New γ-soft rotation in the interacting boson model with $SU(3)$ higher-order interactions*

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Abstract: The interacting boson model with $SU(3)$ higher-order interactions offers a new route to enhance our understanding on γ-soft rotation. In this paper, $U(5)$-like and $O(6)$-like new γ-softness are observed, in which the corresponding energy levels in the ground and quasi-γ bands can be exactly degenerate and have a partial $O(5)$ dynamical symmetry. The spherical-like γ-softness is not related to the classical $O(6)$ dynamical symmetry. The transitional behaviors of $B(E2)$ values of the low-lying levels and quadrupole moment of the 2_1^+ state are also discussed. Spherical-like γ-softness can be used to explain the low-lying spectra and $B(E2)$ values in ^{110}Cd normal states.

Keywords: new γ-softness, $SU(3)$ higher-order interaction, $SU(3)$ degenerate point, partial $O(5)$ dynamical symmetry

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I. INTRODUCTION

In the simplest interacting boson model (IBM-1) [1], valence nucleon-pair coupling to angular momentum $L=0$ and $L=2$ are mapped to s and d bosons. Many features of even-even nuclei can be elegantly described with up to two-body interactions. With a consistent-Q formalism often used in this model [2–4],

$$\hat{H}_1 = c\left[(1-\eta)\hat{n}_d - \frac{\eta}{N}\hat{Q}_\chi\cdot\hat{Q}_\chi\right], \tag{1}$$

it can produce the spectra of spherical ($\eta=0$, $U(5)$ limit), prolate ($\eta=1$, $\chi=-\sqrt{7}/2$, $SU(3)$ limit), oblate ($\eta=1$, $\chi=\sqrt{7}/2$, $\overline{SU(3)}$ limit), and γ-soft ($\eta=1$, $\chi=0$, $O(6)$ limit) nuclei. Here, $\hat{n}_d$ is the d-boson number operator, $\hat{Q}_\chi=[d^\dagger\times\tilde{s}+s^\dagger\times\tilde{d}]^{(2)}+\chi[d^\dagger\times\tilde{d}]^{(2)}$ is the generalized quadrupole operator, N is the total boson number, c is a fitting parameter, and $0\le\eta\le1, -\sqrt{7}/2\le\chi\le\sqrt{7}/2$. This Hamiltonian can be also exploited to describe the shape transitional behaviors between these typical collective excitation modes. An interesting result is that, when $\eta=1$ and χ changes from $-\sqrt{7}/2$ to $\sqrt{7}/2$, γ-soft rotation can be observed as a critical property between the prolate and oblate shapes [5] (Fig. 1). This means that triaxiality may result from the competition between the prolate and oblate shapes.

However, this simple Hamiltonian cannot describe γ-rigid triaxial deformation. This deficiency can be overcome by considering higher-order interactions [6, 7]. $[d^\dagger d^\dagger d^\dagger]^{(L)}\cdot[\tilde{d}\tilde{d}\tilde{d}]^{(L)}$ interactions have been investigated and can result in a stable γ-rigid triaxial deformation for the ground state of a nucleus. Subsequently, $SU(3)$ symmetry-conserving higher-order interactions have also been systematically investigated [8], and a realization of the rigid quantum asymmetric rotor within the $SU(3)$ limit was established [9, 10]. These studies discussed the functions of the $SU(3)$ third-order interactions ($[\hat{Q}\times\hat{Q}\times\hat{Q}]^{(0)}$, $[\hat{L}\times\hat{Q}\times\hat{L}]^{(0)}$) and fourth-order interactions ($(\hat{Q}\cdot\hat{Q})^2$, $[(\hat{L}\times\hat{Q})^{(1)}\times(\hat{L}\times\hat{Q})^{(1)}]^{(0)}$), where $\hat{Q}=\hat{Q}_{-\sqrt{7}/2}$ is the $SU(3)$ quadrupole operator. $SU(3)$ third-order and fourth-order interactions are also discussed in Refs. [11–15]. These interactions can remove the degeneracy in the γ-band and β-band of the spectra within the $SU(3)$ limit and are also intimately connected with the rigid triaxial rotational spectra. Higher-order terms are also important in partial dynamical symmetry [16]. Isacker demonstrated that higher-order interactions $(\hat{Q}_0\times\hat{Q}_0\times\hat{Q}_0)^{(0)}$ can be individually used to present a rotational spectrum [17], where $\hat{Q}_0$ is the quadrupole operator in the $O(6)$ limit. This interesting result was further studied by [18, 19].

Inspired by the relationship between γ-rigid triaxial deformation and higher-order interactions, Fortunato *et al.* investigated triaxiality with a cubic-Q interacting boson model Hamiltonian [20]

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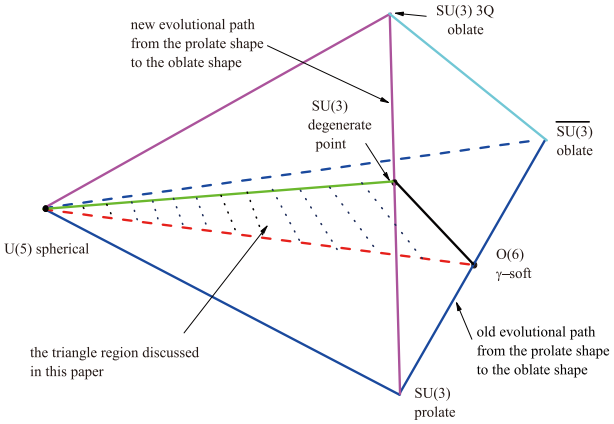

Fig. 1. (color online) Partial phase diagram in the cubic-Q interacting boson model Hamiltonian $\hat{H}_2$.

