Kerr black-bounce solution as an example, whose static and spherically symmetric metric is [21]

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{\sqrt{r^2+l^2}}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\dfrac{2M}{\sqrt{r^2+l^2}}} + \left(r^2+l^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{123}$$

where l is a positive parameter responsible for the regularization of the central singularity. The black-bounce metric is interesting because it is a minimal one-parameter extension of the Schwarzschild metric. This model converts the central singularity of Schwarzschild black holes into the throat of wormholes after the introduction of parameter l, thereby connecting regular black holes and traversable wormholes. Recently, many regular black holes and traversable wormholes have been constructed using the black-bounce proposal [58–60]. However, the key problem is explaining the black-bounce solutions physically, *i.e.*, to determine the theory and matter that can yield such solutions. Now it is known [34] that this theory is the Einstein gravity coupled with matter, where the matter is the combination of a phantom scalar field and a nonlinear electrodynamics field. For the details [34], the action is given by

$$I = \int \sqrt{-g}\,\mathrm{d}^4x\left(\mathcal{R} - 2g^{\mu\nu}\partial_\mu\phi_\mathrm{P}\partial_\nu\phi_\mathrm{P} - 2V(\phi_\mathrm{P}) - \mathcal{L}(\mathcal{F})\right), \tag{124}$$

where $\mathcal{L}(\mathcal{F})$ is the Lagrangian density of gauge-invariant nonlinear electrodynamics with the Faraday electromagnetic invariant, $\mathcal{F} \equiv F^{\mu\nu}F_{\mu\nu}$, and ϕ_P is a phantom scalar field. The Lagrangian density and the potential of a phantom scalar field have the following forms:

$$\mathcal{L}(\mathcal{F})=\frac{12Ml^2}{5\,(2q^2/\mathcal{F})^{5/4}},\tag{125}$$

and

$$V(\phi_{\rm P})=\frac{4M\cos^5\phi_{\rm P}}{5l^3},\tag{126}$$

respectively, where q is magnetic charge of free nonlinear electrodynamics. The Faraday electromagnetic invariant and the phantom scalar field is

$$\mathcal{F}=\frac{2q^2}{(r^2+l^2)^2},\tag{127}$$

and

$$\phi_{\rm P}(r)=\tan^{-1}\frac{r}{l}+{\rm const.},\tag{128}$$

respectively.

The corresponding metric with rotation is given by [46]

$$\begin{aligned}{\rm d}s^2=&-\left(1-\frac{2M\sqrt{r^2+l^2}}{\Sigma}\right){\rm d}t^2+\frac{\Sigma}{\Delta}{\rm d}r^2\\&-\frac{4Ma\sqrt{r^2+l^2}\sin^2\theta}{\Sigma}{\rm d}t{\rm d}\phi\\&+\Sigma{\rm d}\theta^2+\frac{A\sin^2\theta}{\Sigma}{\rm d}\phi^2,\end{aligned}\tag{129}$$

where

$$\Sigma=r^2+l^2+a^2\cos^2\theta,\tag{130a}$$

$$\Delta=r^2+l^2+a^2-2M\sqrt{r^2+l^2},\tag{130b}$$

$$A=(r^2+l^2+a^2)^2-\Delta a^2\sin^2\theta.\tag{130c}$$

Note that the above metric is consistent with Eq. (15) when $\Psi=\Sigma$ is selected. According to $\Delta(r_{\rm H})=0$, we obtain the horizons as

$$r_{\rm H}=\sqrt{(M\pm\sqrt{M^2-a^2})^2-l^2}.\tag{131}$$

When $l\in(0,M]$, the event horizon of an extreme black hole is located at $r_e=\sqrt{M^2-l^2}$, and the rotation parameter satisfies the relation $a=M$. When $l\in(M,2M]$, the event horizon of the ultimate state is located at $r_e=0$, and the rotation parameter satisfies the relation $a=\sqrt{l(l-2M)}$, resulting in the degeneration of the Kerr black-bounce solution into a one-way wormhole.

Next, we consider the necessary conditions for the laws of thermodynamic and weak cosmic censorship conjectures. Owing to

$$\frac{\partial H}{\partial r}\frac{\partial F}{\partial M}=-\frac{4r}{\sqrt{r^2+l^2}}\neq-4,\tag{132}$$

in the range of $r\in[\sqrt{M^2-l^2},\sqrt{4M^2-l^2}]$ (Eq. (104a)), the first law of thermodynamics does not hold for the Kerr black-bounce solution.

