Potential pollution of the $K\pi$-puzzle by the isospin-breaking π^0-η-η' mixing effect*

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Abstract: The influence of the isospin-breaking $\pi^0\!-\!\eta\!-\!\eta'$ mixing effect on the CP-asymmetries of $B\to K\pi$ processes is examined for the first time. It is found that this mixing effect presents significant uncertainty for both the CP-asymmetry sum rule of $B\to K\pi$ processes and the CP-asymmetry difference of $B^+\to K^+\pi^0$ and $B^0\to K^+\pi^-$, potentially obscuring the significance of the $K\pi$-puzzle. As the correction caused by the $\pi^0\!-\!\eta\!-\!\eta'$ mixing effect is highly dependent on four strong non-perturbative phases, a definitive conclusion is currently unavailable.

Keywords: CP violation, B meson decay, $K\pi$ puzzle

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I. BRIEF INTRODUCTION TO THE $K\pi$-PUZZLE

$B\to K\pi$ decay processes provide good probes for new physics (NP) beyond the Standard Model (SM) as tree-level amplitudes are suppressed, increasing their sensitivity to potential NP contributions. Based on the isospin consideration, the amplitudes of $B\to K\pi$ weak decays are related by [1–3]

$$\mathcal{A}_{B^+\to K^0\pi^+}+\sqrt{2}\mathcal{A}_{B^+\to K^+\pi^0}=\mathcal{A}_{B^0\to K^+\pi^-}+\sqrt{2}\mathcal{A}_{B^0\to K^0\pi^0}, \qquad (1)$$

from which a sum rule between the CP asymmetries of $B\to K\pi$ processes is derived [4]:

$$A_{CP}^{K^+\pi^-}+\frac{\mathcal{B}(K^0\pi^+)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^0\pi^+} =\frac{2\mathcal{B}(K^+\pi^0)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^+\pi^0}+\frac{2\mathcal{B}(K^0\pi^0)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^0\pi^0}. \qquad (2)$$

The CP-asymmetry sum rule can be further simplified to a more crude relation between the CP asymmetries of $B^+\to K^+\pi^0$ and $B^0\to K^+\pi^-$ [5]:

$$A_{CP}^{K^+\pi^0}-A_{CP}^{K^+\pi^-}\approx 0, \qquad (3)$$

which clearly contradicts the latest world average of the CP-asymmetry difference between the two processes [6–11]:

$$\Delta A_{CP}(K\pi)\equiv A_{CP}^{K^+\pi^0}-A_{CP}^{K^+\pi^-}=0.115\pm 0.014. \qquad (4)$$

This is essentially a shortened version of the long-standing $K\pi$-puzzle.

The $K\pi$-puzzle has received significant theoretical attention, particularly from studies exploring the possibility of NP [12–20]. Additionally there are studies attempting to understand it within the SM [21–26]. This paper examines the role of the isospin breaking $\pi^0\!-\!\eta\!-\!\eta'$ mixing effect in the $K\pi$-puzzle. The effects of $\pi^0\!-\!\eta\!-\!\eta'$ mixing on the CP asymmetries of B meson decays have been studied in several decay channels, such as $B\to\pi\pi$ [27, 28].

II. POLLUTION OF THE K π -PUZZLE BY THE $\pi^0\!-\!\eta\!-\!\eta'$ MIXING EFFECT

The basic principle is that, as there are π^0s in the final states of $B\to K\pi$ decay processes, the isospin-breaking $\pi^0\!-\!\eta\!-\!\eta'$ mixing effect [29– 33] occurs. This effect initially appears negligible [34], which is perhaps why its influence on the $K\pi$-puzzle has not previously been examined. However, isospin-breaking effects potentially affect CP asymmetries more signficantly than previously assumed [35].