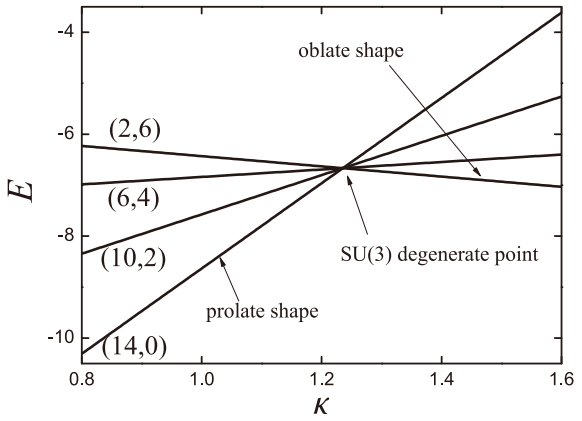

Fig. 2. Evolutional behaviors of the $SU(3)$ irreps (14,0), (10,4), (6,6), (2,8) for $N=7$ in $\hat{H}^{(3)}$. The key observation is that the critical point is also a degenerate point.

$$\hat{H}_2 = c\left[(1-\eta)\hat{n}_d - \eta\left(\frac{\hat{Q}_\chi\cdot\hat{Q}_\chi}{N} + k\frac{[\hat{Q}_\chi\times\hat{Q}_\chi\times\hat{Q}_\chi]^{(0)}}{N^2}\right)\right], \quad (2)$$

where k is the coefficient of the cubic term. They state in their paper, "this looks like the simplest one and it is easy to justify on physical grounds as the first higher-order interaction term in an expansion based on the quadrupole operator". This Hamiltonian is only discussed in the large N limit within the intrinsic state formalism and the phase diagram is explored. One key result is that, when $\chi=-\sqrt{7}/2$ in the $SU(3)$ limit, the cubic term corresponds to an *oblate* shape. Thus, a new evolutional path exists from the prolate shape to the oblate shape (Fig. 1). Based on the results, an analytically solvable prolate-oblate shape phase transitional description within the $SU(3)$ limit was investigated in [21], which offers a finite-N first-order shape transition. Most importantly, the phase transitional point is also a degenerate point [21] (Fig. 2), which may imply a hidden symmetry [22].

Another key result is that this extended Hamiltonian $\hat{H}_2$has only a very small region of rigid triaxiality in the large-N limit at $\chi\approx-\sqrt{7}/2$ when the parameter changes from the $U(5)$ limit to the $SU(3)$ degenerate point (green line in Fig. 1). Thus the shape transitional behaviors from the $U(5)$ limit to the $SU(3)$ degenerate point will differ significantly from the ones from the $U(5)$ limit to the $O(6)$ γ-soft rotation (red line in Fig. 1). Based on these important new findings in Refs. [20, 21], it is very interesting to investigate the spectra of this transitional region for finite-N, which is related to rigid or soft γ-triaxiality.

Some experimental requirements remain to deepen our understanding on γ-soft triaxiality and its relationship with the higher-order interactions. In recent experimental investigations, a cluster of extremely neutron-deficient nuclei ^{168}Os [23], ^{166}W [24], ^{172}Pt [25], and ^{170}Os [26] were observed to have an unpredictably small ratio of reduced transition probabilities $B_{4/2}=B(E2;4_1^+\to2_1^+)/B(E2;2_1^+\to0_1^+)$ along the yrast band. The $B(E2)$ anomaly is considered a puzzle in the field of nuclear structures and cannot be explained by many existing theories. Recently, the author successfully described the $B(E2)$ anomaly of the yrast band in ^{170}Os [27] for the first time via adding two third-order $SU(3)$ conserving interactions into the common Hamiltonian used in IBM-1. In this paper, the spectra of the nucleus is γ-soft. This unexpected discovery makes the $SU(3)$ higher-order interactions more important in the interacting boson models.

Wilets and Jean first provided a γ-soft solution based on the Bohr geometrical model [28]. When the IBM is determined, its $O(6)$ limit can be used to fully describe γ-soft features [29, 30], which is considered a landmark for success of the IBM. However, realistic γ-soft nuclei still cannot be completely described using these models. For example, ^{196}Pt is the first candidate for the $O(6)$ limit (see the spectra of ^{196}Pt in Fig. 3), but its quadrupole moment is not zero [31].

Furthermore, the phonon modes as a major paradigm for description of nuclei near the closed shell have been questioned [32–35] and should be replaced by γ-soft rotation [36]. Figure 3 shows the spectra of the ^{110}Cd normal states, which resembles the spherical spectra in the $U(5)$ limit and is different from the $O(6)$ γ-soft rotation (see the level positions of the 0_2^+ and 0_3^+ in ^{196}Pt and ^{110}Cd normal states). In ^{120}Cd (Fig. 3), it is shown that, the $U(5)$-like spectra is actually an illusion, and there is no 0_3^+ state in the three-phonon level [37]. In recent studies [38, 39] the 0_3^+ state is explained as intruder excitation for the 0_3^+ state as a band-head of a rotational band has been established in ^{112}Cd. Thus, in the normal states of the Cd isotopes, a 0^+ state near the energy level of 6_1^+ state does not exist. The problem is that the special γ-soft spectra of the normal states cannot be reproduced in the previous IBM Hamiltonian.

Recent detailed experimental investigations have revealed that the γ-soft behaviors in $^{126\text{-}132}$Xe [40–42] and $^{98\text{-}102}$Zr [43, 44] cannot also be explained using tradition-

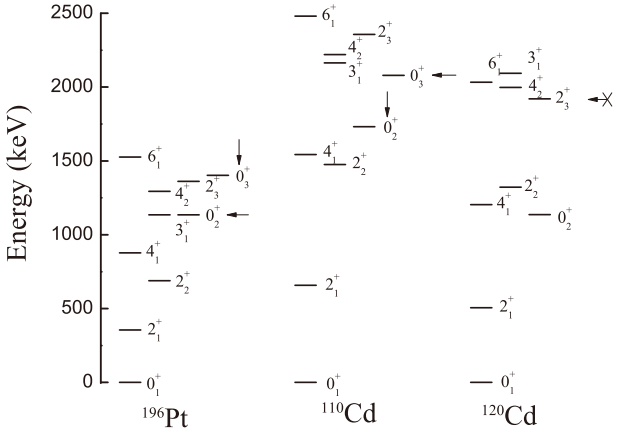

Fig. 3. Partial low-lying energy spectra of ^{196}Pt, ^{110}Cd normal states and ^{120}Cd normal states. ^{196}Pt is a typical example of a γ-soft nucleus described by the $O(6)$ limit, and there is no a 0^+ state near energy levels 4_1^+ and 2_2^+. ^{110}Cd has a typical spherical vibrational spectra, but ^{120}Cd has no the 0_3^+ state in the three-phonon level.

al γ-soft descriptions.

