Electrodynamics with violations of Lorentz and *U*(1) gauge symmetries and their Hamiltonian structures*

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Abstract: This study aims to investigate Lorentz/$U(1)$ gauge symmetry-breaking electrodynamics in the framework of the standard-model extension and analyze the Hamiltonian structure for the theory with a specific dimension $d \leq 4$ of Lorentz breaking operators. For this purpose, we consider a general quadratic action of the modified electrodynamics with Lorentz/gauge-breaking operators and calculate the number of independent components of the operators at different dimensions in gauge invariance and breaking. With this general action, we then analyze how Lorentz/gauge symmetry-breaking can change the Hamiltonian structure of the theories by considering Lorentz/gauge-breaking operators with dimension $d \leq 4$ as examples. We show that the Lorentz-breaking operators with gauge invariance do not change the classes of the theory constrains and the number of physical degrees of freedom of the standard Maxwell electrodynamics. When $U(1)$ gauge symmetry-breaking operators are present, the theories generally lack a first-class constraint and have one additional physical degree of freedom compared to the standard Maxwell electrodynamics.

Keywords: standard model extension, electrodynamics, Lorentz violations

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I. INTRODUCTION

The standard model (SM) successfully describes the fundamental constituents of matter using quarks, leptons, gauge bosons, and Higgs bosons. It effectively explains phenomena involving elementary particles and their interactions. The last predicted elementary particle of the SM, the Higgs boson, was experimentally confirmed in 2012 [1, 2], offering a satisfying conclusion to the development of this great theory. General relativity (GR) links gravity with the curvature of spacetime and provides a highly successful description of gravity-related phenomena. Since its inception, it has been recognized as a great theory and continues to be corroborated through various experiments. Its basic principles and predictions have been confirmed through multiple observations and measurements, establishing its foundational position in modern physics. In 2016, gravitational waves were observed in experiments [3], which further reinforced the concept of GR. These two theories are expected to be unified at the Planck scale and potentially exhibit observable quantum gravity effects at accessible low-energy scales. This signal may be related to Lorentz symmetry breaking and can be described by an effective field theory [4].

To construct a consistent effective field theory that incorporates both GR and the SM, Kostelecký and Colladay proposed an effective field theory known as the SM extension (SME) [5, 6], which features general Lorentz and CPT violations. The measured and derived values of coefficients for Lorentz and CPT violations in the SME can be found in the data table organized by Kostelecký and Russell [7]. In recent years, research on cosmic microwave background (CMB) radiation and ultra high energy cosmic rays (UHECR) has provided new opportunities for studying the pure photon sector of the SME. One reason for this is that any prediction involving the pure photon part that deviates from the SM could potentially indicate Lorentz violation originating from the pure

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photon sector of the SME. A large amount of research has been conducted in this area [8−23]. Compared to conventional Maxwell electrodynamics, the pure-photon sector of the SME includes additional Lorentz-breaking terms, which can be classified as CPT-even and CPT-odd. The inclusion of these terms leads to the emergence of new effects, which has spurred extensive research in this field. Studying the general aspects of the pure photon sector is a challenging task. Ref. [24] proposed a general electrodynamics extension theory with a quadratic action, which can be used to describe many related phenomena, including photon interactions [25], the optical activity of media [26], the Lorentz-invariance-violating (LIV) term [27, 28], Chern-Simons term [29], nonminimal SME [30], and other related phenomena involving photons [31]. Theories with Lorentz-breaking operators of dimension 4 have received considerable attention, including LIV [32], Carroll-Field-Jackiw (CFJ) [33], and Proca electrodynamics [34], where the first two are $U(1)$ gauge invariant theories and the third involves $U(1)$ gauge symmetry breaking. The authors of Refs. [27, 28, 35, 36] conducted Hamiltonian analyses of the above three theories and identified their constraint structures, which motivates us to study the constraint structure of more general theories with a specific dimension $d \leq 4$.

The Lagrangian density of quadratic electrodynamics, given in [24], is a quadratic polynomial in the photon field A_{α_1} and its higher-order derivatives $\partial_{\alpha_3}\ldots\partial_{\alpha_d}A_{\alpha_2}$ with $d \geq 2$. The constant coefficients $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$, which are contracted with $A_{\alpha_1}\partial_{\alpha_3}\ldots\partial_{\alpha_d}A_{\alpha_2}$, remain invariant under coordinate transformations, which leads to a violation of the Lorentz symmetry of the theory. These constant coefficients can be regarded as originating from the vacuum expectation value of an operator in the underlying theory, or the dominant component of dynamic background fields, or an averaged effect. Through dimensional analysis, it can be found that the constants $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ associated with $A_{\alpha_1}\partial_{\alpha_3}\ldots\partial_{\alpha_d}A_{\alpha_2}$ of dimension d must have the dimension $4-d$. Some researchers believe that theories with power series that are renormalizable have the mass dimension $d \leq 4$ [24, 37], and it is these theories with mass dimension $d \leq 4$ that are mainly explored in this study. Typically, there are also theories that contain only the first time derivative of the field A_μ, which is beneficial because this avoids any potential Ostrogradsky instability [38, 39].

In this study, we extend the $U(1)$ gauge-invariant Lagrangian density of quadratic electrodynamics described in [24] to one that includes $U(1)$ gauge symmetry breaking terms. We also perform a Hamiltonian analysis for renormalizable specific dimension cases of $d \leq 4$. Our purpose is to clarify how the $U(1)$ gauge breaking terms affect the constraint structure and physical degrees of freedom of the theory.

Our results indicate that the Lagrangian density of general quadratic electrodynamics is a combination of five terms. The first corresponds to the Lagrangian density of standard electrodynamics, the second is $U(1)$ gauge invariant, the third contains both $U(1)$ gauge invariance and breaking, and the remaining two are $U(1)$ gauge breaking. The Hamiltonian analysis of the general gauge-breaking system with a specific dimension $d \leq 4$ reveals that the system with $d = 4$ requires additional conditions to become a constrained system, whereas systems with $d = 2$ and $d = 3$ do not. It also shows that the Lorentz-breaking operators with gauge invariance do not change the classes of the theory constraints and the number of physical degrees of freedom of the standard Maxwell electrodynamics. When the $U(1)$ gauge symmetry-breaking operators are presented, the theories in general lack a first-class constraint and have one additional physical degree of freedom compared to the standard Maxwell electrodynamics.

The structure of this paper is as follows. The basic theory is discussed in Sec. II, where we give the number of independent components of the Lorentz-breaking operators with $U(1)$ gauge violation and extend it to the general Lagrangian density containing $U(1)$ gauge-breaking terms. Sec. III explores the Hamiltonian structure and degrees of freedom of theories with a specific dimension $d \leq 4$. In Sec. IV, we apply the obtained results to specific models and derive the results for LIV, CFJ, and Proca electrodynamics. Our summary is presented in Sec. V.

For clarity and conciseness, we use two conventions:

1. The Greek indices range from 0 to 3 and the Latin indices range from 1 to 3. The metric of the background spacetime $\eta_{\mu\nu} \equiv (1,-1,-1,-1)$;

2. The time argument of the vector field A_μ is suppressed throughout the study, namely, $A_\mu(\mathbf{x}) \equiv A_\mu(t,\mathbf{x})$.

II. ELECTRODYNAMICS WITH VIOLATIONS OF LORENTZ AND $U(1)$ GAUGE SYMMETRY

In this section, we present a brief introduction of electrodynamics with quadratic action in the pure photon sector in the framework of the SME with both Lorentz and $U(1)$ gauge symmetry breaking. This represents an extension of the Lorentz-violating modified electrodynamics with $U(1)$ gauge invariance in [24] by including the $U(1)$ gauge symmetry-breaking operators in the quadratic action. In the construction of this theory, we also analyze the properties of the coefficients for the Lorentz- and $U(1)$ gauge-violating operators.