The necessary conditions for the second law take the forms

$$\frac{\partial H}{\partial r}\frac{\partial F}{\partial M}=-\frac{4r}{\sqrt{r^2+l^2}}<0,\tag{133}$$

and

$$H\frac{\partial F}{\partial r}\left(\frac{\partial H}{\partial r}\right)^{-1}-\frac{M}{2}\frac{\partial F}{\partial M}=\frac{2M}{\sqrt{r^2+l^2}}=1-F,\tag{134}$$

(Eqs. (105a) and (105b)), indicating that both Eq. (133) and Eq. (134) are valid simultaneously in the range $r\in[\sqrt{M^2-l^2},\sqrt{4M^2-l^2}]$. Therefore, the second law of thermodynamics holds for the Kerr black-bounce solution. This result shows that the area entropy can be used to define the Kerr black-bounce entropy, but the first law of thermodynamics still remains unsatisfied.

Finally, let us examine the third law and weak cosmic censorship conjecture for the Kerr black-bounce solution. When $l\in(0,M]$, according to Eq. (111), *i.e.*,

$$\frac{\partial F(M,r_e)}{\partial M}=-\frac{2}{\sqrt{r^2+l^2}}=-\frac{2}{M},\tag{135}$$

we observe that the horizon of extreme configurations exists. When $l\in(M,2M]$, according to Eq. (111) again, *i.e.*,

$$\frac{\partial F(M,0)}{\partial M}=-\frac{2}{l}\neq-\frac{2}{M},\tag{136}$$

we observe that the horizon of ultimate states disappears. However, the horizon undergoes a natural disappearance, *i.e.*, the Kerr black-bounce solution changes to a one-way wormhole and then to a two-way one. Consequently, the Kerr black-bounce solution satisfies the third law of thermodynamics and weak cosmic censorship conjecture.

2. *Category of H dependent on M*

In this category, several necessary conditions of the laws of thermodynamics cannot be further simplified;

therefore, our verification to the laws depends on the specific forms of metrics. Here, we use rotating loop quantum gravity black holes as an example.

The metric functions of static seed black holes can be expressed as [45]

$$F(r)=\left(1-\frac{2M}{\sqrt{r^2+4\lambda_k^{2/3}M^{2/3}}}\right)\frac{r^2+4\lambda_k^{2/3}M^{2/3}}{r^2+\lambda_k^{2/3}M^{2/3}}, \tag{137a}$$

$$H(r)=r^2+\lambda_k^{2/3}M^{2/3}, \tag{137b}$$

where the quantum parameter λ_k originates from holonomy modifications [22, 23]. This metric is static and spherically symmetric, which is obtained in terms of the effective equation of loop quantum gravity and is considered to be the quantum extension of Schwarzschild black holes. In the rotating metric, the function Δ that determines horizons is

$$\Delta=FH+a^2=r^2+4\lambda_k^{2/3}M^{2/3}-2M\sqrt{r^2+4\lambda_k^{2/3}M^{2/3}}+a^2, \tag{138}$$

and the corresponding horizons are located at

$$r_{\rm H}=\sqrt{\left(M\pm\sqrt{M^2-a^2}\right)^2-4\lambda_k^{2/3}M^{2/3}}. \tag{139}$$

When $\lambda_k\in\left(0,M^2/8\right]$, the ultimate state of rotating loop quantum gravity black holes will remain at its extreme configuration with the horizon: $r_e=\sqrt{M^2-4\lambda_k^{2/3}M^{2/3}}$, and the rotation parameter $a=M$. When $\lambda_k\in\left(M^2/8,M^2\right]$, the ultimate state will be a one-way wormhole with $a=2M^{1/3}\sqrt{\lambda_k^{1/3}M^{2/3}-\lambda_k^{2/3}}$.

Next, we consider the necessary conditions for the laws of thermodynamics. The first necessary condition, Eq. (75a), for the first law requires

$$\frac{4r\left[-5\lambda_k^{2/3}M+\lambda_k^{2/3}\sqrt{4(\lambda_kM)^{2/3}+r^2}-M^{1/3}r^2\right]}{M^{1/3}\left[(\lambda_kM)^{2/3}+r^2\right]\sqrt{4(\lambda_kM)^{2/3}+r^2}}=-4, \tag{140}$$

but it is not satisfied within the range of $r\in[r_e,\sqrt{4M^2-4\lambda_k^{2/3}M^{2/3}}]$. Thus, the first law is invalid for rotating loop quantum gravity black holes.