These results imply that our understanding of the γ-softness in nuclear structure theories is still incomplete. Searching for a new γ-soft rotational description is vital for our understanding of γ-softness and many theoretical deviations from experimental data. The aspects discussed above make a numerical study of the interacting bosom model with higher-order interactions meaningful: (1) The addition of higher-order interactions may induce new unpredicted phenomena, such as the $B(E2)$ anomaly. (2) Rigid triaxiality may be described by this model or its extensions, which is different from the traditional approach discussed in [6, 7] and is important for us to understand γ-rigid triaxial nuclei ^{76,78}Ge [45–47]. (3) Some new insights on γ-soft rotation are required to explain various γ-softness anomalies.

In this paper, new γ-soft rotational behaviors are sought. A similar Hamiltonian with the formalism $\hat{H}_2$ is investigated numerically. $U(5)$-like and $O(6)$-like γ-soft spectra are discovered, which can have large quadrupole moments. An unexpected partial $O(5)$ dynamical symmetry is observed. The $U(5)$-like spectra is applied in ^{110}Cd normal states, which can reproduce the type of the normal states.

II. HAMILTONIAN

Our aim is to search for new γ-soft rotation, which is compared with the traditional γ-soft behaviors in the transitional regions from the $U(5)$ limit to the $O(6)$ limit; therefore, the parameters are selected as in the triangle in Fig. 1. A direct numerical calculation of the cubic-$\hat{Q}_\chi$ interacting boson model Hamiltonian $\hat{H}_2$ [20] is possible, but their spectra appear to lack regularity. To determine the hidden structure associated with the $SU(3)$ degenerate point, a new formalism should be developed. This new Hamiltonian consists of three parts. The first part is the common d boson number operator $H^{(1)} = \hat{n}_d = d^\dagger \cdot \tilde{d}$ in the $U(5)$ limit, which produces the simple harmonic vibration. The second part is the quadrupole-quadrupole interaction $H^{(2)} = -\hat{Q}_0 \cdot \hat{Q}_0/N$ in the $O(6)$ limit, which produces the typical solvable γ-soft spectra. The combination of the two parts can describe the shape transitions from spherical to traditional γ-soft nuclei. The critical point of this transitional region can be described using $E(5)$ dynamical symmetry [48]. A typical feature of γ-softness is the level degeneracy between the ground-band and quasi-γ rotational band due to the common $O(5)$ symmetry, such as 4_1^+, 2_2^+ or 6_1^+, 4_2^+, 3_1^+ (Fig. 3).

When the third part is included, such as the quadrupole-quadrupole interaction $-\hat{Q}\cdot\hat{Q}/N$ in the $SU(3)$ limit, γ-softness is destroyed, and the level degeneracy between the ground-band and quasi-γ band is removed [1]. It appears the framework of the IBM-1 offers no opportunity for a new γ-soft description. Since the emergence of the interacting boson model nearly fifty years ago, the γ-softness is always related to the $O(6)$ dynamical symmetry.

The numerical results in this paper reveal that it is not the case. Here, the third part is a combination of the $SU(3)$ second-order and third-order Casimir operators,

$$\hat{H}^{(3)} = -\frac{\hat{C}_2[SU(3)]}{2N} + \kappa\frac{\hat{C}_3[SU(3)]}{2N^2}, \tag{3}$$

which is investigated in detail in Ref. [21]. The two Casimir operators have relationships with the quadrupole second or third-order interactions in the $SU(3)$ limit as follows:

$$\hat{C}_2[SU(3)] = 2\hat{Q}\cdot\hat{Q} + \frac{3}{4}\hat{L}\cdot\hat{L}, \tag{4}$$

$$\hat{C}_3[SU(3)] = -\frac{4}{9}\sqrt{35}[\hat{Q}\times\hat{Q}\times\hat{Q}]^{(0)} - \frac{\sqrt{15}}{2}[\hat{L}\times\hat{Q}\times\hat{L}]^{(0)}. \tag{5}$$

For a given $SU(3)$ irrep (λ,μ), the eigenvalues of the two Casimir operators under the group chain $U(6)\supset SU(3)\supset O(3)$ are expressed as

$$\langle\hat{C}_2[SU(3)]\rangle = \lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu, \tag{6}$$

$$\langle\hat{C}_3[SU(3)]\rangle = \frac{1}{9}(\lambda-\mu)(2\lambda+\mu+3)(\lambda+2\mu+3). \tag{7}$$

Thus, in the $SU(3)$ limit, $\hat{H}^{(3)}$ is solvable, which has a critical point at $\kappa_0 = (3N)/(2N+3)$, (Fig. 2, when N=7,

$\kappa_0 \approx 1.24$). Note that each irrep (λ,μ) corresponds to a special quadrupole deformation [9, 10]. When $\kappa < \kappa_0$, it presents a prolate shape, and when $\kappa > \kappa_0$, it is (nearly) an oblate shape. This shape phase transition is abrupt at the critical point. Importantly, the critical point is also a multiple-phase coexistence or degenerate point (Fig. 2). At this degenerate point, the $SU(3)$ irreps satisfying the condition $\lambda+2\mu=2N$ are all degenerate, and the degeneracy degree is $(N+2)/2$ for an even N or $(N+1)/2$ for an odd N. This degeneracy observed here is not accidental; thus, it may imply a hidden symmetry [22]. This degenerate point may be related to the important studies of Sebe and Akiyma, which are briefly discussed in Section VI. The Hamiltonian with $\kappa_0=(3N)/(2N+3)$ is adopted to the third part and is denoted as $\hat{H}^{(3)}$.

Now, we discuss the full Hamiltonian for the new γ-soft deformation, which is expressed as

$$\hat{H}=c[(1-\eta)H^{(1)}+\eta((1-\xi)H^{(2)}+\xi H^{(3)})], \tag{8}$$

where $0\le\eta\le1$, and $0\le\xi\le1$. When $\eta=0$, it is the $U(5)$ limit. When $\eta=1,\xi=1$, it represents the degenerate point in the $SU(3)$ limit, and when $\eta=1,\xi=0$, it is the $O(6)$ limit (Fig. 1). The Hamiltonian is numerically investigated using the diagonalization method based on the $SU(3)$ interaction [3, 27].