For this purpose, following a similar construction to that performed in [24], we start with the general quadratic action for Lorentz-violating electrodynamics in the pure photon sector, which can be written as [24]

$$S = \int d^4x\mathcal{L} \tag{1}$$

with

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{d=2}^{\infty}\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}A_{\alpha_1}\partial_{\alpha_3}\ldots\partial_{\alpha_d}A_{\alpha_2}, \tag{2}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ are constant coefficients with the mass dimension $4-d$. One possible explanation for the coefficients $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ originates from non-zero vacuum expectation values to the Lorentz-tensor fields. Note that each term associated with the coefficient $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ violates CPT if d is odd and preserves CPT if d is even.

The symmetry among the indices $\{\alpha_3, \alpha_4,\ldots,\alpha_d\}$ of the tensor $\partial_{\alpha_3}\ldots\partial_{\alpha_d}$ and the use of integration by parts result in two properties of the coefficients $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$. The first is the total symmetry in the $d-2$ indices $\{\alpha_3, \alpha_4,\cdots,\alpha_d\}$, and the second is the symmetry of the two indices $\{\alpha_1,\alpha_2\}$ when d is even and antisymmetry when d is odd. Depending on the specific intrinsic symmetries of the Lorentz-violating operators, the coefficients $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ can be decomposed into five representations [24]. When one imposes the conditions of $U(1)$ gauge invariance on the Lorentz-violating operators, these five representations are reduced to two representations: one corresponds to the CPT-even coefficient and the other corresponds to the CPT-odd coefficient [24].

A. Lorentz-violating electrodynamics with $U(1)$ gauge invariance

Let us first consider Lorentz-violating electrodynamics with $U(1)$ gauge invariance. $U(1)$ gauge invariance is a symmetry of the theory under the $U(1)$ gauge transformation [24]

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu\Lambda, \tag{3}$$

where Λ is an arbitrary function. The variation in the action (1) under this gauge transformation reads as

$$\delta_g S = -\sum_{d=2}^{\infty}\int d^4x\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\cdots\alpha_d}\Lambda \times\partial_{\alpha_3}\cdots\partial_{\alpha_d}(\partial_{[\alpha_1}A_{\alpha_2]\pm} + \frac{1}{2}\partial_{[\alpha_1}\partial_{\alpha_2]\pm}\Lambda) = 0, \tag{4}$$

where "+/−" corresponds to an even/odd dimension, and the brackets $[]_+$ and $[]_-$ indicate symmetrization and antisymmetrization, respectively. Specifically, the terms in the large brackets of the above expression can be written as

$$\partial_{[\alpha_1}A_{\alpha_2]\pm} + \tfrac{1}{2}\partial_{[\alpha_1}\partial_{\alpha_2]\pm}\Lambda = \begin{cases}\partial_{\alpha_1}A_{\alpha_2} + \partial_{\alpha_2}A_{\alpha_1} + \partial_{\alpha_1}\partial_{\alpha_2}\Lambda, & d \text{ is even,}\\ \partial_{\alpha_1}A_{\alpha_2} - \partial_{\alpha_2}A_{\alpha_1}, & d \text{ is odd.}\end{cases} \tag{5}$$

$U(1)$ gauge invariance requires the variation in the action in (4) to vanish. When d is even, because the two first indices α_1 and α_2 are symmetric, to make (4) vanish, both indices α_1 and α_2 must be antisymmetric with one of $\{\alpha_3,\alpha_4,\cdots,\alpha_d\}$ [24]. With these properties, it is straightforward to infer that all the CPT-even operators with $d=2$ are gauge-violating and the gauge invariance requires $d \geq 4$. Using these properties, we can also count the number of independent components of the CPT-even operators, which leads to

$$N_F = (d+1)d(d-3). \tag{6}$$

Similarly, when d is odd, the first two indices α_1 and α_2 are antisymmetric, and the vanishing of (4) requires α_1 and α_2 to be antisymmetric with one of $\{\alpha_3,\alpha_4,\cdots,\alpha_d\}$. For this case, only operators with $d\geq 3$ are allowed and the number of independent components of the CPT-odd operators is

$$N_{AF} = \frac{1}{2}(d+1)(d-1)(d-2). \tag{7}$$

In Table 1, we summarize the number of independent components of the CPT-odd and CPT-even operators.

B. Lorentz-violating electromagnetics with the violations of $U(1)$ gauge symmetry

Now, we consider the case with the breaking of $U(1)$ gauge symmetry. When $U(1)$ gauge symmetry is violated, the variation in the action in (4) does not need to vanish. For this case, extra conditions are not required on the coefficients $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ to ensure the gauge invariance of the theory. For CPT-even operators without gauge invariance, as mentioned before, while the indices

Table 1. Number of independent components of the Lorentz-violating operators with/without $U(1)$ gauge invariance.

d	CPT	Gauge invariance	Number of independent components
2	even	no	10
even, ≥ 4	even	yes	$(d+1)d(d-3)$
		no	$\frac{2}{3}(d+2)(d+1)d$
odd, ≥ 3	odd	yes	$\frac{1}{2}(d+1)(d-1)(d-2)$
		no	$\frac{1}{2}(d+2)(d+1)(d-1)$

$\{\alpha_3,\alpha_4,\cdots,\alpha_d\}$ in the CPT-even coefficients $\mathcal{K}_{(d)}^{\alpha_1\alpha_2\alpha_3\ldots\alpha_d}$ are symmetric, the first two indices α_1 and α_2 are also symmetric. Similarly, for CPT-odd operators without gauge invariance, the indices $\{\alpha_3,\alpha_4,\cdots,\alpha_d\}$ are symmetric, whereas the first two indices α_1 and α_2 are antisymmetric. We can then count the number of independent components of the CPT-even and CPT-odd operators with a specific dimension d, as summarized in Table 1.

For later convenience, we can decompose the Lagrangian density (2) into two parts, the $U(1)$ gauge invariance and gauge-violating parts. This involves rewriting the Lorentz-breaking coefficients in five different forms with distinct symmetries. The specific representation of the decomposition of these five forms can be found in [24]. Subsequently, the Lagrangian density (2) can be rewritten as

$$\begin{aligned}\mathcal{L}= & -\frac{1}{4}F_{\alpha\nu}F^{\alpha\nu}\\ & +\sum_{\text{even } d=2}^{\infty}\mathcal{K}_{(d)}^{(1)\alpha_1\alpha_2\alpha_3\ldots\alpha_d}A_{\alpha_1}\partial_{\alpha_3}\ldots\partial_{\alpha_d}A_{\alpha_2}\\ & +\sum_{d=3}^{\infty}\mathcal{K}_{(d)}^{(2)\alpha_1\mu\nu\alpha_2\ldots\alpha_{d-2}}A_{\alpha_1}\partial_\nu\partial_{\alpha_2}\ldots\partial_{\alpha_{d-2}}A_\mu\\ & +\sum_{d=3}^{\infty}\mathcal{K}_{(d)}^{(3)\mu\alpha_1\nu\alpha_2\ldots\alpha_{d-2}}A_\mu\partial_\nu\partial_{\alpha_2}\ldots\partial_{\alpha_{d-2}}A_{\alpha_1}\\ & +\sum_{d=4}^{\infty}\mathcal{K}_{(d)}^{(4)\mu\nu\rho\sigma\alpha_1\ldots\alpha_{d-4}}A_\mu\partial_\rho\partial_\sigma\partial_{\alpha_1}\ldots\partial_{\alpha_{d-4}}A_\nu\\ & +\sum_{\text{odd } d=3}^{\infty}\mathcal{K}_{(d)}^{(5)\mu\nu\rho\alpha_1\ldots\alpha_{d-3}}A_\mu\partial_\rho\partial_{\alpha_1}\ldots\partial_{\alpha_{d-3}}A_\nu,\end{aligned}\tag{8}$$

