Linear seesaw model with a modular S_4 flavor symmetry*

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Abstract: We discuss a linear seesaw model with a field content as minimum as possible, introducing a modular S_4 using gauged $U(1)_{B-L}$ symmetries. Owing to the rank two neutrino mass matrix, we obtain a vanishing neutrino mass eigenvalue, and only the normal mass hierarchy of neutrinos is favored via the modular S_4 symmetry. In our numerical $\Delta\chi^2$ analysis, we especially determine a relatively sharp prediction on the sum of neutrino masses to be approximately 60 meV, in addition to other predictions.

Keywords: neutrino mass, linear seesaw mechanism, modular S_4 flavor symmetry

DOI: 10.1088/1674-1137/ac4975

I. INTRODUCTION

The neutrino sector is theoretically unconfirmed in the standard model (SM), because only two mass squared differences and three mixings are experimentally found and the scale of mass is extremely minuscule compared to the other three sectors in the SM. Hence, several scientists expect the neutrino sector to possess new physics. A gauged $B-L$ (baryon number minus lepton number) $U(1)$ symmetry; $U(1)_{B-L}$, is a promising prescription for generating such tiny neutrino masses, introducing three right-handed neutrinos with rather heavy masses ($M_{\rm R}$), and is called a canonical seesaw model [1-4]. Because the mass scale is sometimes expected to be that of a grand unified theory ($M_{\rm GUT}\sim 10^{15}$ GeV) to be small neutrino masses, its scale cannot be verified by our current experiments, and the spontaneous $U(1)_{B-L}$ symmetry breaking scale is naturally expected to be the same energy as the cut-off scale; $M_{\rm R}\sim M_{\rm GUT}$.

To achieve a successful neutrino mass model within our scale (~TeV), heavy neutral fermions ($S_{\rm L}$) with left-handed chirality are introduced along this line of thought. Inverse seesaw model [5, 6] also requests both of $N_{\rm R}$ and $S_{\rm L}$ and the neutrino mass could be realized within TeV scale. But this model may not require GUT scale. Currently, $N_{\rm R}$ and $S_{\rm L}$ can be embedded into the middle scale with $SU(3)_{\rm C}\otimes SU(2)_{\rm L}\otimes SU(2)_{\rm R}\otimes U(1)_{B-L}$ [6-8], which is included in $SO(10)$ group [9]. Then, the neutrino mass matrix would not be suppressed by the middle scale, but by $M_{\rm GUT}$ scale when appropriate charge assignments are assigned for each fields in a supersymmetric theory [9]. Hence, we can test phenomenologies with our current experiments, supposing the middle scale breaking can occur at the TeV scale. This type of model is called the "linear seesaw".[1)] Because these models typically require more free parameters than the other three sectors in the SM fermion, flavor symmetries are also frequently introduced in these models, to reduce the parameters and obtain predictions (if possible).

In 2017, attractive flavor symmetries were proposed in Refs. [10, 11], in which the authors applied modular non-Abelian discrete flavor symmetries to quark and lepton sectors. A remarkable advantage is that any dimensionless coupling can also be transformed as non-trivial representations under these symmetries. Therefore, we do not need so many scalars to determine a predictive mass matrix. Another advantage is that we have a modular weight from the modular origin that can play a role in stabilizing DM when appropriate charge assignments are distributed to each of the fields of the models. Along this

Received 23 December 2021; Accepted 11 January 2022; Published online 25 March 2022

* Supported by an appointment to the JRG Program at the APCTP through the Science and Technology Promotion Fund and Lottery Fund of the Korean Government

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1) Notice here that our model has a different mechanism from the one of original linear seesaw, even though our neutrino model can be realized within TeV scale. Therefore, our neutrino mass matrix directly depends on the scale of $U(1)_{B-L}$ breaking and ratio between vacuum expectation values (VEVs) of two Higgs doublet model as can be seen in the main text, while we expect higher energy scale such as GUT in order that the modular field must break and get VEV denoted by τ in our literature.

line of thought, significantly relevant references have recently appeared in the literature, e.g., A_4 [10, 12-48], S_3 [49-54], S_4 [55-66], A_5 [60, 67, 68], double covering of A_5 [69-72], larger groups [73], multiple modular symmetries [74], double covering of A_4 [75-77], S_4 [78, 79], and other types of groups [80-85] in which masses, mixing, and CP phases for the quark and/or lepton have been predicted. For interested readers, we provide some literature reviews, which are useful to understand the non-Abelian group and its applications to flavor structure [86-93]. Moreover, a systematic approach to understanding the origin of CP transformations has been discussed in Ref. [94]; in addition, CP/flavor violation in the models with modular symmetry was discussed in Refs. [95-97], and a possible correction from the Kähler potential was discussed in Ref. [98]. Furthermore, a systematic analysis of the fixed points (stabilizers) was discussed in Ref. [99]. It would be interesting to consider a linear see-saw model with the local $U(1)_{B-L}(\subset SO(10))$ under modular symmetry because these symmetries can be obtained from a string theory. Moreover, the nature of the modular symmetry can be used to realize the linear seesaw mechanism, in addition to constraining the flavor structure.