III. ANALYSIS OF γ-SOFT SPECTRA

For a new understanding on the spectra in Fig. 3 and their deviations from experimental data, the nine low-lying levels of $'2_1^+$', $'4_1^+,2_2^+,0_2^+$', and $'6_1^+,4_2^+,3_1^+,2_3^+,0_3^+$' are emphasized. In the $U(5)$ limit, they correspond to the one, two, and three phonon excited states, respectively. When moving to the $O(6)$ limit (Fig. 4(a) ξ=0.0), the transitional behavior is very familiar to us. For the common $O(5)$ symmetry, both $'4_1^+,2_2^+$' and $'6_1^+,4_2^+,3_1^+,0_{2or3}^+$' are always degenerate. The energy levels of 0_2^+ and 2_3^+ (corresponding to the $U(5)$ limit) shift upwards.

However, when $\xi\neq0.0$, the mixing of the $SU(3)$ degenerate point cannot break the *exact* degeneracy between the ground and quasi-γ rotational bands. In the total parameter triangle, the degeneracy of $'4_1^+,2_2^+$' is not broken and $'6_1^+,4_2^+,3_1^+$' are also degenerate (Fig. 4(b), (c) and Fig. 5(a), (b)). The results shown here are for $N=7$; however, this degenerate behavior can hold for any η,ξ and any N. This result is astonishingly unexpected and resembles a partial $O(5)$ dynamical symmetry [16]. Figure 4(c) shows the transitional behaviors from the spherical vibrational modes to the $SU(3)$ degenerate point, which is the main aim of this paper. The key finding is that the energy level of 0_2^+ state (blue line) is always between the degenerate $'4_1^+,2_2^+$' double states and degenerate $'6_1^+,4_2^+,3_1^+$' triple states. The energy level of 2_3^+ is slightly above those of the $'6_1^+,4_2^+,3_1^+$' triple states, while the 0_3^+ state is moved higher up (red line), and an abrupt change occurs. These $U(5)$-like spectra were not expected before this study, and this new γ-softness is not connected with the $O(6)$ dynamical symmetry. Figure 5(a) shows the transitional behaviors from the traditional $O(6)$ γ-soft ro-

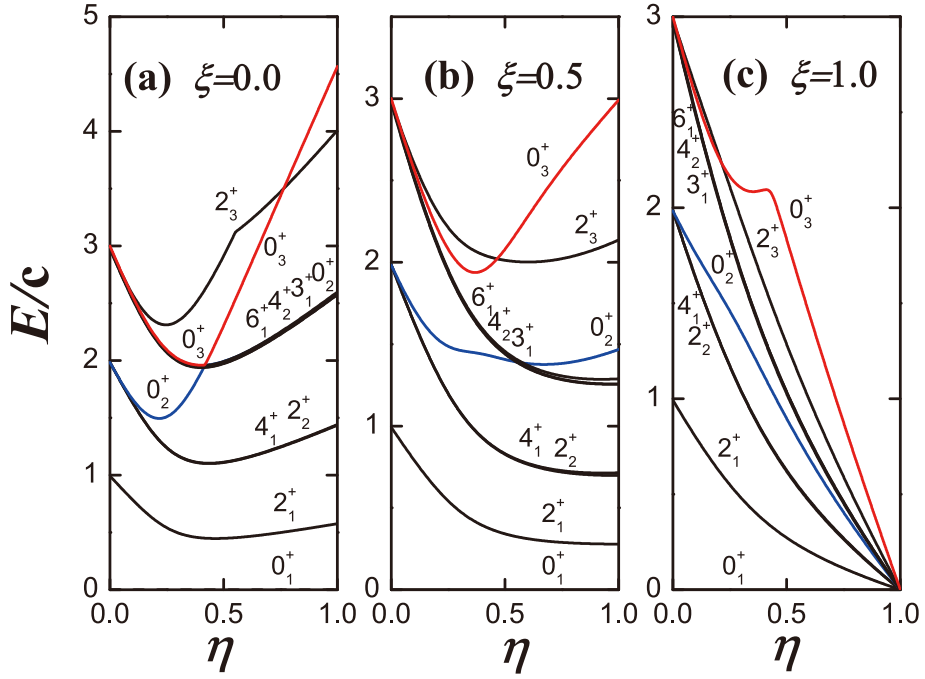

Fig. 4. (color online) Evolutional behaviors of partial low-lying levels when η changes from 0.0 to 1.0 and (a)$\xi=0.0$, (b) $\xi=0.5$, and (c) $\xi=1.0$ for $N=7$. It is clearly shown that, in the total parameter triangle, both $'4_1^+,2_2^+$' and $'6_1^+,4_2^+,3_1^+$' are always degenerate.

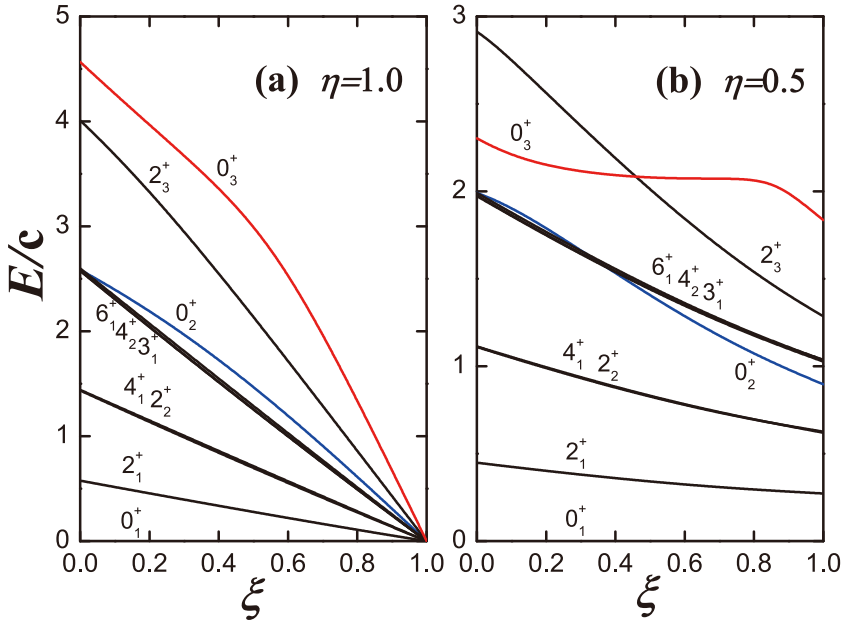

Fig. 5. (color online) Evolutional behaviors of partial low-lying levels when ξ changes from 0.0 to 1.0 and (a)$\eta = 1.0$ and (b) $\eta = 0.5$ for $N = 7$. The spectra in (a) resemble the one in the $O(6)$ limit.