where the five coefficients $\mathcal{K}_{(d)}^{(i)\alpha_1\alpha_2\alpha_3\cdots\alpha_d}$ with $i=1,2,3,4,5$ are distinguished by their distinct symmetries among the indices $\{\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_d\}$. The coefficients $\mathcal{K}_{(d)}^{(1)\alpha_1\alpha_2\alpha_3\cdots\alpha_d}$ are all CPT-even with $d\geq 2$, whereas their indices $\{\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_d\}$ are all symmetric. We can clearly infer from (4) that these coefficients violate the $U(1)$ gauge symmetry of the theory. The coefficients $\mathcal{K}_{(d)}^{(2)\alpha_1\mu\nu\alpha_2\cdots\alpha_{d-2}}$ are either CPT-even or CPT-odd with $d\geq 3$. Except for the two indices μ and ν, which are antisymmetric, the remaining indices $\{\alpha_1,\alpha_2,\cdots,\alpha_{d-2}\}$ are all symmetric. Note that these coefficients are also gauge-violating. Similarly, $\mathcal{K}_{(d)}^{(3)\mu\alpha_1\nu\alpha_2\cdots\alpha_{d-2}}$ can also be either CPT-even or CPT-odd with $d\geq 3$. The two indices of this coefficient, μ and ν, are antisymmetric, and the remaining indices $\{\alpha_1,\alpha_2,\cdots,\alpha_{d-2}\}$ are all symmetric. We can verify that these coefficients also break the $U(1)$ gauge symmetry of the theory. Then, for the indices of the coefficients $\mathcal{K}_{(d)}^{(4)\mu\nu\rho\sigma\alpha_1\ldots\alpha_{d-4}}$, the indices μ and ρ and the indices ν and σ are both antisymmetric. We also note that these coefficients are symmetric upon interchanging the two pairs of indices (μ,ρ) and (ν,σ). By inspecting the variation in the action (4) with these coefficients, we find that the CPT-odd operators with $\mathcal{K}_{(d)}^{(4)\mu\nu\rho\alpha_1\cdots\alpha_{d-4}}$ violate gauge invariance, whereas the CPT-even ones are gauge invariant. The final coefficients $\mathcal{K}_{(d)}^{(5)\mu\nu\rho\alpha_1\ldots\alpha_{d-3}}$ are CPT-odd with $d\geq 3$, and their three indices $\{\mu,\nu,\rho\}$ are antisymmetric. These coefficients preserve the $U(1)$ gauge symmetry of the theory.

To simplify the later handling of the Hamiltonian analysis of the theory, we can rewrite the Lagrangian density of the theory in a compact form:

$$\begin{aligned}\mathcal{L}= & -\frac{1}{4}F_{\alpha\nu}F^{\alpha\nu}+A_{\alpha_1}\hat{\mathcal{K}}^{(1)\alpha_1\alpha_2}A_{\alpha_2}+\frac{1}{2}A_{\alpha_1}\hat{\mathcal{K}}^{(2,3)\mu\nu\alpha_1}F_{\nu\mu}\\ & -\frac{1}{4}F_{\mu\rho}\hat{\mathcal{K}}^{(4)\mu\rho\nu\sigma}F_{\nu\sigma}+\frac{1}{2}\epsilon^{\kappa\mu\nu\rho}A_\mu\hat{\mathcal{K}}_\kappa^{(5)}F_{\nu\rho},\end{aligned}\tag{9}$$

where

$$\hat{\mathcal{K}}^{(1)\alpha_1\alpha_2}\equiv\sum_{d=\text{even}}\mathcal{K}_{(d)}^{(1)\alpha_1\alpha_2\alpha_3\ldots\alpha_d}\partial_{\alpha_3}\ldots\partial_{\alpha_d},\tag{10}$$

$$\hat{\mathcal{K}}^{(4)\mu\rho\nu\sigma}\equiv\sum_{d=4}^{\infty}\mathcal{K}_{(d)}^{(4)\mu\nu\rho\sigma\alpha_1\ldots\alpha_{d-4}}\partial_{\alpha_1}\ldots\partial_{\alpha_{d-4}},\tag{11}$$

$$\hat{\mathcal{K}}_\kappa^{(5)}\equiv\frac{\epsilon_{\kappa\mu\nu\rho}}{6}\sum_{d=\text{odd}}\mathcal{K}_{(d)}^{(5)\mu\nu\rho\alpha_1\ldots\alpha_{d-3}}\partial_{\alpha_1}\ldots\partial_{\alpha_{d-3}},\tag{12}$$

and

$$\begin{aligned}\hat{\mathcal{K}}^{(2,3)\mu\nu\alpha_1}\equiv & \sum_{d=3}^{\infty}\left[\mathcal{K}_{(d)}^{(2)\alpha_1\mu\nu\alpha_2\ldots\alpha_{d-2}}+(-1)^d\mathcal{K}_{(d)}^{(3)\mu\alpha_1\nu\alpha_2\ldots\alpha_{d-2}}\right]\\ & \times\partial_{\alpha_2}\ldots\partial_{\alpha_{d-2}}.\end{aligned}\tag{13}$$

The index symmetries of the operators in (9) are shown in Table 2. The indices enclosed in the same brackets {} of the second and third columns represent the symmetry and antisymmetry between them, respectively. $\{\mu\rho,\nu\sigma\}$ in the second column indicates that the corresponding operators are symmetric when the two pairs (μ,ρ) and (ν,σ) are exchanged. The fourth column displays the conditions under which each class of operators appears.

III. HAMILTONIAN STRUCTURE OF THE THEORYS

In this section, we perform a Hamiltonian analysis on the Lorentz-violating electromagnetics with and without $U(1)$ gauge invariance using the Dirac-Bergmann proced-

Table 2. Symmetries of the indices of the operator coefficients in (9).

Coefficient	Symmetry	Antisymmetry	d
$\hat{\mathcal{K}}^{(1)\alpha_1\alpha_2}$	$\{\alpha_1,\alpha_2\}$	$\cdots$	even, ≥ 2
$\hat{\mathcal{K}}^{(2,3)\mu\nu\alpha_1}$	$\cdots$	$\{\mu,\nu\}$	≥ 3
$\hat{\mathcal{K}}^{(4)\mu\rho\nu\sigma}$	$\{\mu\rho,\nu\sigma\}$	$\{\mu,\rho\},\{\nu,\sigma\}$	≥ 4
$\hat{\mathcal{K}}^{(5)}_{\kappa}$	$\cdots$	$\cdots$	odd, ≥ 3

ure [40–44]. For simplicity, we focus on the theory with Lagrangian densities with a specific dimension $d\leq 4$ of Lorentz/gauge-breaking operators.

A. $d=4$

We start the Hamiltonian analysis with $d=4$. For this case, the Lagrangian density of the theory reduces to

$$\mathcal{L}_{(4)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}U^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma - \frac{1}{2}V^{\mu\nu\rho\sigma}F_{\nu\mu}\partial_\sigma A_\rho - \frac{1}{4}W^{\mu\rho\nu\sigma}F_{\mu\rho}F_{\nu\sigma}, \tag{14}$$

where

$$U^{\mu\nu\rho\sigma} = 2\mathcal{K}^{(1)\mu\nu\rho\sigma}_{(4)}, \tag{15}$$

$$V^{\mu\nu\rho\sigma} = \mathcal{K}^{(2)\rho\mu\nu\sigma}_{(4)} + \mathcal{K}^{(3)\mu\rho\nu\sigma}_{(4)}, \tag{16}$$

$$W^{\mu\rho\nu\sigma} = \mathcal{K}^{(4)\mu\nu\rho\sigma}_{(4)}. \tag{17}$$

The terms in $\mathcal{L}_{(4)}$ with coefficients $U^{\mu\nu\rho\sigma}$ and $V^{\mu\nu\rho\sigma}$ break the $U(1)$ gauge symmetry. The four indices $\{\mu,\nu,\rho,\sigma\}$ in $U^{\mu\nu\rho\sigma}$ are totally symmetric, whereas $V^{\mu\nu\rho\sigma}$ are antisymmetric in $\{\mu,\nu\}$ and symmetric in $\{\rho,\sigma\}$. The terms with coefficients $W^{\mu\rho\nu\sigma}$ are gauge invariant, and the corresponding indices of the coefficients $W^{\mu\rho\nu\sigma}$ have the same symmetric properties as the Riemann tensor, i.e., the first pair $\{\mu\rho\}$ and last pair $\{\nu\sigma\}$ of $W^{\mu\rho\nu\sigma}$ are both antisymmetric but symmetric upon interchanging the two pairs. By varying the action (1) with the above Lagrangian with respect to the field A_μ, we obtain the equation of motion of the electromagnetics, i.e.,

$$\partial_\mu\Big(F^{\mu\nu} + U^{\mu\nu\rho\sigma}\partial_\rho A_\sigma + V^{\nu\mu\rho\sigma}\partial_\sigma A_\rho + \frac{1}{2}V^{\rho\sigma\nu\mu}F_{\sigma\rho} + W^{\rho\sigma\mu\nu}F_{\rho\sigma}\Big) = 0. \tag{18}$$