For the second law of thermodynamics, the necessary condition, Eq. (79b), requires

$$\frac{2r(\lambda_kM)^{2/3}\left[4M-\sqrt{4(\lambda_kM)^{2/3}+r^2}\right]}{\left[(\lambda_kM)^{2/3}+r^2\right]\sqrt{4(\lambda_kM)^{2/3}+r^2}}=0. \tag{141}$$

It can only be satisfied when $\lambda_k=0$, in which case the rotating loop quantum gravity black holes revert to Kerr black holes. Therefore, the second law of thermodynamics is not satisfied by the rotating loop quantum gravity black holes.

Finally, we examine the evolution of rotating loop quantum gravity black holes near an ultimate state. When the ultimate state is an extreme configuration, the condition, Eq. (103), becomes

$$-\frac{6\lambda_k^{2/3}}{M^{1/3}}=0, \tag{142}$$

which is unattainable under the circumstances, $M>0$ and $\lambda_k\neq0$. Therefore, this case leads to the disappearance of horizons when incident scalar particles are incoming, indicating a breakdown of both the third law and weak cosmic censorship conjecture. If the final state is a one-way wormhole, the condition, Eq. (103), becomes

$$7\lambda_k^{1/3}-8M^{2/3}=0. \tag{143}$$

It is only applicable to $\lambda_k=(8/7)^3M^2$, but λ_k falls outside its reasonable range. In this scenario, the rotating loop quantum gravity black holes may eventually transform into a two-way wormhole. Thus, all the first, second, and third laws of thermodynamics are invalid in loop quantum gravity black holes.

C. Summary for above applications

In summary, we list validity or invalidity for three black hole models to obey the laws of thermodynamics and weak cosmic censorship conjecture in Table 1. The violation of the first law is evident in the three models. Meanwhile, both the Hayward and loop quantum gravity black holes fail to satisfy all the laws, indicating that rotating regular black holes may require new definitions of thermodynamic quantities and new laws of thermodynamics. For Kerr black-bounce solutions, while the first

Table 1. This table shows whether the three specific models of rotating regular black holes satisfy the laws of thermodynamics and weak cosmic censorship conjecture, where "WCCC" is the abbreviation of weak cosmic censorship conjecture.

	Hayward	Kerr black-bounce	Loop quantum gravity
First law	no	no	no
Second law	no	yes	no
Third law	no	yes	no
WCCC of extreme BHs	no	yes	no
WCCC of one-way wormholes		no	no

law is violated, both the second and third laws are still well fulfilled. This suggests that the area entropy has a certain degree of thermodynamic self-consistency. The primary challenge lies in modifying the first law of thermodynamics to achieve overall self-consistency for all the laws of thermodynamics.

D. Attempts to recover the laws of thermodynamics for rotating regular black holes

The analyses in the above three subsections show that rotating regular black holes break the laws of thermodynamics deduced from singular black holes. In practice, static regular black holes behave similarly, where two modified approaches are mainly adopted: One is to modify the definition of entropy [61−63], and the other is to extend phase spaces by treating regularized parameters as variables [54, 55, 64]. In the following, we attempt to employ the two methods in the recovery of the laws of thermodynamics for rotating regular black holes.

1. *Modification of entropy*

Modifying the definition of entropy aims to establish the entropy that conforms to the first law of thermodynamics. As previously mentioned, if the first law of thermodynamics deduced from singular black holes holds for regular black holes, the second law also holds for regular black holes. Therefore, it can ensure the validity of the two laws to discover a suitable definition for entropy. Let us assume that the entropy in question denoted by S_M satisfies the first law of thermodynamics,

$$T\mathrm{d}S_M = \mathrm{d}M - \Omega_{\rm H}\mathrm{d}J, \tag{144}$$

and it can be obtained through integration in the (M,J) plane. Note that the entropy must be independent of the choice of integration paths. Therefore, according to the path independence of curve integrals, the sufficient and necessary condition of a suitable entropy is

$$\frac{\partial}{\partial J}\left(\frac{1}{T}\right) = -\frac{\partial}{\partial M}\left(\frac{\Omega_{\rm H}}{T}\right). \tag{145}$$

This condition is comparatively more flexible than the condition described by Eqs. (104a) and (104b) because the latter requires not only a path-independent entropy but also an area entropy.

Now, we apply the condition Eq. (145) to the three specific models discussed above. By defining

$$Di \equiv \left|\frac{\partial}{\partial J}\left(\frac{1}{T}\right) + \frac{\partial}{\partial M}\left(\frac{\Omega_{\rm H}}{T}\right)\right|, \tag{146}$$

we plot the diagrams shown in Fig. 2 in which Di changes with respect to the angular momentum for the three modes with a given black hole mass $M=1$ but different regularization parameters. Based on this figure, we conclude that no appropriate entropy exists for the fulfillment of the first law of thermodynamics because Di is non-vanishing in the (M,J) plane. Hence, require rotating regular black holes to satisfy the first law of thermodynamics solely by modifying entropy is not feasible.