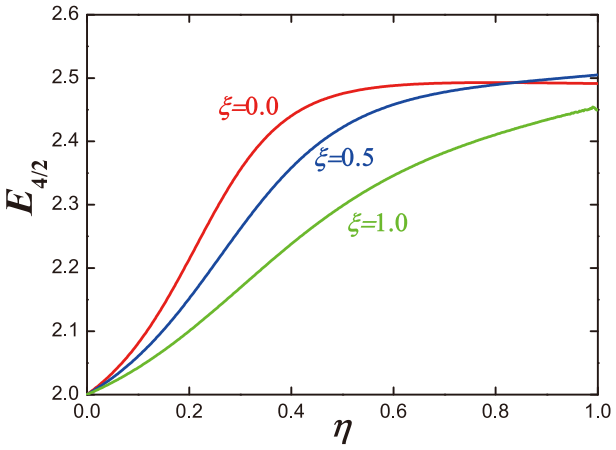

Fig. 6. (color online) Evolutional behaviors of $E_{4/2}$ when η changes from 0.0 to 1.0 and $\xi = 0.0$ (red line), $\xi = 0.5$ (blue line), $\xi = 1.0$ (green line) for $N = 7$.

tation to the $SU(3)$ degenerate point, which shows that the spectra are all similar to the $O(6)$ γ-soft ones. The transitional behaviors of the energy ratios $E_{4/2}$ between $E(4_1^+)$ and $E(2_1^+)$ change from 2 to 2.5 in a similar style when moving from the $U(5)$ limit to the $O(6)$ limit and other various $O(6)$-like locations (Fig. 6). When $\xi = 1.0$ and $\eta = 0.5$, the energy ratio $E_{4/2}$ is approximately 2.3, slightly smaller than 2.5. This result is nearly the same as the ratio of the ^{110}Cd nucleus (2.34).

These results offer a new understanding for γ-soft rotations. Since the classical research on γ-soft spectra in [28], it has been a major paradigm of collective nuclear structure for more than sixty years. In this paper, we show that the simplest IBM-1 still has significantly more space for γ-soft rotational behaviors, except for the classical $O(6)$ description via introducing higher-order interactions. The key aspect is the exact degeneracy in $'4_1^+, 2_2^+'$ and $'6_1^+, 4_2^+, 3_1^+'$, which is the typical feature of γ-softness. This type of degeneracy can hold for higher energy levels, such as $'8_1^+, 6_2^+, 5_1^+, 4_3^+'$, which are not shown here.

IV. ANALYSIS OF $B(E2)$ AND QUADRUPOLE MOMENT

The $B(E2)$ values are necessary for us to understand the γ-softness. The operator is defined as

$$\hat{T}(E2) = e\hat{Q}_\chi, \tag{9}$$

where e is the boson effective charge and $\chi = -\sqrt{7}/2\xi$. The evolutional behaviors of the $B(E2; 2_1^+ \to 0_1^+)$, $B(E2; 0_2^+ \to 2_1^+)$, $B(E2; 0_2^+ \to 2_2^+)$, $B(E2; 0_3^+ \to 2_1^+)$, and $B(E2; 0_3^+ \to 2_2^+)$ values are plotted in Fig. 7 when η changes from 0.0 to 1.0 and $\xi = 0.0$ (a), $\xi = 0.5$ (b), and $\xi = 1.0$ (c). The transitional behavior of the $B(E2; 2_1^+ \to 0_1^+)$ value is similar to the one from the $U(5)$ limit to $O(6)$ limit. The values of the $B(E2; 0_2^+ \to 2_1^+)$ are reduced whereas the ones of the $B(E2; 0_2^+ \to 2_2^+)$ are promoted. The values of the $B(E2; 0_3^+ \to 2_2^+)$ are also reduced. These are all the typical trends for the γ-soft rotation (Fig. 8).

Now, we discuss the transitional pattern for the quadrupole moment of the 2_1^+ state, which is shown in Fig. 9. This quantity in the $O(6)$ limit is absolutely zero. If a $SU(3)$ two-body interaction is mixed, a large value can be obtained, but the γ-soft degeneracy will be significantly destroyed. This confusion has existed for many years.