Now, to perform the Hamiltonian analysis, it is convenient to define the conjugate momentum

$$\pi^\mu \equiv \frac{\partial\mathcal{L}_{(4)}}{\partial\dot{A}_\mu} = -F^{0\mu} - U^{0\mu\nu\rho}\partial_\nu A_\rho - V^{\mu0\nu\rho}\partial_\rho A_\nu - \frac{1}{2}V^{\nu\rho\mu0}F_{\rho\nu} - W^{0\mu\nu\rho}F_{\nu\rho}, \tag{19}$$

which sets the fundamental Poisson brackets (PB) as

$$\{A_\mu(\mathbf{x}),\pi^\nu(\mathbf{y})\} = \delta^\nu_\mu\delta(\mathbf{x}-\mathbf{y}). \tag{20}$$

From the conjugate momentum, the canonical Hamiltonian of the system can be expressed as

$$\mathcal{H}_{(4)} \equiv \dot{A}_\mu\pi^\mu - \mathcal{L}_{(4)}. \tag{21}$$

In the Hamiltonian analysis, a significant portion of the work involves computing the Poisson bracket between the Hamiltonian and functions in the phase space. In this context, it is important to express the Hamiltonian as a function of the conjugate momenta and coordinates. To do so, let us write the time and spatial components of the conjugate momentum for (19),

$$\pi^0 = -U^{00\mu\nu}\partial_\mu A_\nu - \frac{1}{2}V^{\mu\nu00}F_{\nu\mu}, \tag{22}$$

$$\pi^k = D^k{}_i F^{0i} + N^k, \tag{23}$$

where the matrices $D^k{}_i$ and N^k are defined as

$$D^k{}_i \equiv -\delta^k{}_i - V_{i0}{}^{k0} - V^{k0}{}_{0i} - 2W^{0k}{}_{0i} - U^{0k}{}_{0i}, \tag{24}$$

$$N^k \equiv -\Big(\frac{1}{2}V^{jik0} + W^{0kij}\Big)F_{ij} - (U^{0k00} + V^{k000})\partial_0 A_0 - 2(U^{0k0i} + V^{k00i})\partial_i A_0 - (U^{0kij} + V^{k0ij})\partial_i A_j. \tag{25}$$

From the expression of $D^k{}_i$, it is easy to obtain $D^k{}_i = D_i{}^k$. Further analysis is required to determine the constraint structure of the system. For this purpose, we assume that $D^k{}_i$ is a non-degenerate matrix, just like in gauge invariance [27, 28], such that

$$F^{0i} = (D^{-1})^i{}_k(\pi^k - N^k). \tag{26}$$

After some manipulations, we can finally express the canonical Hamiltonian density as a function of the field

$$\mathcal{H}_{(4)} = \pi^k\partial_k A_0 + \frac{1}{2}(D^{-1})_{jl}(\pi^l - N^l)\left[\pi^j - N^j + (V^{0j}{}_{0k} - V_{k0}{}^{j0})(D^{-1})^k{}_i(\pi^i - N^i) - 2(U^{0j00} + V^{j000})\partial_0 A_0\right]$$
$$+ \frac{1}{2}F_{ij}(\frac{1}{2}F^{ij} + \frac{1}{2}W^{ijkl}F_{kl} + 2V^{ji0k}\partial_k A_0 + V^{jikl}\partial_l A_k) + (\frac{1}{2}U^{ijkl}\partial_i A_j + 2U^{0jkl}\partial_j A_0)\partial_k A_l$$
$$+ 2U^{00kl}\partial_k A_0\partial_l A_0 - \frac{1}{2}U^{0000}\partial_0 A_0\partial_0 A_0, \tag{27}$$

where we keep the term $\partial_0 A_0$ explicit because when studying the constrained system with $d=4$ in the following, we must impose some constraints on the Lorentz-breaking coefficients. A option here is exactly $U^{0000}=0$, which can be considered as throwing away the term with respect to $\partial_0 A_0$.

Following the Dirac-Bergmann algorithm, our next step is to write the total Hamiltonian density of the system, which is composed of the canonical Hamiltonian density and primary constraints. However, for the system in $d=4$, the existence of primary constraints requires certain additional conditions.

1. *Conditions for the existence of the constraints*

Let us consider the system described by (14) and find which of its conditions contain constraints. According to the Dirac-Bergmann procedure, the presence of constraints in the system with Lagrangian density $\mathcal{L}_{(4)}$ can be determined by checking whether the Hessian matrix $\frac{\partial^2\mathcal{L}}{\partial\dot{A}_\mu\partial\dot{A}_\nu}$ is degenerate. Therefore, to make the system with Lagrangian density $\mathcal{L}_{(4)}$ a constrained system, it is sufficient to provide the condition that the matrix $\frac{\partial^2\mathcal{L}_{(4)}}{\partial\dot{A}_\mu\partial\dot{A}_\nu}$ is degenerate. The Hessian matrix takes the form

$$\frac{\partial^2\mathcal{L}_{(4)}}{\partial\dot{A}_\mu\partial\dot{A}_\nu} = -(\eta^{\mu\nu} - \eta^{0\mu}\delta_0{}^\nu) - 2W^{0\mu0\nu} - U^{0\mu0\nu}$$
$$- V^{\mu0\nu0} - V^{\nu0\mu0}. \tag{28}$$

When the Hessian matrix is non-degenerate, the system does not possess any constraint that is not of interest. It is easy to observe that, for the gauge invariant case, because

$$U^{\mu\nu\rho\sigma} = 0 = V^{\mu\nu\rho\sigma}, \tag{29}$$

the Hessian matrix is identically degenerate. This indicates that the theory with $U(1)$ gauge symmetry always has constraints. However, when $U(1)$ gauge symmetry is broken, whether the theory possesses constraints depends on the specific forms of the gauge-breaking coefficients $U^{\mu\nu\rho\sigma}$ and $V^{\mu\nu\rho\sigma}$.

For our purposes, we intend to consider the case when the Hessian matrix is degenerate, either with gauge invariance or gauge breaking. Considering the complexity of the Lagrangian density in the $d=4$ case, for simplicity, let us only focus on the following simple case to make the Hessian matrix degenerate:

$$U^{0000} = 0, \quad U^{000i} = -V^{i000}. \tag{30}$$

As shown in the above equation, this constraint condition only restricts the gauge-breaking coefficients and the first row and column of the Hessian matrix that are 0, which allows the subsequent conclusions to be fully applicable to gauge-invariant systems. For clarity of discussion, similar to the gauge invariant case [27, 28], we assume that the 3×3 matrix $\frac{\partial^2\mathcal{L}_{(4)}}{\partial\dot{A}_i\partial\dot{A}_j}$ is non-degenerate.

Thus, for the $d=4$ case, hereafter, we analyze the system with

$$\mathcal{L}_{(4)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}U^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma$$
$$-\frac{1}{2}V^{\mu\nu\rho\sigma}F_{\nu\mu}\partial_\sigma A_\rho - \frac{1}{4}W^{\mu\rho\nu\sigma}F_{\mu\rho}F_{\nu\sigma}, \tag{31}$$

with conditions $U^{0000}=0$ and $U^{000i} = -V^{i000}$.