In this study, we propose a linear seesaw model under modular S_4, using the $U(1)_{B-L}$ symmetry, in which we try to construct the predictive model as minimum as possible. Owing to the rank two neutrino mass matrix, we obtain a vanishing neutrino mass eigenvalue. Furthermore, only the normal mass hierarchy of neutrinos is favored via the modular S_4 symmetry. In our numerical analysis, we perform $\Delta\chi^2$ analysis in the neutrino sector, considering non-unitarity constraint.

This remainder of this paper is organized as follows. In Sec. II, we review our model, constructing renormalizable Lagrangian and mass matrices in the lepton sector. Then, we formulate the neutrino mass matrix with rank two, in which we estimate the structure of the neutrino mass matrix in the expansion of modulus. In addition, we derive several observables in the lepton sector. At the end of this section, we discuss the non-unitarity bound. In Sec. III, we perform $\Delta\chi^2$ analysis in the lepton sector, and present some predictions using our model. In Sec. IV, we present a summary and discussion. In the Appendix, we elucidate the modular S_4 symmetry.

II. MODEL

A. Model review

In this section, we review our model framework for the linear seesaw mechanism, introducing the $B-L$ local Abelian symmetry; $U(1)_{B-L}$ and modular A_4 symmetry. Regarding the fermion sector, we add two left-handed neutral fermions $\{S_{L_1}, S_{L_2}\}$ that belong to the isospin singlet, where they exhibit a zero charge under $U(1)_{B-L}$, $\{\mathbf{1},\mathbf{1'}\}$ under S_4, and -1 under $-k_I$, respectively. In addition, we introduce three right-handed neutral fermions $\overline{N_R}$ that belong to the isospin singlet, where they exhibit 1 charge under $U(1)_{B-L}$, $\mathbf{3}$ under S_4, and -3 under $-k$, respectively. The SM left-handed leptons $L_L \equiv [L_{L_e}, L_{L_\mu}, L_{L_\tau}]^T$ belong to -1 charge under $U(1)_{B-L}$, $\mathbf{3'}$ under S_4, and -1 under $-k$, respectively. However, the SM right-handed leptons $\{\overline{e_R}, [\overline{\mu_R}, \overline{\tau_R}]\}$ belong to the $+1$ charge under $U(1)_{B-L}$, $\{\mathbf{1},\mathbf{2}\}$ under S_4, and $\{-1,-3\}$ under $-k_I$, respectively.

Regarding the scalar sector, we adopt two Higgs doublets, H_1, H_2, and an isospin singlet field φ. The isospin singlet φ has a -1 charge under $U(1)_{B-L}$. Here, H_1 is the SM-like Higgs that has a zero charge under $U(1)_{B-L}$ and $-k_I$ while H_2 has 1 charge under $U(1)_{B-L}$, and zero under $-k_I$. We denote each of the vacuum expectation values (VEVs) to be $\langle H_{1,2}\rangle \equiv [0, v_{1,2}/\sqrt{2}]^T$, and $\langle\varphi\rangle \equiv v_\varphi/\sqrt{2}$. We summarize our particle content and assignments in Table 1.

Then the valid lepton Yukawa Lagrangian is expressed as

$$-\mathcal{L}_{\rm lepton} = \mathcal{L}_{M_\ell} + \mathcal{L}_{M_D} + \mathcal{L}_{M'_D} + \mathcal{L}_{M_{NS}}, \qquad (1)$$

where $\mathcal{L}_{M_\ell}$ denotes the charged lepton Yukawa Lagrangian. $\mathcal{L}_{M_D}$ belongs to $[\overline{N_R} L_L \tilde{H}_1]$ where $\tilde{H} \equiv i\sigma_2 H^*$. $\mathcal{L}_{M'_D}$ belongs to $[\overline{L_L^C} S_L H_2]$. $\mathcal{L}_{M_{NS}}$ belongs to $[\overline{N_R} S_L \varphi]$. $[\cdots]$ implies that the concrete flavor structures are manifolded. Each of their structures are presented below.