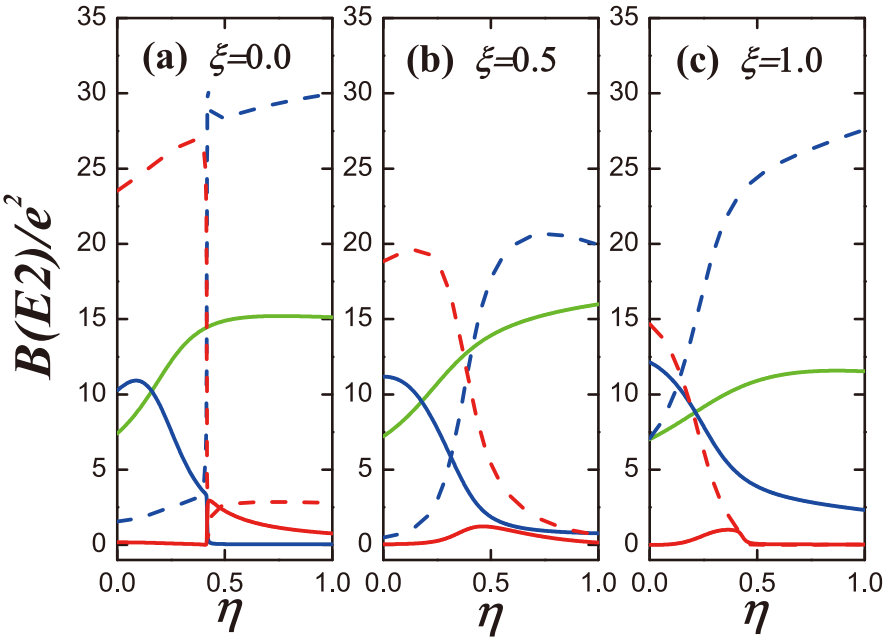

Fig. 7. (color online) Evolutional behaviors of $B(E2;2_1^+ \to 0_1^+)$ (green solid line), $B(E2;0_2^+ \to 2_1^+)$ (blue solid line), $B(E2;0_2^+ \to 2_2^+)$ (blue dashed line), $B(E2;0_3^+ \to 2_1^+)$ (red solid line), and $B(E2;0_3^+ \to 2_2^+)$ (red dashed line) when η changes from 0.0 to 1.0 and (a) $\xi = 0.0$, (b) $\xi = 0.5$, and (c) $\xi = 1.0$ for $N = 7$.

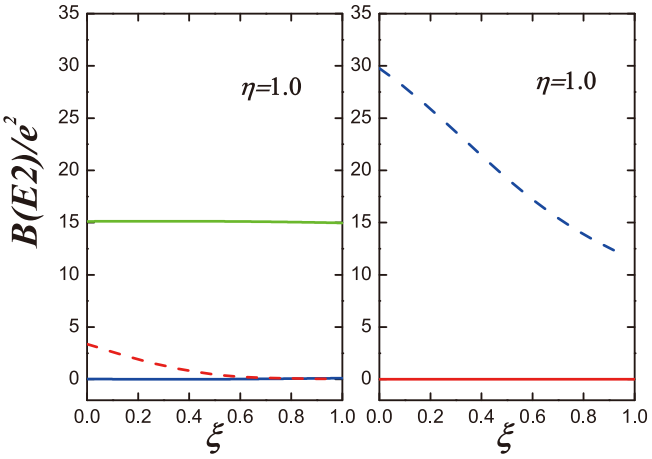

Fig. 8. (color online) Evolutional behaviors of $B(E2;2_1^+ \to 0_1^+)$ (green solid line), $B(E2;0_2^+ \to 2_1^+)$ (blue solid line), $B(E2;0_2^+ \to 2_2^+)$ (blue dashed line), $B(E2;0_3^+ \to 2_1^+)$ (red solid line), and $B(E2;0_3^+ \to 2_2^+)$ (red dashed line) when ξ changes from 0.0 to 1.0 and $\eta = 1.0$ for $N = 7$.

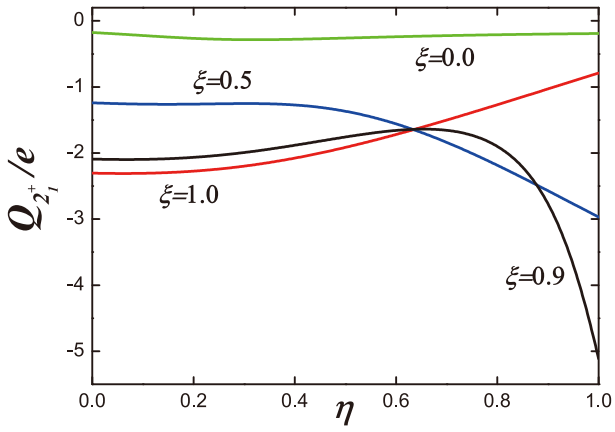

Fig. 9. (color online) Evolutional behaviors of the quadrupole moment of the 2_1^+ state when η changes from 0.0 to 1.0 and $\xi = 0.0$ (green), $\xi = 0.5$ (blue), $\xi = 0.9$ (black), and $\xi = 1.0$ (red) for $N = 7$.

Specifically, this quantity is related to the first excited state, which is vital for a successful model [31]. Fig. 9 shows that the values increase when ξ changes from 0 to 1 for $\eta = 1.0$, which may resolve the debates between γ-softness and large quadrupole moment.

V. THEORETICAL FITTING OF ^{110}Cd NORMAL STATES

The $U(5)$-like softness can be used to fit the spectra of ^{110}Cd normal states, and the $\hat{L}^2$ term is also incorporated. The Hamiltonian is

$$\hat{H}' = \alpha H^{(1)} + \beta H^{(3)} + \gamma \hat{L}^2, \tag{10}$$

here $\xi = 1.0, \alpha = c(1-\eta), \beta = c\eta$, and γ is the coefficient of the $\hat{L}^2$ term. Fig. 10 shows the partial low-lying levels for ^{110}Cd normal states. The fitting results reproduce the style of experimental data; however, for higher-levels, the theoretical values are larger than the experimental ones. To reduce the energies of the higher-levels, Pan *et al.* presented a new method and observed an excellent fitting result

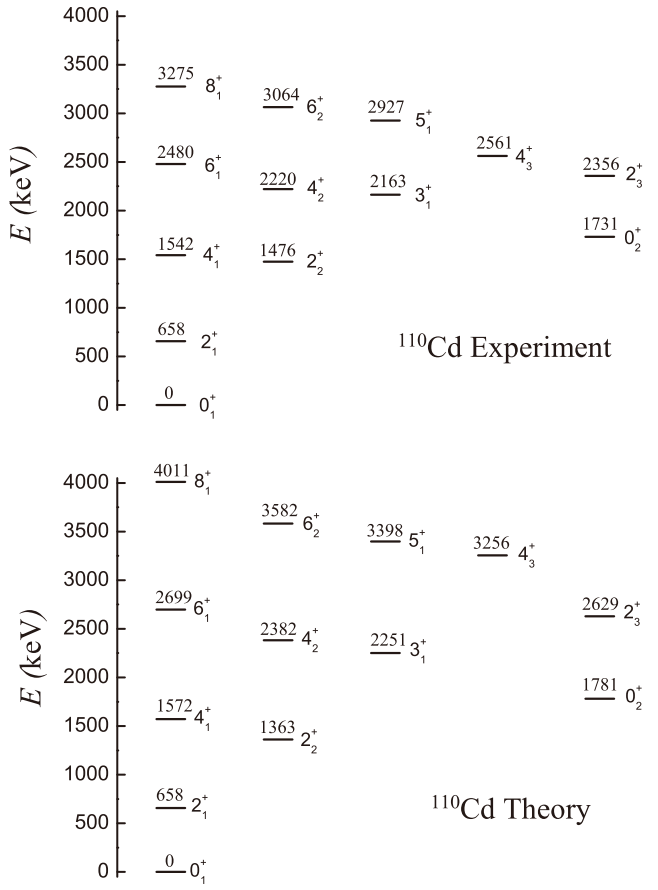

Fig. 10. Partial low-lying levels for ^{110}Cd normal states. The values above are the experimental data and the ones below are the results of theoretical fitting.