2. *Canonical analysis*

After obtaining the constrained system with a Lagrangian density of (31), we analyze the constraint structure of the system. Under the condition of (30), the conjugate momentum in (23) can be written as

$$\pi^0 = -(U^{00ij} + V^{ji00})\partial_i A_j - 2U^{000i}\partial_i A_0, \tag{32}$$

$$\pi^k = D^k{}_i F^{0i} + N^k, \tag{33}$$

where

$$N^k = -(\frac{1}{2}V^{jik0} + W^{0kij})F_{ij} - 2(U^{0k0i} + V^{k00i})\partial_i A_0$$
$$-(U^{0kij} + V^{k0ij})\partial_i A_j. \tag{34}$$

The rank of matrix $\frac{\partial^2\mathcal{L}_{(4)}}{\partial\dot{A}_\mu\partial\dot{A}_\nu}$ is 3, giving rise to a unique primary constraint

$$\phi_1^{(4)}=\pi^0+(U^{00ij}+V^{ji00})\partial_i A_j+2U^{000i}\partial_i A_0\approx 0. \tag{35}$$

The symbol "≈" is known as the weak equality symbol, which implies that the equation only holds at the constraint surfaces but not throughout phase space. With the expressions of (30), the canonical Hamiltonian density (27) also becomes

$$\begin{aligned}\mathcal{H}_{(4)}=&\pi^k\partial_k A_0+\frac{1}{2}(D^{-1})_{jl}(\pi^l-N^l)\\&\times\left(\pi^j-N^j+(V^{0j}{}_{0k}-V_{k0}{}^{j0})(D^{-1})^k{}_i(\pi^i-N^i)\right)\\&+\frac{1}{2}F_{ij}\Big(\frac{1}{2}F^{ij}+\frac{1}{2}W^{ijkl}F_{kl}+2V^{ji0k}\partial_k A_0+V^{jikl}\partial_l A_k\Big)\\&+\left(\frac{1}{2}U^{ijkl}\partial_i A_j+2U^{0jkl}\partial_j A_0\right)\partial_k A_l+2U^{00kl}\partial_k A_0\partial_l A_0,\end{aligned} \tag{36}$$

and the total Hamiltonian density can be written as

$$\mathcal{H}_{(4)T}=\mathcal{H}_{(4)}+u^{(4)}\phi_1^{(4)}, \tag{37}$$

which gives the total Hamiltonian in the form

$$H_{(4)T}=\int\mathcal{H}_{(4)T}(\mathbf{x})\mathrm{d}^3x, \tag{38}$$

where $u^{(4)}$ is an arbitrary Lagrangian multiplier. Note that $\partial_t A_0(\mathbf{x})=\{A_0(\mathbf{x}),H_{(4)T}\}=u^{(4)}(\mathbf{x})$ because the Poisson bracket between $A_0(\mathbf{x})$ and $\mathcal{H}_{(4)}(\mathbf{y})$ is zero, and that between $A_0(\mathbf{x})$ and other terms in $\phi_1^{(4)}(\mathbf{y})$, except for $\pi(\mathbf{y})$, is also zero. This gives meaning to the coefficient $u^1(\mathbf{x})$: the time derivative of $A_0(\mathbf{x})$.

Following the standard Dirac-Bergmann procedure, we then analyze the requirement for the preservation of the primary constraint. Such a requirement is also known as the consistency condition of the primary constraint, which requires that the time derivative of this constraint also vanishes. The consistency condition of the primary constraint, i.e.,

$$\dot{\phi}_1^{(4)}(\mathbf{x})=\{\phi_1^{(4)}(\mathbf{x}),H_{(4)T}\}\approx 0, \tag{39}$$

gives rise to a secondary constraint of the system,

$$\begin{aligned}\phi_2^{(4)}=&\partial_k\pi^k+M^m{}_i(\partial_m\pi^i-\partial_m N^i)\\&+(5U^{00ij}+V^{ji00})\partial_i\partial_j A_0\\&+V^{ji0k}\partial_k F_{ij}+2U^{0jkl}\partial_j\partial_k A_l\approx 0,\end{aligned} \tag{40}$$

where

$$\begin{aligned}M^m{}_i=&\frac{1}{2}\Big[(D^{-1})_{ij}+(D^{-1})_{ji}\\&+2(V^{0l}{}_{0k}-V_{k0}{}^{l0})(D^{-1})_{li}(D^{-1})^k{}_j\Big]\\&\times(3U^{0j0m}+2V^{j00m}+V^{jm00}),\end{aligned} \tag{41}$$

which indicates that the structure of Gauss's law is influenced by the gauge-breaking term.

Similar to the primary constraint, the secondary constraint $\phi_2\approx 0$, being a constraint itself, also has a corresponding consistency condition, which is

$$\dot{\phi}_2^{(4)}(\mathbf{x})=\{\phi_2^{(4)}(\mathbf{x}),H_{(4)T}\}\approx 0. \tag{42}$$

This condition then leads to

$$\begin{aligned}&O_1^{ik}{}_j\partial_i\partial_k(\pi^j-N^j)+O_2^{ijk}\partial_i\partial_j\partial_k A_0+O_3^{jilk}\partial_j\partial_i\partial_l A_k\\&+O_4^{klij}\partial_k\partial_l F_{ij}+T^{ij}\partial_i\partial_j u^{(4)}\approx 0,\end{aligned} \tag{43}$$

where

$$O_3^{jilk}\equiv(\delta^j{}_m+M^j{}_m)\Big(\frac{1}{2}V^{mikl}+U^{imlk}\Big), \tag{44}$$

$$O_4^{klij}\equiv\frac{1}{2}(\delta^k{}_m+M^k{}_m)(\eta^{li}\eta^{mj}+W^{ijlm}+V^{jiml}), \tag{45}$$

$$\begin{aligned}O_1^{ik}{}_j\equiv&\frac{1}{2}\Big[(\delta^i{}_m+M^i{}_m)\big(V^{mkl0}+2W^{0lkm}+U^{0lkm}+V^{l0km}\big)\\&+M^i{}_m(V^{lkm0}+2W^{0mkl}+U^{0mkl}+V^{m0kl})\\&+2(V^{li0k}+U^{0ikl})\Big]\Big[(D^{-1})_{jl}+(D^{-1})_{lj}\\&+2(V^{0i_1}{}_{0k_1}-V_{k_10}{}^{i_10})(D^{-1})_{i_1j}(D^{-1})^{k_1}{}_l\Big],\end{aligned} \tag{46}$$

$$O_2^{ijk}\equiv(\delta^i{}_l+M^i{}_l)(V^{lj0k}+2U^{0jkl})+M^i{}_l(U^{0ljk}+V^{l0jk})+2U^{0ijk}, \tag{47}$$

and

$$T^{ij}\equiv 6U^{00ij}+(3U^{00jk}+V^{kj00}+2V^{k00j})M^i{}_k. \tag{48}$$

For the gauge invariant case, $T^{ij}=0$, and we can check that the condition (43) is satisfied identically. When the $U(1)$ gauge symmetry of the theory is violated, T^{ij} are generally nonzero; therefore, (43) represents a restriction on the Lagrange multiplier $u^{(4)}$. One special case is $T^{ij}=0$ for some specific combination of the gauge breaking coefficients, for which a new constraint arises:

$$\phi_3^{(4)} = O_1^{ik}{}_j(\partial_i\partial_k\pi^j - \partial_i\partial_k N^j) + O_2^{ijk}\partial_i\partial_j\partial_k A_0 + O_3^{jilk}\partial_j\partial_i\partial_l A_k + O_4^{klij}\partial_k\partial_l F_{ij} \approx 0. \tag{49}$$

With this constraint, by repeating the above procedure, its consistency condition may produce more constraints under certain conditions. It is worth mentioning that according to Table 1, the number of independent coefficients is finite, and the emergence of new constraints will give a set of limiting equations on the coefficients. Just as in the presence of $\phi_3^{(4)} \approx 0$, $T^{ij} = 0$ gives 9 equations between the coefficients; therefore, the number of independent coefficients is reduced. Detailed analysis shows that each time a new constraint is present, the number of limiting equations for the coefficients rapidly increases compared to the previous constraint, eventually stopping the generation of possible constraints at a certain step. This results in a closed Poisson bracket and a limited number of constraints. According to the analysis in [45, 46], such constraint may also lead to an unphysical half degree of freedom. However, this requires a very special choice of gauge-breaking coefficients. For simplicity, we do not explore these specific cases in detail in this study.

Before we go further, we would like to summarize the main results obtained so far for the $d=4$ case. For both the gauge invariant and gauge breaking cases with the degenerate condition (30), the theory can have one primary constraint $\phi_1^{(2)}$ and one secondary constraint $\phi_2^{(4)}$.