The scalar potential of our model is expressed as

$$\begin{aligned} V =& m_\varphi^2 \varphi^*\varphi + m_1^2 H_1^\dagger H_1 + m_2^2 H_2^\dagger H_2 - \mu_{12}(H_1^\dagger H_2 \varphi + {\rm h.c.}) \\ &+ \lambda_1 (H_1^\dagger H_1)^2 + \lambda_2 (H_2^\dagger H_2)^2 + \lambda_\varphi (\varphi^*\varphi)^2 \\ &+ \lambda_3 (H_1^\dagger H_1)(H_2^\dagger H_2) + \lambda_4 (H_1^\dagger H_2)(H_2^\dagger H_1) \\ &+ \lambda_{\varphi H_1} (H_1^\dagger H_1)(\varphi^*\varphi) + \lambda_{\varphi H_2} (H_2^\dagger H_2)(\varphi^*\varphi), \end{aligned} \qquad (2)$$

where h.c. stands for Hermitian conjugate. We consider that φ develops a VEV at a significantly larger scale than $H_{1,2}$. Then, after the φ developes a VEV, the scalar poten-

Table 1. Lepton and boson particle contents and their charge assignments under $SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times S_4 \times (-k_I)$ where $L_L \equiv [L_{L_e}, L_{L_\mu}, L_{L_\tau}]^T$ k_I represents the number of the modular weight.

	Fermions				Scalars		
	L_L	$\overline{e_R}, [\overline{\mu_R}, \overline{\tau_R}]$	$\overline{N_R}$	S_{L_1}, S_{L_2}	H_1	H_2	φ
$SU(2)_L$	**2**	**1**	**1**	**1**	**2**	**2**	**1**
$U(1)_Y$	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$U(1)_{B-L}$	-1	1	1	0	0	1	-1
S_4	**3'**	**1**, **2**	**3**	**1**, **1'**	**1**	**1**	**1**
$-k_I$	-1	$-1, -3$	-3	-1	0	0	0

tial matches with that of two Higgs doublet model (THDM) without the $(H_1^\dagger H_2)^2$ term owing to the $U(1)_{B-L}$ symmetry. In addition, the Yukawa couplings associated with the two Higgs doublet belong to the type-I THDM because only H_1 can couple to the SM fermions. Hence, in our analysis below, we do not discuss the THDM part further, but focus on the neutrino mass.

B. Valid Lagrangians

Before discussing the valid Lagrangians for the lepton sector, we define Yukawa couplings under the modular S_4 symmetry as follows: the S_4 doublets with $-k_I = 2$ and 4 are denoted by $Y_{\mathbf{2}}^{(2)} \equiv [y_1, y_2]^T$ and $[y'_1, y'_2]^T$, respectively. The S_4 triplet with $-k_I = 2$ is $Y_{\mathbf{3}'}^{(2)} \equiv [y_3, y_4, y_5]^T$, while the S_4 triplets with $-k_I = 4$ are $Y_{\mathbf{3}}^{(4)} \equiv [y'_3, y'_4, y'_5]^T$ and $Y_{\mathbf{3}'}^{(4)} \equiv [y''_3, y''_4, y''_5]^T$. Each of these structures are explicitly presented in the Appendix.

Charged lepton mass matrix:

The renormalizable Lagrangian for the charged-lepton sector is expressed as

$$-\mathcal{L}_{M_\ell} = \alpha_\ell \overline{e_{\rm R}}(y_3 L_{{\rm L}_e} + y_4 \bar{L}_{{\rm L}_\tau} + y_5 \bar{L}_{{\rm L}_\mu})\tilde{H}_1 + \beta_\ell\left[\frac{\sqrt{3}}{2}\overline{\mu_{\rm R}}(y'_4 L_{{\rm L}_\mu} + y'_5 L_{{\rm L}_\tau}) + \overline{\tau_{\rm R}}\left(-y'_3 L_{{\rm L}_e} + \frac{1}{2}(y'_4 L_{{\rm L}_\tau} + y'_5 L_{{\rm L}_\mu})\right)\right]\tilde{H}_1$$
$$+ \gamma_\ell\left[\frac{\sqrt{3}}{2}\overline{\tau_{\rm R}}(y''_4 L_{{\rm L}_\mu} + y''_5 L_{{\rm L}_\tau}) + \overline{\mu_{\rm R}}\left(y''_3 L_{{\rm L}_e} - \frac{1}{2}(y''_4 L_{{\rm L}_\tau} + y''_5 L_{{\rm L}_\mu})\right)\right]\tilde{H}_1 + \text{h.c.}, \quad (3)$$

where $\{\alpha_\ell, \beta_\ell, \gamma_\ell\}$ are real parameters without loss of generality. Subsequently, the charged-lepton mass matrix after the spontaneous symmetry breaking is expressed as

$$(M_\ell)_{\rm RL} = \frac{v_1}{\sqrt{2}}\begin{pmatrix} \alpha_\ell y_3 & \alpha_\ell y_5 & \alpha_\ell y_4 \\ \gamma_\ell y''_3 & \frac{\sqrt{3}}{2}\beta_\ell y'_4 - \frac{1}{2}\gamma_\ell y''_5 & \frac{\sqrt{3}}{2}\beta_\ell y'_5 - \frac{1}{2}\gamma_\ell y''_4 \\ -\beta_\ell y'_3 & \frac{\sqrt{3}}{2}\gamma_\ell y''_4 + \frac{1}{2}\beta_\ell y'_5 & \frac{\sqrt{3}}{2}\gamma_\ell y''_5 + \frac{1}{2}\beta_\ell y'_4 \end{pmatrix}. \quad (4)$$