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Within the context of the π^0–η–η' mixing effect, the π^0 meson can be expressed as an admixture of the isospin eigenstate π_3 and the mass eigenstates η and η',

$$|\pi^0\rangle = |\pi_3\rangle + \epsilon|\eta\rangle + \epsilon'|\eta'\rangle, \tag{5}$$

where ϵ and ϵ' are small parameters accounting for the mixing between π^0 and η and η', respectively [30, 31]. Both the branching ratios and the CP asymmetry parameters of $B^+\to K^+\pi^0$ and $B^0\to K^0\pi^0$ require corrections because of the π^0–η–η' mixing effect. Taking the decay process $B^+\to K^+\pi^0$ as an example, the full decay amplitude can be expressed as

$$\mathcal{A}_{B^+\to K^+\pi^0} = \mathcal{A}_{B^+\to K^+\pi_3} + \mathcal{A}_{B^+\to K^+\eta}\epsilon + \mathcal{A}_{B^+\to K^+\eta'}\epsilon', \tag{6}$$

accounting for the π^0–η–η' mixing effect. Up to $O(\epsilon)$ and $O(\epsilon')$, the branching ratio of $B^+\to K^+\pi^0$ can be expressed as

$$\begin{aligned}\mathcal{B}_{B^+\to K^+\pi^0} =&\mathcal{B}_{B^+\to K^+\pi_3}\left[1+2\Re\left(\frac{\mathcal{A}_{B^+\to K^+\eta}}{\mathcal{A}_{B^+\to K^+\pi^0}}\right)\epsilon\right.\\ &\left.+2\Re\left(\frac{\mathcal{A}_{B^+\to K^+\eta'}}{\mathcal{A}_{B^+\to K^+\pi^0}}\right)\epsilon'\right],\end{aligned} \tag{7}$$

where $\mathcal{B}_{B^+\to K^+\pi_3}$ is the branching ratio of $B^+\to K^+\pi_3$, which is also the branching ratio of $B^+\to K^+\pi^0$ in the absence of the π^0–η–η' mixing effect.

The CP asymmetry of $B^+\to K^+\pi^0$ is modified to[1)]

$$A_{CP}^{K^+\pi^0} = A_{CP}^{K^+\pi_3} + \Delta_{\rm IB}^{K^+\pi^0} \tag{8}$$

because of the π^0–η–η' mixing effect, where $A_{CP}^{K^+\pi_3}$ is the CP asymmetry of the decay channel $K^+\pi_3$, taking the form $A_{CP}^{K^+\pi_3}\equiv\dfrac{\left|\mathcal{A}_{B^-\to K^-\pi_3}\right|^2-\left|\mathcal{A}_{B^+\to K^+\pi_3}\right|^2}{\left|\mathcal{A}_{B^-\to K^-\pi_3}\right|^2+\left|\mathcal{A}_{B^+\to K^+\pi_3}\right|^2}$. $\Delta_{\rm IB}^{K^+\pi^0}$ represents the correction of the π^0–η–η' mixing effect to the CP asymmetry of $B^+\to K^+\pi^0$. Up to $O(\epsilon)$ and $O(\epsilon')$, this can be expressed as

$$\begin{aligned}\Delta_{\rm IB}^{K^+\pi^0} =&\Re\left[\left(\frac{\mathcal{A}_{B^-\to K^-\eta}}{\mathcal{A}_{B^-\to K^-\pi^0}}-\frac{\mathcal{A}_{B^+\to K^+\eta}}{\mathcal{A}_{B^+\to K^+\pi^0}}\right)\epsilon\right.\\ &\left.+\left(\frac{\mathcal{A}_{B^-\to K^-\eta'}}{\mathcal{A}_{B^-\to K^-\pi^0}}-\frac{\mathcal{A}_{B^+\to K^+\eta'}}{\mathcal{A}_{B^+\to K^+\pi^0}}\right)\epsilon'\right].\end{aligned} \tag{9}$$

To account for the potential relative phases between the decay amplitudes, which could arise from effects such as final state interactions, two strong phases θ and θ' are introduced to Eq. (9), which becomes

$$\begin{aligned}\Delta_{\rm IB}^{K^+\pi^0} =&\Re\left[\left(\frac{\mathcal{A}_{B^-\to K^-\eta}}{\mathcal{A}_{B^-\to K^-\pi^0}}-\frac{\mathcal{A}_{B^+\to K^+\eta}}{\mathcal{A}_{B^+\to K^+\pi^0}}\right)e^{i\theta}\epsilon\right.\\ &\left.+\left(\frac{\mathcal{A}_{B^-\to K^-\eta'}}{\mathcal{A}_{B^-\to K^-\pi^0}}-\frac{\mathcal{A}_{B^+\to K^+\eta'}}{\mathcal{A}_{B^+\to K^+\pi^0}}\right)e^{i\theta'}\epsilon'\right].\end{aligned} \tag{10}$$