3. *Counting the degrees of freedom*

After obtaining all the primary and secondary constraints of the system, we identify the first- and second-class constraints of the system by analyzing the Poisson bracket of the constraints. In general, first-class constraints are associated with the gauge symmetry of the theory. They are gauge generators, which generate gauge transformations that do not alter the physical state. Second-class constraints cannot generate gauge transformations because the transformation generated by the second-class does not preserve all the constraints, which violates the consistency condition, but are important in the definition of the Dirac bracket, which plays a key role in the transition from classical to quantum theory [43]. Specifically, first-class constraints are those whose Poisson bracket with every constraint vanishes weakly; others are second-class constraints.

The Poisson bracket of the primary constraint $\phi_1^{(4)}$ and secondary constraint $\phi_2^{(4)}$ gives

$$\{\phi_2^{(4)}(y), \phi_1^{(4)}(x)\} = (C_1)^{ij}\partial'_i\partial_j\delta(x-y) + (C_2)^{ij}\partial'_i\partial'_j\delta(x-y). \tag{50}$$

where ∂' denotes the partial derivative with respect to y, and

$$(C_1)^{ij} = -(U^{00ji} + (U^{00jk} + V^{kj00})M^i{}_k), \tag{51}$$

$$(C_2)^{ij} = 2M^i{}_k(U^{0k0j} + V^{k00j}) + 5U^{00ij}. \tag{52}$$

It is clear that for the gauge invariant case, $C_1^{ij} = 0 = C_2^{ij}$; therefore, $\phi_1^{(4)}$ and $\phi_2^{(4)}$ are both first-class constraints. For the case without the gauge symmetry, the gauge breaking coefficients $U^{\mu\nu\rho\sigma}$ and $V^{\mu\nu\rho\sigma}$ are generally nonzero; thus, the theory does not possess any first-class constraints. In this case, $\phi_1^{(4)}$ and $\phi_2^{(4)}$ are both second-class constraints. This result is expected because first-class constraints can only exist when the theory has gauge symmetry, and thus a theory without gauge symmetry should not have first-class constraints.

With all first- and second-class constraints, we can count the number of physical degrees of freedom ($N_{\rm DOF}$) using the following formula [47]:

$$N_{\rm DOF} = \frac{1}{2}(N_{\rm var} - 2{\rm NOF} - {\rm NOS}), \tag{53}$$

where "$N_{\rm var}$" represents the total number of canonical variables, "NOF" is the number of constraints of the first class, and "NOS" represents the number of second-class constraints. Thus, for the gauge invariant case, the number of degrees of freedom is

$$N_{\rm DOF} = \frac{1}{2}(8 - 2\times 2) = 2, \tag{54}$$

which is the same as that in the standard Maxwell electrodynamics, whereas in the case of gauge violation,

$$N_{\rm DOF} = \frac{1}{2}(8-2) = 3. \tag{55}$$

Therefore, the violation of $U(1)$ gauge symmetry induces one additional physical degree of freedom compared to the standard Maxwell electrodynamics and the theory with gauge invariance. This additional physical degree of freedom results in a third state of polarization, corresponding to a new particle known as the longitudinal photon [48]. This represents a new type of electromagnetic radiation that may alter the radiation spectra of many sources with nonzero temperature [48]. However, its phenomenological effects in both experiments and astrophysical observations are expected to be too small to be detected for now [48−50].

B. $d = 3$

In this subsection, we analyze the constraint structure of the theory with Lorentz/gauge-breaking operators with dimension $d=3$. The Lagrangian density of the considered theory can be written in the form of

$$\mathcal{L}_{(3)} = -\frac{1}{4}F_{\alpha\nu}F^{\alpha\nu} + \frac{1}{2}S^{\mu\nu\rho}A_\rho F_{\nu\mu} + \frac{1}{2}\epsilon^{\kappa\mu\nu\rho}(k_{AF})_\kappa A_\mu F_{\nu\rho}. \tag{56}$$

where

$$(k_{AF})_\kappa = \frac{1}{3!}\epsilon_{\kappa\mu\nu\rho}\mathcal{K}^{(5)\mu\nu\rho}_{(3)}, \tag{57}$$

$$S^{\mu\nu\rho} = \mathcal{K}^{(2)\rho\mu\nu}_{(3)} - \mathcal{K}^{(3)\mu\rho\nu}_{(3)}. \tag{58}$$

Note that the terms with $(k_{AF})_\kappa$ are gauge-invariant and the terms with $S^{\mu\nu\rho}$ are gauge-breaking. Varying in the action with respect to A_μ, we obtain field equation

$$\epsilon^{\kappa\nu\mu\rho}(k_{AF})_\kappa F_{\mu\rho} + \frac{1}{2}S^{\mu\rho\nu}F_{\rho\mu} + \partial_\mu(F^{\mu\nu} - S^{\nu\mu\rho}A_\rho) = 0. \tag{59}$$

The corresponding conjugate momentum of this theory reads as

$$\pi^\mu = \frac{\partial\mathcal{L}_{(3)}}{\partial\dot{A}_\mu} = -F^{0\mu} + (\epsilon^{\beta\nu0\mu}k_{AF\beta} + S^{\mu0\nu})A_\nu. \tag{60}$$

To analyze the constraint structure of the theory, it is convenient to write down the time and spatial components of the above conjugate momentum (60) as

$$\pi^0 = 0, \tag{61}$$

$$\pi^k = -F^{0k} + (\epsilon^{j\mu0k}k_{AF\,j} + S^{k0\mu})A_\mu. \tag{62}$$

We can conclude that in the specific dimension $d=3$, gauge-breaking and gauge-invariant Lorentz-breaking electrodynamics have the same form for the conjugate momentum π^0 of the photon field A_0. From these properties, the only primary constraint of the system is

$$\phi^{(3)}_1 = \pi^0 \approx 0. \tag{63}$$

Then, the canonical and total Hamiltonian densities are given by

$$\mathcal{H}_{(3)} = \pi^k\dot{A}_k - \mathcal{L}_{(3)}, \tag{64}$$

and

$$(\mathcal{H}_{(3)})_T = \mathcal{H}_{(3)} + u^{(3)}\phi^{(3)}_1. \tag{65}$$

Here, similar to the case of $d=4$, the coefficient $u^{(3)}$ is the time derivative of A_0.

Once again, the consistency condition of the primary constraint gives a secondary constraint

$$\begin{aligned}\phi^{(3)}_2 = \dot{\phi}^{(3)}_1 &= \partial_k\pi^k + \frac{1}{2}F_{ik}(\epsilon^{j0ik}(k_{AF})_j + S^{ki0}) \\ &\quad + S_k{}^{00}(\epsilon^{j\mu0k}(k_{AF})_j + S^{k0\mu})A_\mu - \pi^k S_k{}^{00} \\ &\approx 0.\end{aligned} \tag{66}$$

Similarly, this suggests that the form of Gauss's law must be modified when considering the presence of the gauge-breaking term. The consistency condition of $\phi^{(3)}_2$ imposes a restriction on $u^{(3)}$:

$$\begin{aligned}\dot{\phi}^{(3)}_2 = {} & S_l{}^{00}S^{l00}u^{(3)} - \left(S_i{}^{0k} + S_i{}^{k0}\right)\partial_k\pi^i \\ & + S_k{}^{00}\left(2\epsilon^{jk0}{}_i(k_{AF})_j + S_i{}^{0k} - S^{k0}{}_i\right)\pi^i \\ & + S_i{}^{00}\left[2\epsilon^{l0i}{}_k(k_{AF})_l + S^{i0}{}_k - S_k{}^{0i}\right] \\ & \times[\epsilon^{jn0k}(k_{AF})_j + S^{k0n}]A_n \\ & + \frac{1}{2}\left(-2S^{j00}\eta^{ki} + S^{jik}\right)\partial_k F_{ij} \\ & - \frac{1}{2}(\epsilon^{0kij}(k_{AF})_0 + S^{jik})S_k{}^{00}F_{ij} \\ & + 2S_l{}^{00}(S^{l0k} + S^{lk0})\partial_k A_0 \\ & + S_i{}^{00}\left[2\epsilon^{l0i}{}_k(k_{AF})_l + S^{i0}{}_k - S_k{}^{0i}\right]S^{k00}A_0 \\ & + \Big[S_k{}^{00}(\epsilon^{0jik}(k_{AF})_0 + S^{kij}) \\ & + (S_k{}^{0i} + S_k{}^{i0})[\epsilon^{lj0k}(k_{AF})_l + S^{k0j}]\Big]\partial_i A_j \approx 0.\end{aligned} \tag{67}$$

For the gauge invariant case, all the gauge breaking coefficients $S^{\mu\nu\rho} = 0$; thus, the above consistency condition is satisfied identically. In this case, the theory only has two constraints, the primary constraint $\phi^{(3)}_1$ and secondary constraint $\phi^{(3)}_2$.