The charged-lepton mass eigenvalues are obtained by diagonalizing ${\rm diag}[m_e, m_\mu, m_\tau] = V_{{\rm R}_\ell}^\dagger M_\ell V_{{\rm L}_\ell}$, where $V_{{\rm L}_\ell,{\rm R}_\ell}$ represent unitary matrices. In our numerical analysis, we will determine the free parameters $\{\alpha_\ell, \beta_\ell, \gamma_\ell\}$, to fit the three charged-lepton mass eigenstates after providing all the numerical values, by applying the following relationship. Here, we fix $\alpha_\ell, \beta_\ell, \gamma_\ell$, to determine the experimental three charged-lepton masses by applying the following relationship:

$$\begin{aligned}
&{\rm Tr}[{M_\ell}^\dagger M_\ell] = |m_e|^2 + |m_\mu|^2 + |m_\tau|^2, \\
&{\rm Det}[{M_\ell}^\dagger M_\ell] = |m_e|^2 |m_\mu|^2 |m_\tau|^2, \\
&({\rm Tr}[{M_\ell}^\dagger] M_\ell)^2 - {\rm Tr}[({M_\ell}^\dagger M_\ell)^2] \\
&= 2(|m_e|^2|m_\mu|^2 + |m_\mu|^2|m_\tau|^2 + |m_e|^2|m_\tau|^2).
\end{aligned} \quad (5)$$

Neutral fermion mass matrices:

First, we construct the valid Lagrangian of the Dirac mass matrix $\mathcal{L}_{M_{\rm D}}$. This is given by

$$\mathcal{L}_{M_{\rm D}} = \alpha_{\rm D}\left[y'_3\left(\overline{N_{{\rm R}_3}} L_{{\rm L}_\mu} - \overline{N_{{\rm R}_2}} L_{{\rm L}_\tau}\right) + y'_4\left(\overline{N_{{\rm R}_1}} L_{{\rm L}_\tau} - \overline{N_{{\rm R}_3}} L_{{\rm L}_e}\right) + y'_5\left(\overline{N_2} L_{{\rm L}_e} - \overline{N_{{\rm R}_1}} L_{{\rm L}_\mu}\right)\right]\tilde{H}_1$$
$$+ \beta_{\rm D}\left[y''_3\left(\overline{N_{{\rm R}_3}} L_{{\rm L}_\tau} - \overline{N_{{\rm R}_2}} L_{{\rm L}_\mu}\right) + y''_4\left(-\overline{N_{{\rm R}_1}} L_{{\rm L}_\mu} - \overline{N_{{\rm R}_2}} L_{{\rm L}_e}\right) + y''_5\left(\overline{N_{{\rm R}_1}} L_{{\rm L}_\tau} + \overline{N_{{\rm R}_3}} L_{{\rm L}_e}\right)\right]\tilde{H}_1$$
$$+ \gamma_{\rm D}\left[\frac{\sqrt{3}}{2} y'_1(\overline{N_{{\rm R}_2}} L_{{\rm L}_\mu} + \overline{N_{{\rm R}_3}} L_{{\rm L}_\tau}) + y'_2\{-\overline{N_{{\rm R}_1}} L_{{\rm L}_e} + \frac{1}{2}\left(\overline{N_{{\rm R}_2}} L_{{\rm L}_\tau} + \overline{N_{{\rm R}_3}} L_{{\rm L}_\mu}\right)\}\right]\tilde{H}_1 + \text{h.c.}, \quad (6)$$

where we suppose $\alpha_{\rm D}$ to be real while $\beta_{\rm D}$, $\gamma_{\rm D}$ are complex after rephasing the fields. Similar to the charged-lepton sector, we determine the Dirac mass matrix as follows:

$$(M_{\rm D})_{\rm RL} = \frac{v_1}{\sqrt{2}}\begin{pmatrix} -\gamma_{\rm D} y'_2 & -\alpha_{\rm D} y'_5 - \beta_{\rm D} y''_4 & \alpha_{\rm D} y'_4 + \beta_{\rm D} y''_5 \\ \alpha_{\rm D} y'_5 - \beta_{\rm D} y''_4 & -\beta_{\rm D} y''_3 + \frac{\sqrt{3}}{2}\gamma_{\rm D} y'_1 & -\alpha_{\rm D} y'_3 + \frac{1}{2}\gamma_{\rm D} y'_2 \\ -\alpha_{\rm D} y'_4 + \beta_{\rm D} y''_5 & \alpha_{\rm D} y'_3 + \frac{1}{2}\gamma_{\rm D} y'_2 & \beta_{\rm D} y''_3 + \frac{\sqrt{3}}{2}\gamma_{\rm D} y'_1 \end{pmatrix}. \quad (7)$$

For convenience, to analyze the neutrino oscillation, we redefine $M_{\rm D} \equiv \frac{v_1}{\sqrt{2}}\tilde{M}_{\rm D}$.