Table 1. Absolute $B(E2)$ values in W.u. for $E2$ transitions from the low-lying normal states in ^{110}Cd [36, 50] and present results when $\alpha = 956.54$ keV, $\beta = 672.28$ keV, and $\gamma = 14.41$ keV with an effective charge of $e = 1.609$ (W.u.)$^{1/2}$.

L_i	L_f	Expt.	Results
2^+_1	0^+_1	27.0(8)	27.0
4^+_1	2^+_1	42(9)	39.0
2^+_2	2^+_1	30(5); 19(4)[a]	39.8
	0^+_1	1.35(20); 0.68(14)[a]	0.17
0^+_2	2^+_1	< 7.9[a]	12.2
	2^+_2	<1680[a]	59.5
6^+_1	4^+_1	40(30); 62(18)[a]	44.3
	4^+_2	< 5[a]	3.08
4^+_2	4^+_1	12^{+4}_{-6}; $10.7^{+4.9}_{-4.8}$[a]	21.1
	2^+_1	$0.20^{+0.06}_{-0.09}$; 0.14(6)[a]	0.04
	2^+_2	32^{+10}_{-14}; 22(10)[a]	23.2
3^+_1	4^+_1	$5.9^{+1.8}_{-4.6}$; $2.4^{+0.9}_{-0.8}$[a]	12.9
	2^+_1	$1.1^{+0.3}_{-0.8}$; 0.85(25)[a]	0.11
	2^+_2	32^{+8}_{-24}; 22.7(69)[a]	31.5
2^+_3	4^+_1	< 5[a]	5.95
	2^+_1	$2.8^{+0.6}_{-1.0}$	0.08
	2^+_2	$0.7^{+0.5}_{-0.6}$[a]	3.62
	0^+_2	24.2(22)[a]	24.2

[a] From Ref. [36].

for ^{194}Pt [49], which may be considered in future research to improve the fitting precision. The theoretical energy of the 0^+_3 state is 3411.21 keV, which is significantly larger than the energy of the 6^+_1 state, smaller than the 6^+_2 state, and nearly twice the energy of the 0^+_2 state. Experimentally, a 0^+ state exists with an energy of 2662 keV, which is between the energies of the 6^+_1 state (2480 keV) and 6^+_2 state (2877 keV).

The $B(E2)$ values of the lowest levels are compared with the experimental results of the normal states in ^{110}Cd, which are listed in Table 1. This γ-soft description can produce a good consistency with the empirical data qualitatively, which is different from the phonon descriptions. The quadrupole moment of the 2^+_1 state is -0.36 eb, which is also close to the experimental value of -0.39 eb. When weak mixing with the intruder states is considered, more reasonable results may be obtained [36]. Thus, our Hamiltonian may be also vital for understanding the Cd puzzle [51].

VI. SOME DISCUSSIONS

If "vibrational behavior at low energy is refuted" [51], deformation has a dominant role when describing the collective nature of nuclei, and the description of triaxiality has a more important role than before. In IBM-1, protons and neutrons are not distinguished. If the rigid triaxial deformation for the ground state of a nucleus is included, the higher-order interactions must be considered [6]. However, in IBM-2 [1], protons and neutrons can be considered separately, and the triaxial shape can be described even with up to two-body interactions [52–55]. Three-body interactions $[d^\dagger d^\dagger d^\dagger]^{(L)} \cdot [\tilde{d}\tilde{d}\tilde{d}]^{(L)}$ are also used in IBM-2 to investigate the γ triaxiality [56]. In the sdg-IBM, $L = 4$ g bosons can be introduced and hexadecapole deformation can be discussed [57]. Except for the interacting boson models, triaxial shapes have also been investigated using many existing nuclear models [4, 51, 58–64].

In our paper, a new γ-soft rotational mode is observed, which is different from previous triaxial descriptions. In IBM-1 in particular, this finding can complement the existing $O(6)$ γ-soft description. This γ-softness is related to the $SU(3)$ degenerate point. Degeneracy implies symmetry [22]. Numerical results show that this symmetry is a $O(5)$ partial dynamical symmetry. This finding is somewhat beyond expectation. We can ob-

serve that the degenerate point is generated by the states with the $SU(3)$ irreps satisfying the condition $\lambda+2\mu=2N$. This degenerate condition may provide some clues. In IBM-1, nuclear states can be classified by $U(6)$ symmetry [1]. In the $SU(3)$ limit, symmetry reduction chain $U(6)\supset SU(3)\supset SO(3)$ is used. In 1968, Sebe indicated that for the special type of states with irreps $\lambda+2\mu=2N$, the $U(6)\supset SU(3)$ reduction coefficient is identical to that for $U(3)\supset SO(3)$ [65]. This can be used to explain the emergence of the degenerate point[1)]. In 1985, when studying the sdg IBM in the $SU(3)$ scheme, Akiyama observed that the $SU(3)$ representations (λ,μ) satisfying $\lambda+2\mu=4N$ belong to a special class. The reduction coefficient for the $U(15)\supset SU(3)$ chain is the same as that for $U(5)\supset SO(5)\supset SO(3)$ [66]. If the sd IBM-1 can be considered a reduction of the sdg IBM, the $U(6)\supset SU(3)$ reduction may correspond to parts of the $U(5)\supset SO(5)\supset SO(3)$ reduction, and $O(5)$ partial dynamical symmetry may occur. A detailed discussion on this is required in the future. A numerical study on the transitional behaviors for our similar Hamiltonians from the $U(5)$ limit to the $SU(3)$ degenerate point in the sdg IBM would be extraordinarily interesting [67].