When gauge symmetry is violated, the gauge breaking coefficients $S^{\mu\nu\rho}$ are generally nonzero. For this case, the above consistency condition leads to a specific form of $u^{(3)}$,

$$\begin{aligned}u^{(3)} \approx {} & -\frac{1}{S_m{}^{00}S^{m00}} \\ & \times\Big[S_i{}^{00}\left[2\epsilon^{l0i}{}_k(k_{AF})_l + S^{i0}{}_k - S_k{}^{0i}\right] \\ & \times[\epsilon^{jn0k}(k_{AF})_j + S^{k0n}]A_n - \left(S_i{}^{0k} + S_i{}^{k0}\right)\partial_k\pi^i\end{aligned}$$

$$+S_k{}^{00}\big(2\epsilon^{jk0}{}_i(k_{AF})_j+S_i{}^{0k}-S^{k0}{}_i\big)\pi^i$$
$$+\frac{1}{2}\big(-2S^{j00}\eta^{ki}+S^{jik}\big)\partial_k F_{ij}$$
$$-\frac{1}{2}(\epsilon^{0kij}(k_{AF})_0+S^{jik})S_k{}^{00}F_{ij}$$
$$+2S_l{}^{00}(S^{l0k}+S^{lk0})\partial_k A_0$$
$$+S_i{}^{00}(2\epsilon^{l0i}{}_k(k_{AF})_l+S^{i0}{}_k-S_k{}^{0i})S^{k00}A_0$$
$$+\Big[S_k{}^{00}(\epsilon^{0jik}(k_{AF})_0+S^{kij})$$
$$+(S_k{}^{0i}+S_k{}^{i0})[\epsilon^{lj0k}(k_{AF})_l+S^{k0j}]\Big]\partial_i A_j\Big]. \quad (68)$$

This indicates that the theory for this case does not have any additional constraints. Similar to the gauge invariant case, there are only two constraints, one primary constraint $\phi_1^{(3)}$ and one secondary constraint $\phi_2^{(3)}$. Here, we would like to mention that under certain conditions on the gauge breaking coefficients, such that $S_l{}^{00}S^{l00}=0$, the theory may produce additional constraints. However, this requires a very special choice of gauge-breaking coefficients. For simplicity, we do not explore this specific case in detail in this study and focus on the case with $S_l{}^{00}S^{l00}\neq 0$ when gauge symmetry is violated.

Then, let us consider the Poisson bracket of the primary constraint $\phi_1^{(3)}$ and secondary constraint $\phi_2^{(3)}$, which is

$$\{\phi_1^{(3)}(\mathbf{x}),\phi_2^{(3)}(\mathbf{y})\}=-S_k{}^{00}S^{k00}\delta(\mathbf{x}-\mathbf{y}). \quad (69)$$

It is clear that for the gauge invariant case, because $S^{\mu\nu\rho}=0$, the above Poisson bracket vanishes and the constraints $\phi_1^{(3)}$ and $\phi_2^{(3)}$ are both first-class. Thus, the number of the physical degrees of freedom is

$$N_{\rm DOF}=\frac{1}{2}(8-2\times 2)=2, \quad (70)$$

which is the same as that in the standard Maxwell electrodynamics.

For the case with gauge violation, because generally $S_k{}^{00}S^{k00}\neq 0$, $\phi_1^{(3)},\phi_2^{(3)}$ are both second-class constraints. Thus, the number of the physical degrees of freedom is

$$N_{\rm DOF}=\frac{1}{2}(8-2)=3. \quad (71)$$

Similar to the case with $d=4$, the violation of $U(1)$ gauge symmetry induces one extra physical degree of freedom compared to the standard Maxwell electrodynamics and the case with gauge invariance.

C. $d = 2$

After completing the constraint structure analysis of a specific dimension $d=3$ and $d=4$, we analyze the structure of $d=2$, the Lagrangian density of which is

$$\mathcal{L}_{(2)}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}U^{\mu\nu}A_\mu A_\nu, \quad (72)$$

with

$$U^{\mu\nu}=2\mathcal{K}_{(2)}^{(1)\mu\nu}, \quad (73)$$

which are gauge-breaking terms. Varying the action with respect to A_μ, we obtain

$$U^{\nu\mu}A_\mu+\partial_\mu F^{\mu\nu}=0. \quad (74)$$

The conjugate momenta now read as

$$\pi^\mu=\frac{\partial\mathcal{L}_{(2)}}{\partial\dot{A}_\mu}=-F^{0\mu}. \quad (75)$$

One difference from $d=3$ and $d=4$ is that at this point, the conjugate momenta π^μ are not affected by the gauge-breaking coefficients, $U^{\mu\nu}$; hence, they are the same as in Maxwell electrodynamics because the time derivative of the photon field A_μ only appears in the Maxwell term, i.e., the first term in (72). Therefore, the only primary constraint is

$$\phi_1^{(2)}=\pi^0\approx 0. \quad (76)$$

The canonical Hamiltonian density differs by only one term of $U^{\mu\nu}$ from that in Maxwell electrodynamics:

$$\mathcal{H}_{(2)}=\pi^k\partial_k A_0-\frac{1}{2}\pi^k\pi_k+\frac{1}{4}F_{ik}F^{ik}-\frac{1}{2}U^{\mu\nu}A_\mu A_\nu. \quad (77)$$

This gives the total Hamiltonian density

$$\mathcal{H}_{(2)T}=\mathcal{H}_{(2)}+u^{(2)}\phi_1^{(2)}. \quad (78)$$

The influence of the gauge-breaking terms are reflected in the secondary constraint because it originates from the Poisson bracket between the primary constraint and the total Hamiltonian, which is now gauge-breaking. It reads as

$$\phi_2^{(2)}=\partial_k\pi^k+U^{0\mu}A_\mu\approx 0. \quad (79)$$

The consistency condition of ϕ_2 is

$$\dot{\phi}_2^{(2)} = U^{\mu k}\partial_k A_\mu + U^{0k}\partial_k A_0 - U^{0k}\pi_k + u^{(2)}U^{00} \approx 0, \tag{80}$$

which provides the constraint on $u^{(2)}$ when $U^{00} \neq 0$:

$$u^{(2)} \approx (U^{00})^{-1}[U^{0k}(\pi_k - \partial_k A_0) - U^{\mu k}\partial_k A_\mu]. \tag{81}$$

Noting the absence of π^0 in $\mathcal{H}_{(2)}$, we obtain $\dot{A}_0(\mathbf{x}) = \{A_0(\mathbf{x}), H_{(2)T}\} = u^{(2)}(\mathbf{x})$. As a consequence, the meaning of $u^{(2)}$ is the time derivative of A_0. Combining (81), the time component A_0 of the photon field can be determined by the first-order differential equation.

Finally, the Poisson bracket of $\phi_1^{(2)}$ and $\phi_2^{(2)}$ is

$$\{\phi_1^{(2)}(\mathbf{x}), \phi_2^{(2)}(\mathbf{y})\} = -U^{00}\delta(\mathbf{x}-\mathbf{y}). \tag{82}$$

When the gauge breaking coefficients $U^{\mu\nu}$ are nonzero, $\phi_1^{(2)}, \phi_2^{(2)}$ are both second class constraints. Similar to the cases of $d=4$ and $d=3$, we do not explore the case with $U^{00}=0$ in this study, which may induce more constraints. Then, the number of physical degrees of freedom is

$$N_{\rm DOF} = \frac{1}{2}(8-2) = 3. \tag{83}$$

Again, compared to the standard Maxwell electrodynamics, because $d=2$ operators always break the $U(1)$ gauge symmetry, it induces one additional physical degree of freedom.