2. *Extension of phase spaces*

We consider the regularization parameters, $(Y_1, Y_2, Y_3, \cdots, Y_n)$, as variables that vary during the process of particle incidence to extend a phase space. Based on the existence conditions of black hole event horizons before and after a particle incident process, we express the relationship among the first order derivatives of variables,

$$\mathrm{d}r_{\rm H} = -\frac{\partial\Delta}{\partial M}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}\Big|_{r=r_{\rm H}}\mathrm{d}M - \frac{\partial\Delta}{\partial J}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}\Big|_{r=r_{\rm H}}\mathrm{d}J - \sum_{i=1}^{n}\frac{\partial\Delta}{\partial Y_i}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}\Big|_{r=r_{\rm H}}\mathrm{d}Y_i, \tag{147}$$

and derive the corresponding change of entropy:

$$\mathrm{d}S_{\rm H} = \frac{\partial S_{\rm H}}{\partial M}\mathrm{d}M + \frac{\partial S_{\rm H}}{\partial J}\mathrm{d}J + \frac{\partial S_{\rm H}}{\partial r_{\rm H}}\mathrm{d}r_{\rm H} + \sum_{i=1}^{n}\frac{\partial S_{\rm H}}{\partial Y_i}\mathrm{d}Y_i. \tag{148}$$

By combining Eqs. (147) and (148), we obtain

$$\mathrm{d}S_{\rm H} = \tilde{A}\mathrm{d}M + \tilde{B}\mathrm{d}J + \sum_{i=1}^{n}\tilde{C}_i\mathrm{d}Y_i, \tag{149}$$

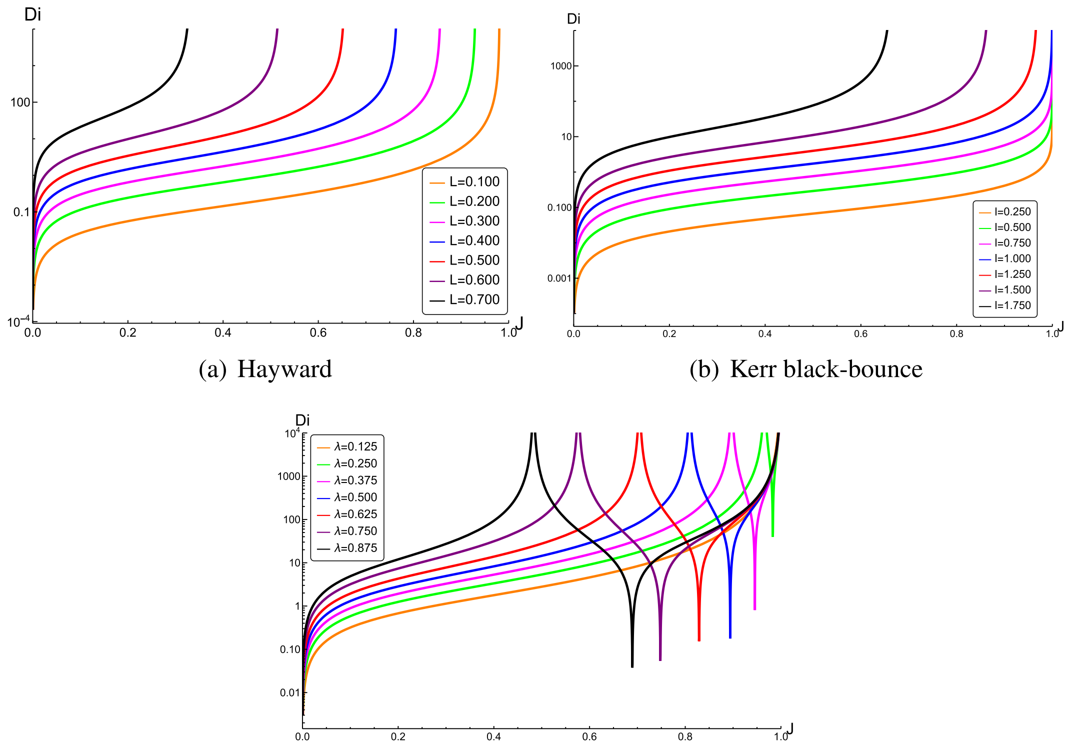
where

$$\tilde{A} = \frac{\partial S_{\rm H}}{\partial M} - \frac{\partial S_{\rm H}}{\partial r_H}\frac{\partial\Delta}{\partial M}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}, \tag{150a}$$

$$\tilde{B} = \frac{\partial S_{\rm H}}{\partial J} - \frac{\partial S_{\rm H}}{\partial r_H}\frac{\partial\Delta}{\partial J}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}, \tag{150b}$$

$$\tilde{C}_i = \frac{\partial S_{\rm H}}{\partial Y_i} - \frac{\partial S_{\rm H}}{\partial r_H}\frac{\partial\Delta}{\partial Y_i}\left(\frac{\partial\Delta}{\partial r}\right)^{-1}. \tag{150c}$$