Triaxiality can be soft [28] or rigid [68]. The relationship between the two scenarios is an interesting topic under nuclear structures [56, 64]. In the large-N limit, Ref. [20] demonstrated that a rigid triaxial shape can exist from the $U(5)$ limit to the $SU(3)$ degenerate point. Our discussion directly corresponds to this interval. Numerical results have revealed that the triaxial deformation is γ-soft for small boson numbers. Thus, our model can connect the γ-soft and γ-rigid scenarios. Otsuka *et al.* demonstrated that, in finite systems, the soft and rigid triaxial descriptions can be identical [69], and they also provided a transitional description from γ-soft to γ-rigid shapes when the boson number N is increased [70]. With our model, it would also be interesting to study the transitional behaviors when N increases.

Since the Cd puzzle was proposed [32, 33, 51], some new attempts have been made to explain the experimental data. Garrett *et al.* indicated that no evidence exists for the strong mixing of the normal and intruder states, and the decay pattern is a signature of a γ-soft nucleus [36]. Nomura *et al.* conducted constrained self-consistent mean-field calculations to obtain the deformation energy surface, and a prolate global minimum was observed for $^{108\text{-}116}$Cd [71]. An attractive explanation was provided by Leviatan *et al.* with $U(5)$ partial dynamical symmetry [72]. The $U(5)$ dynamical symmetry was partly broken, and large anharmonicity for some nonyrast states was introduced. The two-phonon 0^+ state still exists, but at a higher excitation energy. This theory can excellently reproduce the data except for the 0^+_4 state (two-phonon state). Our concept is similar to this; however, the $U(5)$ dynamical symmetry is not kept and only $O(5)$ dynamical is partly broken. The γ-soft mode can still exist. In the theory of Leviatan *et al.*, the 0^+ two-phonon state can have enhanced $B(E2)$ values for decay to the 2^+_1 state, which is not supported by existing experimental results [39]. This deficiency may be overcome in our theory.

Recently, Garrett *et al.* promoted the developments on the spherical nucleus puzzle [38, 39]. In their experiments, very weak decay branches from nonyrast states were observed, and the 0^+_4 band was identified for ^{112}Cd. Within this band, $(E_{4^+_1}-E_{0^+_1})/(E_{2^+_1}-E_{0^+_1})=2.95$, which is a signature of large deformation. These experimental data were interpreted using beyond-mean-field calculations, and they suggested the Cd isotopes exhibit a multiple shape coexistence. The ground state had a prolate shape. However, their explanation was insufficient for the intruder states quantitatively.

Finally, some subtle connections in some Zn-Ge isotopes has been observed to date. In ^{70}Zn and ^{72,74}Ge, the shape coexistence has been observed [73, 74]. ^{76,78}Ge are observed to have a rigid triaxial deformation for their ground states [45– 47]. Interestingly, ^{72,74}Zn have an $B(E2)$ anomaly [75]. Thus, a self-consistent nuclear model is required to explain these nuclei. A microscopic calculation based on the nuclear density functional theory predicted that most of the observed γ-soft Ge nuclei exhibit some features between the γ-rigid and γ-unstable rotor limits [76]. In our opinion, including the $SU(3)$ higher-order interactions into the common IBM-1 may be a simple and effective method.

VII. CONCLUSIONS

New γ-soft triaxial rotation is observed in the interacting boson model with $SU(3)$ conserving higher-order interactions. This attempt appears meaningful. With the aid of introducing $SU(3)$ higher-order terms, the yrast $B(E2)$ anomaly has been described successfully. In this paper, following the studies in Refs. [20, 21], the transitional behaviors from the $U(5)$ limit to the $SU(3)$ degenerate point is numerically explored and compared with the transitional behaviors from the $U(5)$ limit to the $O(6)$ limit. In particular, spherical-like γ-soft triaxial rotational spectra actually exists, which may be what is required to solve the spherical nucleus puzzle [36]. $O(6)$-like γ-soft spectra with large quadrupole moment are also shown. Importantly, in the parameter triangle, the degeneracy of the corresponding energy levels between the ground and quasi-γ bands are not broken, which is a partial $O(5)$ dynamical symmetry [16]. This symmetry is related to the studies of Sebe and Akiyama [65, 66], and they will be discussed in the future.

1) This point is pointed out by one of the anonymous reviewers.

The new γ-soft triaxial spectra may be intimately connected with the realistic spectra in Cd isotopes, and it may be an effective description of the spherical-like nuclei [34, 35, 51]. A further discussion for multiple shape coexistence in ^{110}Cd and ^{112}Cd [38, 39] will be conducted. Additionally, many new experimental γ-soft anomalies in $^{126\text{-}132}$Xe [40–42] and $^{98\text{-}102}$Zr [43, 44] will be investigated. The relationship between γ-rigid triaxial deformation in ^{76,78}Ge [45–47] and higher-order quadrupole interactions also require further discussion in the future.

In this paper, we show that $SU(3)$ higher-order interaction may be more important than expected before. Here, only the $SU(3)$ cubic interaction $[\hat{Q}\times\hat{Q}\times\hat{Q}]^{(0)}$ is considered. In Ref. [27], two $SU(3)$ third-order interactions $[\hat{Q}\times\hat{Q}\times\hat{Q}]^{(0)}$, $[\hat{L}\times\hat{Q}\times\hat{L}]^{(0)}$ were included to explain the $B(E2)$ anomaly successfully. In future discussions on fourth-order interactions, $(\hat{Q}\cdot\hat{Q})^2$ should be also considered for it is related to the rigid triaxial rotation [9, 10], and any type of quadrupole deformation can be described in a consistent manner.

The partial $O(5)$ dynamical symmetry revealed in this paper can be further used to discuss quantum chaos in nuclear physics [77–79], which may provide a new regular region in the interacting boson model [80, 81]. This will be discussed in the future. In addition, the numerical results for large N are also required to study the quantum phase transition of this transitional region from the $U(5)$ limit to the $SU(3)$ degenerate point and the rigid triaxial rotation in the large-N limit described in Ref. [20].

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