Another Dirac Lagrangian is induced via $\mathcal{L}_{M'_{\rm D}}$, which is given by

$$\mathcal{L}_{M'_{\rm D}} = \alpha'_{\rm D}\left[y_3\overline{L^{\rm C}_{{\rm L}_e}} + y_4\overline{L^{\rm C}_{{\rm L}_\tau}} + y_5\overline{L^{\rm C}_{{\rm L}_\mu}}\right]S_{{\rm L}_1}H_2 + {\rm h.c.}, \tag{8}$$

Subsequently, we determine another Dirac mass matrix as follows:

$$(M'_{\rm D})_{L_{\rm L}S_{\rm L}} = \frac{v_2}{\sqrt{2}}\alpha'_{\rm D}\begin{pmatrix} y_3 & 0 \\ y_5 & 0 \\ y_4 & 0 \end{pmatrix}. \tag{9}$$

Being the same as the reason for $M_{\rm D}$, we redefine $M'_{\rm D} \equiv \frac{v_2}{\sqrt{2}}\alpha'_{\rm D}\tilde{M}'_{\rm D}$.

The third term of the Lagrangian $\mathcal{L}_{M_{NS}}$ is given by

$$\begin{aligned}\mathcal{L}_{M_{NS}} =& \alpha_{NS}\left[y'_3\overline{N_{{\rm R}_e}} + y'_4\overline{N_{{\rm R}_\tau}} + y'_5\overline{N_{{\rm R}_\mu}}\right]S_{{\rm L}_1}\varphi \\ &+ \beta_{NS}\left[y''_3\overline{N_{{\rm R}_e}} + y''_4\overline{N_{{\rm R}_\tau}} + y''_5\overline{N_{{\rm R}_\mu}}\right]S_{{\rm L}_2}\varphi + {\rm h.c.},\end{aligned} \tag{10}$$

where α_{NS}, β_{NS} are real without loss of generality. Accordingly, we obtain the mass matrix

$$M_{NS} = \frac{v_\varphi}{\sqrt{2}}\begin{pmatrix} y'_3 & y''_3 \\ y'_5 & y''_5 \\ y'_4 & y''_4 \end{pmatrix}\begin{pmatrix} \alpha_{NS} & 0 \\ 0 & \beta_{NS} \end{pmatrix} \equiv \frac{v_\varphi}{\sqrt{2}}\tilde{M}_{NS}. \tag{11}$$

In basis of $[\nu_{\rm L}, N^{\rm C}_{\rm R}, S_{\rm L}]^T$, the neutral fermion mass matrix is given by

$$M_N = \begin{pmatrix} 0_{3\times3} & M^T_{\rm D} & M'_{\rm D} \\ M_{\rm D} & 0_{3\times3} & M_{NS} \\ m'^T_{\rm D} & M^T_{NS} & 0_{2\times2} \end{pmatrix}. \tag{12}$$

Hence, the active neutrino mass matrix is expressed as

$$\begin{aligned} m_\nu =& M'_{\rm D}(M^T_{NS}M_{NS})^{-1}M^T_{NS}M_{\rm D} \\ &+ [M'_{\rm D}(M^T_{NS}M_{NS})^{-1}M^T_{NS}M_{\rm D}]^T \\ =& \frac{v_1v_2}{\sqrt{2}v_\varphi}\Big(\tilde{M}'_{\rm D}(\tilde{M}^T_{NS}\tilde{M}_{NS})^{-1}\tilde{M}^T_{NS}\tilde{M}_{\rm D} \\ &+ [\tilde{M}'_{\rm D}(\tilde{M}^T_{NS}\tilde{M}_{NS})^{-1}\tilde{M}^T_{NS}\tilde{M}_{\rm D}]^T\Big) = \kappa\tilde{m}_\nu, \end{aligned} \tag{13}$$