IV. MAP TO SEVERAL SPECIFIC MODELS

The Lorentz-violating electrodynamics presented in this paper provide a unifying framework for describing possible violations of Lorentz and $U(1)$ gauge symmetries in the electromagnetic interaction. In this section, we present several specific modified electrodynamics by expressing their actions in the form of (1) and summarize their Hamiltonian structures from our general analysis. We consider three specific theories, LIV, CFJ, and Proca electrodynamics. The first two theories only break the Lorentz symmetry of the theory, whereas the third only breaks $U(1)$ gauge symmetry.

A. LIV electrodynamics

We first consider LIV electrodynamics, which is proposed in [27, 28], and the Lagrangian density is given by [27, 28]

$$\mathcal{L}_{LIV} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}W^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{84}$$

which corresponds to the case in our model where the gauge-breaking coefficients $U^{\mu\nu\rho\sigma}$ and $V^{\mu\nu\rho\sigma}$ are set as zero for $d=4$. Using the result from (35) and (40), we obtain the constraint structure in this case as

$$(\phi_1)_{LIV} = \pi^0 \approx 0, \tag{85}$$

$$(\phi_2)_{LIV} = \partial_k \pi^k \approx 0. \tag{86}$$

We must verify that the consistency of $(\phi_2)_{LIV}$ gives no new constraints. Noting that $T^{ij} = M^{ij} = 0$ using (41) and (48) because $V^{\mu\nu\rho\sigma} = U^{\mu\nu\rho\sigma} = 0$, we only need to check whether $\phi_3^{(4)}$ in (49) is identically zero throughout the phase space, not just on the constraint surface. In this case, the only possible non-zero coefficients remaining in (49) are $O_{1\ j}^{ik}$ and O_4^{klij}. According to (33), as a result, $\phi_3^{(4)}$ now changes to

$$W^{0lik}((D^{-1})_{jl} + (D^{-1})_{lj})\partial_k\partial_i(\pi^j - N^j) + \frac{1}{2}(\eta^{li}\eta^{kj} + W^{ijlk})\partial_k\partial_l F_{ij}. \tag{87}$$

Because the last two indices of W^{ijkl} are antisymmetric, and i, j in $\eta^{li}\eta^{kj}\partial_k\partial_l$ are symmetric, (87) is equal to 0. This proves that $(\phi_2)_{\rm LIV}$ does not give new constraints. In addition, because the theory has gauge symmetry, both the constraints $(\phi_1)_{\rm LIV}$ and $(\phi_2)_{\rm LIV}$ are first-class; thus, the theory has two degrees of freedom, the same as that in the standard Maxwell electrodynamics. Our result is consistent with the results in LIV [27, 28].

B. CFJ electrodynamics

For CFJ electrodynamics, the Lagrangian density is given by [35]

$$\mathcal{L}_{\rm CFJ} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon^{\kappa\mu\nu\rho}(k_{AF})_\kappa A_\mu F_{\nu\rho}, \tag{88}$$

with the caveat that the coefficient $(k_{AF})_\kappa$ differs by a factor of $-\frac{1}{2}$ compared to that in [35]. The Lagrangian density (88) can be obtained in our model by setting $S^{\mu\nu\rho} = 0$ to zero in $d=3$. Similarly, according to the results of (63) and (66), we have two constraints for this theory,

$$(\phi_1)_{\rm CFJ} = \pi^0 = F^{00} \approx 0, \tag{89}$$

$$(\phi_2)_{\rm CFJ} = \partial_k \pi^k + \frac{1}{2}F_{ik}\epsilon^{j0ik}(k_{AF})_j \approx 0, \tag{90}$$

where $(\phi_1)_{\rm CFJ}$ is the primary constraint and $(\phi_2)_{\rm CFJ}$ is the secondary constraint. According to (67), it can be concluded that $(\dot{\phi}_2)_{\rm CFJ}$ is always zero in the phase space in this case, which does not generate new constraints. This result confirms that in [35]; however, note that the coefficient k_{AF} here differs by a factor of $-1/2$ compared to that in [35]. Note that both $(\phi_1)_{\rm CFJ}$ and $(\phi_2)_{\rm CFJ}$ are first-class, and the number of degrees of freedom is the same as that in the standard Maxwell electrodynamics.

C. Proca electrodynamics

When $d=2$, if we set $U^{\mu\nu}=m^2\eta^{\mu\nu}$, the Lagrangian density $\mathcal{L}_{(2)}$ will return to the case of Proca electrodynamics [36]:

$$\mathcal{L}_{\rm Proca}=-\frac{1}{4}F_{\alpha\nu}F^{\alpha\nu}+\frac{1}{2}m^2A_\mu A^\mu, \tag{91}$$

where m is the mass of the photon. Because of this mass term, Proca electrodynamics break the $U(1)$ gauge symmetry of the theory. Replacing the constant coefficients $U^{\mu\nu}$ with the product of the metric tensor $\eta^{\mu\nu}$ and m^2 is the reason why Proca electrodynamics does not break Lorentz symmetry, because the product is also a tensor. For the same reason as $d=3,4$, at this point, the theory also has two constraints,

$$(\phi_1)_{\rm Proca}=\pi^0\approx 0, \tag{92}$$

$$(\phi_2)_{\rm Proca}=\partial_k\pi^k+m^2A^0\approx 0. \tag{93}$$

Moreover, $(\phi_2)_{\rm Proca}$ gives no new constraints. They are both second-class because $\{(\phi_1)_{\rm Proca}(\mathbf{x}),(\phi_2)_{\rm Proca}(\mathbf{y})\}=m^2\delta(\mathbf{x}-\mathbf{y})$. Thus, this theory propagates three physical degrees of freedom, which is different from the two degrees of freedom in the standard Maxwell electrodynamics and the cases with gauge invariance. These results are consistent with those in [36], with the only difference being that their coefficient m^2 differs from ours by a factor of 1/2.

V. SUMMARY AND DISCUSSION

In this study, we perform an extended analysis of modified electrodynamics with the violations of both Lorentz symmetry and $U(1)$ gauge symmetry in the framework of the SME. This represents an extension of the previous construction of Lorentz-violating electrodynamics with gauge invariance [24]. For our purposes, by following the procedure in [24], we construct the quadratic Lagrangian density of electrodynamics by allowing the violations of both the Lorentz and $U(1)$ gauge symmetries. The Lorentz- and gauge-violating effects in the quadratic Lagrangian are represented by new operators with a specific dimension $d\le 4$. With the constructed quadratic Lagrangian, we calculate in detail the number of independent components of the Lorentz-violating operators at different dimensions in both the gauge invariance and gauge violation cases.

Table 3. Number of first-class and second-class constraints, and the number of physical degrees of freedom for each case with a specific dimension $d=2$, $d=3$, and $d=4$.

d	Gauge invariance	Number of first-class constraints	Number of first-class constraints	Number of DOF
2	no	0	2	3
3	yes	2	0	2
	no	0	2	3
4	yes	2	0	2
	no	0	2	3

We then perform a Hamiltonian analysis of the general theory by considering Lorentz and gauge-breaking operators with dimension $d\le 4$ as examples. Specifically, we perform the analysis for $d=4$, $d=3$, and $d=2$. It is shown that the Lorentz-breaking operators with gauge invariance do not change the classes of the theory constrains and have the same number of physical degrees of freedom as that in the standard Maxwell electrodynamics. When the $U(1)$ gauge symmetry-breaking operators are presented, the theories generally lack a first-class constraint and have one additional physical degree of freedom compared to the standard Maxwell electrodynamics. The results of the Hamiltonian structure and the corresponding number of degrees of freedom are presented in Table 3.

Finally, we map our general analysis to several specific modified electrodynamics, including LIV, CFJ, and Proca electrodynamics. While the first two theories represent two specific examples of Lorentz-violating theories with gauge invariance, the third is a theory that breaks $U(1)$ gauge symmetry but still maintains Lorentz symmetry. We show that our general results are consistent with the existing Hamiltonian analysis in the literature for these specific examples.

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