Similarly, the CP asymmetry of $B^0\to K^0\pi^0$ is modified to

$$A_{CP}^{K^0\pi^0} = A_{CP}^{K^0\pi_3} + \Delta_{\rm IB}^{K^0\pi^0}, \tag{11}$$

where the correction term is expressed as

$$\begin{aligned}\Delta_{\rm IB}^{K^0\pi^0} =&\Re\left[\left(\frac{\mathcal{A}_{\overline{B^0}\to \overline{K^0}\eta}}{\mathcal{A}_{\overline{B^0}\to \overline{K^0}\pi^0}}-\frac{\mathcal{A}_{B^0\to K^0\eta}}{\mathcal{A}_{B^0\to K^0\pi^0}}\right)e^{i\tilde{\theta}}\epsilon\right.\\ &\left.+\left(\frac{\mathcal{A}_{\overline{B^0}\to \overline{K^0}\eta'}}{\mathcal{A}_{\overline{B^0}\to \overline{K^0}\pi^0}}-\frac{\mathcal{A}_{B^0\to K^0\eta'}}{\mathcal{A}_{B^0\to K^0\pi^0}}\right)e^{i\tilde{\theta}'}\epsilon'\right].\end{aligned} \tag{12}$$

The different behaviors of the π^0–η–η' mixing effect in the branching ratios and the CP asymmetry parameters can be easily determined through a comparison of Eqs. (7) and (8). The π^0–η–η' mixing effect is clearly negligible for the branching ratios in Eq. (7). However, its contribution to the CP asymmetry parameters, $\Delta_{\rm IB}^{K^+\pi^0}$, cannot be neglected, as a small CP asymmetry parameter $A_{CP}^{K^+\pi_3}$ is potentially comparable with $\Delta_{\rm IB}^{K^+\pi^0}$. Moreover, with a large difference between the amplitudes of the pair of CP-conjugate processes $B^\pm\to K^\pm\eta^{(\prime)}$, $\Delta_{\rm IB}^{K^+\pi^0}$ can be further enlarged. This is the case for $B^\pm\to K^\pm\eta$, as the CP asymmetry of this process is as large as -0.37 ± 0.08 [36], indicating a large difference between the corresponding decay amplitudes of the CP-conjugate processes.

The influence of the π^0–η–η' mixing effect in Eqs. (2) and (3) can now be studied. As these two equations are obtained under the ignorance of the the π^0–η–η' mixing effect, the π^0s are the isospin eigenstates π_3s. Consequently, Eqs. (2) and (3) are rewritten as

$$\begin{aligned}&A_{CP}^{K^+\pi^-}+\frac{\mathcal{B}(K^0\pi^+)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^0\pi^+}\\ =&\frac{2\mathcal{B}(K^+\pi_3)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^+\pi_3}+\frac{2\mathcal{B}(K^0\pi_3)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^0\pi_3},\end{aligned} \tag{13}$$

and

$$A_{CP}^{K^+\pi_3}-A_{CP}^{K^+\pi^-}\approx 0, \tag{14}$$

respectively. It should be noted that all the branching ra-

1) An extra factor 1-$A_{CP}^{K^+\pi_3^2}$ in front of $\Delta_{\rm IB}^{K^+}\pi^0$ is omitted as it is numerically very close to one

tios and CP asymmetries containing π_3 in the final state are not strictly physically observable. However, this issue can be resolved using Eqs. (8) and (11) and by rewriting Eqs. (13) and (14) in terms of the physical branching ratios and CP asymmetries with π^0s contained in the final states. The CP-asymmetry sum rule and the CP-asymmetry difference are now expressed as

$$A_{CP}^{K^+\pi^-}+\frac{\mathcal{B}(K^0\pi^+)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^0\pi^+} =\frac{2\mathcal{B}(K^+\pi^0)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^+\pi^0}+\frac{2\mathcal{B}(K^0\pi^0)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}A_{CP}^{K^0\pi^0}-\Delta_{\rm IB}, \tag{15}$$