Here $\tilde{A}$ and $\tilde{B}$ play the same role as $\hat{A}$ and $\hat{B}$ in Eq. (70), but $\tilde{C}_i$ is non-vanishing. Comparing Eq. (70) with Eq. (149), we deduce that the extension of phase spaces cannot revise the first law. In conclusion, neither the modification of entropy nor extension of phase spaces can render the laws of thermodynamics for rotating regular black holes, indicating that further research is required to explore correct laws of thermodynamics for rotating regu-



(a) Hayward (b) Kerr black-bounce

(c) Loop quantum gravity

Fig. 2. Relationship between Di and the angular momentum under different regularization parameters in the three specific black hole models, where the black hole mass $M = 1$.

lar black holes.

VI. CONCLUSION

In this paper, we establish criteria for the validity of thermodynamic laws and weak cosmic censorship conjecture in rotating regular black holes by examining the process of the incidence of neutral scalar particles into a rotating regular black hole. In this process, we calculate mass, charge, and other conserved quantities of black holes in general and provide the formula of Hawking temperature for rotating regular black holes. By examining the relationships among these quantities, we evaluate whether rotating regular black holes satisfy the laws of thermodynamics deduced from singular black holes. Moreover, we oberve that a complementary relationship between the third law of thermodynamics and weak cosmic censorship conjecture exists only when the ultimate state of a rotating regular black hole is an extreme configuration. Alternatively, the ultimate state of a rotating regular black hole is a one-way wormhole, such that the rotating regular black hole evolves into a two-way wormhole without violating the third law but leading to disappearance of event horizons, i.e., leading to invalidity of the weak cosmic censorship conjecture. To provide a deeper understanding of such an evolution, we must establish the thermodynamics of wormholes in a manner that connects the thermodynamic states of rotating regular black holes before evolution to those states after evolution.

In addition, for three specific models of rotating regular black holes, we verify their compliance with the thermodynamic laws and weak cosmic censorship conjecture. However, as shown in Table 1, two of the three models do not satisfy all the laws of thermodynamics, suggesting that a reestablishment of thermodynamics is necessary for them. Fortunately, the Kerr black-bounce solution satisfies the second and third laws of thermodynamics. In the attempt to recover the laws of thermodynamics for rotating regular black holes, we eomploy the redefinition of entropy and extension of phase spaces. However, these two methods fail, indicating the unusual property of rotating regular black holes. Such an unusual property implies the necessity of employing alternative approaches that are distinct from those employed in static and spherically symmetric regular black holes, to recover the thermodynamic laws applicable to rotating regular black holes. Meanwhile, if a black hole is treated as a thermodynamic system, the self-consistency of the first, second, and third laws must fully be guaranteed. The significance of our results is that we may establish the self-consistent thermodynamic laws by considering the inconsistencies we have revealed.

Finally, we propose some ideas that are useful for a deeper understanding of rotating regular black holes.

• To redefine the conserved quantities of a rotating regular black hole. The aforementioned conserved quantities, such as mass and angular momentum, are defined in an entirety of spacetime. If we consider only the region inside a horizon of a rotating regular black hole as a thermodynamic system, we may constrain the scope of integration within a horizon, which would affect the configurations of conserved quantities and thus the corresponding thermodynamic system.

• To obtain a new algorithm for the construction of a rotating regular black hole. As a mathematical technique, the NJA transforms a static and spherically symmetric black hole into a rotating and axially symmetric one. Owing to its nonphysical defects, such as a complex radial coordinate, we may require a more suitable algorithm for constructing a rotating regular black hole.

• To consider the reaction caused by particle incidence. Our present discussions neglect the spacetime reaction during particle incidence. If we consider the impact of particle incidence on field equations, the original relationships among conserved quantities may be altered, leading to modifications in thermodynamic laws.

• To introduce the quantum correction in black hole models. Studies have shown that the first law of thermodynamics for some rotating black holes can be self-consistent by limiting the running Newton coupling under the asymptotically safe gravity [65] . This method provides an alternative route to recover the first law of thermodynamics for the rotating regular black holes we have analyzed. An interesting aspect is extending the method to the recovery of the second and third laws.

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