where $\kappa \equiv \frac{v_1v_2}{\sqrt{2}v_\varphi}$ and we assume the mass hierarchies among $M_{\rm D}, M'_{\rm D} \ll M_{NS}$. Mass hierarchies is dynamically achieved in Refs. [100, 101]. The neutrino mass eigenvalues are obtained as follows: $D_\nu = \kappa D_\nu = U^T_\nu m_\nu U_\nu = \kappa U^T_\nu \tilde{m}_\nu U_\nu$, where U_ν is a unitary matrix. Then, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is given by $U_{\rm PMNS} \equiv V^\dagger_{{\rm L}_\ell}U_\nu$. Notice that m_ν is rank two; hence, the lightest neutrino mass is zero. Here κ is described by one experimental value and dimensionless neutrino mass eigenstates as:

$$({\rm NH}):\ \kappa^2 = \frac{|\Delta m^2_{\rm atm}|}{\tilde{D}^2_{\nu_3}}, \quad ({\rm IH}):\ \kappa^2 = \frac{|\Delta m^2_{\rm atm}|}{\tilde{D}^2_{\nu_2}}, \tag{14}$$

where $\Delta m^2_{\rm atm}$ denotes the atmospheric neutrino mass-squared difference and NH (IH) stands for normal (inverted) ordering. Subsequently, the solar mass difference squared can be expressed in terms of κ as follows:

$$({\rm NH}):\ \Delta m^2_{\rm sol} = \kappa^2\tilde{D}^2_{\nu_2}, \quad ({\rm IH}):\ \Delta m^2_{\rm sol} = \kappa^2(\tilde{D}^2_{\nu_2} - \tilde{D}^2_{\nu_1}), \tag{15}$$

which can be compared to the observed value. In other words, we explicitly express the mass eigenvalues in terms of $\Delta m^2_{\rm atm}$ and $\Delta m^2_{\rm sol}$ as:

$$({\rm NH}):\ D^2_{\nu_1} = 0, \quad D^2_{\nu_2} = \Delta m^2_{\rm sol}, \quad D^2_{\nu_3} = \Delta m^2_{\rm atm}, \tag{16}$$

$$({\rm IH}):\ D^2_{\nu_1} = \Delta m^2_{\rm atm} - \Delta m^2_{\rm sol}, \quad D^2_{\nu_2} = \Delta m^2_{\rm sol}, \quad D^2_{\nu_3} = 0, \tag{17}$$

which implies that NH is hierarchical, but IH is degenerate, as $\Delta m^2_{\rm sol}/\Delta m^2_{\rm atm} << 1$. Here, we expand $|\tilde{m}_\nu|^2$ in terms of $q \equiv {\rm e}^{2p\pi i\tau} << 1$. Then, the mass matrix is given by

$$|\tilde{m}_\nu|^2 \sim \begin{pmatrix} O(1) & O(q) & 0 \\ O(q) & O(1) & O(q^2) \\ 0 & O(q^2) & 0 \end{pmatrix}. \tag{18}$$

The ratio between two nonzero squared eigenvalues R is estimated by

$$R = O(q) << 1. \tag{19}$$

This suggests that the neutrino mass eigenvalues tend to be hierarchical; hence, NH is favored. In fact, we would not obtain the allowed region within 3σ for IH in our numerical analysis. Therefore, we focus on NH hereafter.

In our model, the PMNS matrix is parametrized by three mixing angles $\theta_{ij}(i,j = 1,2,3; i < j)$, one CP violating Dirac phase $\delta_{\rm CP}$, and one Majorana phase α_{21} as follows:

$$U_{\rm PMNS}=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{20}$$

where c_{ij} and s_{ij} represent $\cos\theta_{ij}$ and $\sin\theta_{ij}$, respectively. Then, these mixings are given in terms of the components of $U_{\rm PMNS}$ as follows:

$$\sin^2\theta_{13}=|(U_{\rm PMNS})_{13}|^2,\quad \sin^2\theta_{23}=\frac{|(U_{\rm PMNS})_{23}|^2}{1-|(U_{\rm PMNS})_{13}|^2},$$
$$\sin^2\theta_{12}=\frac{|(U_{\rm PMNS})_{12}|^2}{1-|(U_{\rm PMNS})_{13}|^2}. \tag{21}$$

Furthermore, we compute the Jarlskog invariant $J_{\rm CP}$ derived from PMNS matrix elements as follows:

$$J_{\rm CP}={\rm Im}[U_{e1}U_{\mu2}U^*_{e2}U^*_{\mu1}]=s_{23}c_{23}s_{12}c_{12}s_{13}c^2_{13}\sin\delta_{\rm CP}. \tag{22}$$

Majorana phase is estimated in terms of other invariant I_1 as follows:

$$I_1={\rm Im}[U^*_{e1}U_{e2}]=c_{12}s_{12}c^2_{13}\sin\left(\frac{\alpha_{21}}{2}\right). \tag{23}$$

In addition, the effective mass for the neutrinoless double beta decay is expressed as:

$$({\rm NH}):\ \langle m_{ee}\rangle=\kappa|\tilde D_{\nu_2}s^2_{12}c^2_{13}e^{i\alpha_{21}}+\tilde D_{\nu_3}s^2_{13}e^{-2i\delta_{CP}}|, \tag{24}$$

$$({\rm IH}):\ \langle m_{ee}\rangle=\kappa|\tilde D_{\nu_1}c^2_{12}c^2_{13}+\tilde D_{\nu_2}s^2_{12}c^2_{13}e^{i\alpha_{21}}|, \tag{25}$$

where its value could be measured by KamLAND-Zen in the future [102]. We will adopt the neutrino experimental data at the 3σ interval in Nufit 5.0 [103, 104].