and

$$A_{CP}^{K^+\pi^0}-A_{CP}^{K^+\pi^-}\approx\Delta_{\rm IB}, \tag{16}$$

respectively, where $\Delta_{\rm IB}$ accommodates the π^0–η–η' mixing correction and takes the form

$$\Delta_{\rm IB}=\frac{2\mathcal{B}(K^+\pi^0)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}\Delta_{\rm IB}^{K^+\pi^0}+\frac{2\mathcal{B}(K^0\pi^0)}{\mathcal{B}(K^+\pi^-)}\frac{\tau_0}{\tau_+}\Delta_{\rm IB}^{K^0\pi^0}. \tag{17}$$

Note that the π^0–η–η' mixing corrections from the branching ratios $\mathcal{B}(K^+\pi^0)$ and $\mathcal{B}(K^0\pi^0)$, which are proportional to $A_{CP}^{K^{+,0}\pi^0}\epsilon^{(\prime)}$, are neglected in $\Delta_{\rm IB}$ as $A_{CP}^{K^{+,0}\pi^0}$ are significantly less than 1.

An interesting behavior of this modification is that, although Eq. (3) relates only the CP asymmetries of $B^0\to K^+\pi^-$ and $B^+\to K^+\pi^0$, the isospin-breaking correction term $\Delta_{\rm IB}$ in Eq. (16) contains not only the contribution of the process $B^+\to K^+\pi^0$, but also that of the process $B^0\to K^0\pi^0$. The latter is of greater numerical importance in $\Delta_{\rm IB}$. The reason for the presence of $\Delta_{\rm IB}^{K^0\pi^0}$ in Eq. (16) is that it is not obtained by substituting Eq. (8) into Eq. (14). If that were the case, Eq. (16) would read as $A_{CP}^{K^+\pi^0}-A_{CP}^{K^+\pi^-}\approx\Delta_{\rm IB}^{K^+\pi^0}$. Instead, Eq. (16) is derived from Eq. (15), which follows a similar logic to obtaining Eq. (3) from Eq. (2). In this respect, the approximate relation of Eq. (16) is more reasonable, as its isospin-breaking contributions are more complete.

The impact of the the isospin-breaking correction term $\Delta_{\rm IB}$ on the $K\pi$-puzzle can now be estimated. The amplitudes in Eqs. (10) and (12) can be calculated theoretically via various approaches, such as QCD factorization [37, 38], perturbative QCD factorization [39, 40], soft collinear effective theory [41], or strategies with the aid of experimental data [23, 25]. However, the four strong phases θ, θ', $\tilde{\theta}$, and $\tilde{\theta}'$ are non-perturbative, which prevents an accurate prediction of $\Delta_{\rm IB}$. Nevertheless, it is possible to obtain an approximate estimation of the possible range of $\Delta_{\rm IB}$ by treating the four strong phases as free parameters, varying from 0 to 2π independently. Based on this strategy, using the amplitudes from Ref. [23], the latest world average values of the branching ratios, and the life times from Particle Data Group [37], $\Delta_{\rm IB}$ is estimated as[1)]

$$\Delta_{\rm IB}=(-0.37,+0.37)\times\epsilon+(-1.16,+1.16)\times\epsilon'. \tag{18}$$

In Table 1, using different values of ϵ and ϵ' from different studies, the corresponding allowable range of $\Delta_{\rm IB}$ is calculated via Eq. (18). Note that the mixing parameters ϵ and ϵ' take significantly different values in different studies, resulting in different ranges of $\Delta_{\rm IB}$, as presented in the last column of the table. From Table 1 it can be seen that the isospin-breaking-correction term $\Delta_{\rm IB}$ ranges from 0.2 to 2.2% for different fitting values of ϵ and ϵ'. As this range is quite large, a definitive conclusion is unobtainable. However, there is still a good chance that $\Delta_{\rm IB}$ can be as high as several percent, which indicates that the influence of the π^0–η–η' mixing effect on the CP-asymmetry sum rule of $B\to K\pi$ and the CP-asymmetry difference of $B^+\to K^+\pi^0$ and $B^0\to K^+\pi^-$ cannot be ignored. Taking the CP-asymmetry difference between $B^+\to K^+\pi^0$ and $B^0\to K^+\pi^-$ as an example and combining Eq. (16) with the CP-asymmetry difference in Eq. (14), which is approximately zero in the isospin limit, provides the expression