Non-unitarity:

Here, let us briefly discuss the non-unitarity matrix $U'_{\rm PMNS}$. This is typically parametrized by the form

$$U'_{\rm PMNS}\equiv\left(1-\frac{1}{2}F^\dagger F\right)U_{\rm PMNS}, \tag{26}$$

where $F\equiv(M^T_{NS}M_{NS})^{-1}M^T_{NS}M_{\rm D}$ is a hermitian matrix and $U'_{\rm PMNS}$ represents the deviation from the unitarity. Applying global constraints [105], one finds [106]

$$|FF^\dagger|\le\begin{bmatrix}2.5\times10^{-3} & 2.4\times10^{-5} & 2.7\times10^{-3}\\ 2.4\times10^{-5} & 4.0\times10^{-4} & 1.2\times10^{-3}\\ 2.7\times10^{-3} & 1.2\times10^{-3} & 5.6\times10^{-3}\end{bmatrix}. \tag{27}$$

In our case, $F\equiv(M^T_{NS}M_{NS})^{-1}M^T_{NS}M_{\rm D}=\frac{v_1}{v_\varphi}(\tilde M^T_{NS}\tilde M_{NS})^{-1}\times \tilde M^T_{NS}\tilde M_{\rm D}$. Because v_φ is freely considered to be large (while $v_1=O(100)$ GeV at most), we easily satisfy this bound.

III. NUMERICAL ANALYSIS

In this section, we perform a numerical $\Delta\chi^2$ analysis to determine the parameters that satisfy the neutrino oscillation data and non-unitarity constraint; in addition, we present our predictions, where we adopt best fit values of charged-lepton masses. Here, we focus on NH because IH is disfavored by the analytical estimation, as can be observed in the previous section.

In our numerical analysis, we randomly scan free parameters in following ranges

$$\{\alpha_{\rm D},|\beta_{\rm D}|,|\gamma_{\rm D}|,\alpha_{NS},\beta_{NS}\}\in[10^{-5},1.0],\quad t_\beta\in[10,100], \tag{28}$$

where τ runs over the fundamental region, $t_\beta\equiv v_1/v_2$, and $\sqrt{v_1^2+v_2^2}=246$ GeV. We perform numerical analysis under the above regions. Figure 1 illustrates the correlation between the real and imaginary parts of τ, where the blue, green, and red points are allowed within 2, 3, and 5 of the $\Delta\chi^2$ analysis, respectively, for five accurately known observables $\Delta m^2_{\rm atm},\Delta m^2_{\rm sol},s^2_{12},s^2_{23},s^2_{13}$ in Nufit 5.0 [103, 104]. The real part runs through the entire range, but the imaginary part is localized at the $[1.35-1.5]$.

Figure 2 illustrates the correlation between the Majorana phase α_{21} and Dirac CP phase $\delta_{\rm CP}$. The legend is the

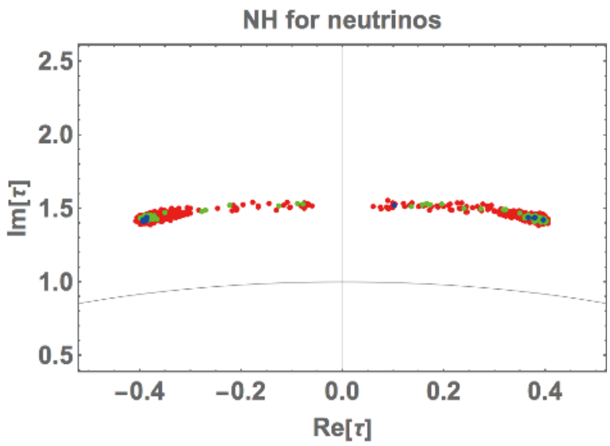

Fig. 1. (color online) Allowed region of modulus τ, where the blue, green, and red points are allowed within 2, 3, and 5 of the $\Delta\chi^2$ analysis, respectively.