$$A_{CP}^{K^+\pi_3}-A_{CP}^{K^+\pi^-}\approx\left(A_{CP}^{K^+\pi^0}-A_{CP}^{K^+\pi^-}\right)+\Delta_{\rm IB}, \tag{19}$$

where the CP-asymmetry difference in parentheses is the measured $\Delta A_{CP}(K\pi)$ in Eq. (4). Treating the $\Delta_{\rm IB}$-term as an uncertainty, with the mixing parameters from Ref. [33], the CP-asymmetry difference is now expressed as

$$A_{CP}^{K^+\pi_3}-A_{CP}^{K^+\pi^-}=0.115\pm0.014\pm0.041_{\pi^0\text{-}\eta\text{-}\eta'}=0.115\pm0.043, \tag{20}$$

Table 1. π^0–η–η' mixing parameters from different references and their corresponding range of $\Delta_{\rm IB}$. Only the central values of ϵ and ϵ' are used when obtaining the range of $\Delta_{\rm IB}$.

Reference	ϵ(%)	ϵ'(%)	$\Delta_{\rm IB}$(%)
Kroll [42]	1.7 ± 0.2	0.4 ± 0.1	$(-1.2,+1.2)$
Escribano *et al.* [43]	0.98 ± 0.03	0.025 ± 0.015	$(-0.4,+0.4)$
Escribano & Royo [32]	0.1 ± 0.9	3.5 ± 0.9	$(-4.1,+4.1)$
Benayoun *et al.* [44]	4.16 ± 0.20	1.05 ± 0.05	$(-3.0,+3.0)$

1) There are no big differences between the numerical values of $\Delta_{\rm IB}$ obtained with weak decay amplitudes via different theoretical approaches, as the main uncertainty of $\Delta_{\rm IB}$ comes form the two strong phases.

from which it is clear that the uncertainty caused by the π^0–η–η' mixing effect can be larger than the other experimental uncertainties combined. Hence, the significance of the nonzeroness of this CP difference is considerably reduced to less than three standard deviations.

It should be noted that the exact values of the π^0–η–η' mixing correction term, $\Delta_{\rm IB}$, depend highly on the values of the four strong phases. Here, the possible maximum pollution of the π^0–η–η' mixing effect on the $K\pi$-puzzle has been estimated by treating the four phases as free varying parameters. However, when the limits of all four strong phases are zero, the value of $\Delta_{\rm IB}$ will be reduced significantly. More specifically, if ϵ and ϵ' take the four different sets of values in Table 1, $\Delta_{\rm IB}$ would take the values of −0.1%, −0.02%, −1.0%, and −0.3%, respectively, under the zero-value limit for these four phases. Under this limit, the corrections of the π^0–η–η' mixing effect are not sufficiently large to explain the $K\pi$-puzzle. All four values are negative, which can bring the central value of Eq. (20) closer to zero, thus slightly reducing the significance of the nonzeroness of this equation. In conclusion, further study on the determination of the values of the four strong phases is crucial to assess whether the π^0–η–η' mixing effect provides considerable corrections for the CP asymmetries of channels involved in the $K\pi$-puzzle.

III. CONCLUSION

In conclusion, the contribution of the isospin-breaking π^0–η–η' mixing effect to the CP asymmetries of $B\to K\pi$ was investigated in this paper. It was found that there is a high chance that the π^0–η–η' mixing effect pollutes the $K\pi$-puzzle, to a greater degree than previously expected. This pollution is imbedded in the parameter $\Delta_{\rm IB}$, which can present high uncertainty for the CP-asymmetry sum rule of $B\to K\pi$ and the CP-asymmetry difference of $B^+\to K^+\pi^0$ and $B^0\to K^+\pi^-$. Although a definitive conclusion on the pollution from the π^0–η–η' mixing effect is unobtainable, the analysis in this paper suggests that the implications on the $K\pi$-puzzle should be carefully considered.

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