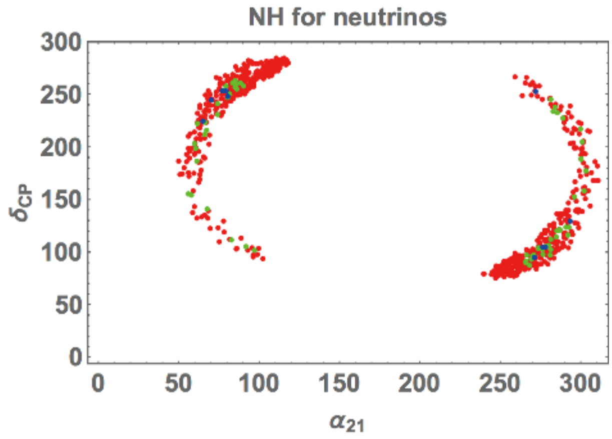

Fig. 2. (color online) Correlation between a Majorana phase α_{21} and $\delta_{\rm CP}$, where the legend is the same as the case of Fig. 1.

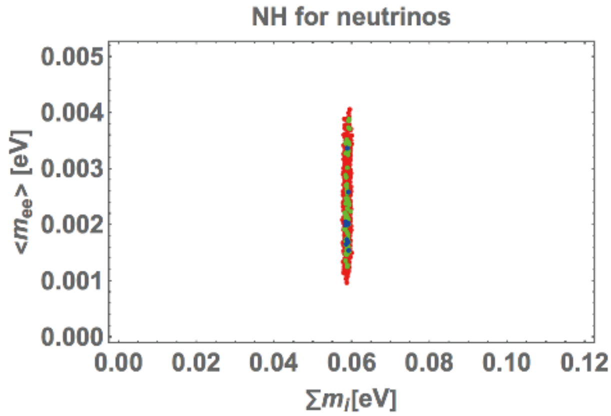

Fig. 3. (color online) Predicted correlation between the effective mass for the neutrinoless double beta decay $\langle m_{ee}\rangle$ and the sum of neutrino masses $\sum m_i$ in [eV], where the legend is the same as the case of Fig. 1.

same as in the case of Fig. 1. We identify a clear feature in the region where α_{21} is within $\{0°-120°, 240°-310°\}$, while $\delta_{\rm CP}$ is within $\{70°-290°\}$.

The upper figures in Fig. 3 present the correlation between the effective mass for the neutrinoless double beta decay $\langle m_{ee}\rangle$ and the sum of neutrino masses $\sum m_i$ in the [eV] unit, where the legend is the same as the case of Fig. 1. We determine that $\langle m_{ee}\rangle$ is allowed within $[0.001-0.004]$ eV. However, $\sum m_i$ is restricted to be approximately 0.06 eV, which would be a sharp prediction of this model.

Finally, we present a benchmark in Table 2, which is selected, such that $\sqrt{\Delta\chi^2}$ is minimum. The mass matrices for the dimensionless neutrino and charged-lepton are determined as

$$\tilde{m}_\nu = \begin{bmatrix} 37.1331-17.9304i & -16.2576+55.2032i & 35.2235+24.3462i \\ -16.2576+55.2032i & -40.0223-38.1947i & -55.9229+44.0753i \\ 35.2235+24.3462i & -55.9229+44.0753i & -1.23231+74.8642i \end{bmatrix}, \tag{29}$$

$$M_\ell = \begin{bmatrix} 0.00232+0.000467i & -0.00111+0.00314i & -0.00453+0.00274i \\ -0.282+0.102i & -0.0524-0.469i & 0.214-0.559i \\ 0.336-0.277i & 0.362+0.706i & -0.160+1.23i \end{bmatrix}. \tag{30}$$

Table 2. Benchmark point of our input parameters and observables, which is selected, such that $\sqrt{\Delta\chi^2}$ is minimum.

parameter	valve
τ	$0.113762+1.43906i$
t_β	98.6
$[\alpha_\ell, \gamma_\ell, \beta_\ell]$	$[1.72\times10^{-5},\ 6.15\times10^{-4},\ 8.06\times10^{-4}]$
$[\alpha_{\rm D}, \alpha_{NS}, \beta_{NS}]$	$[-0.0112907,\ -0.00078512,\ 0.0203379]$
$[\beta_{\rm D}, \gamma_{\rm D}]$	$[-0.01432-0.00360i, -1.90\times10^{-5}-2.34\times10^{-6}i]$
$\Delta m^2_{\rm atm}$	$2.53\times10^{-3}{\rm eV}^2$
$\Delta m^2_{\rm sol}$	$7.48\times10^{-5}{\rm eV}^2$
$\sin^2\theta_{12}$	0.289
$\sin^2\theta_{23}$	0.565
$\sin^2\theta_{13}$	0.02207
$[\delta_{\rm CP},\ \alpha_{21}]$	$[248°, 80.3°]$
$\sum m_i$	58.5 meV
$\langle m_{ee}\rangle$	1.65 meV
$\sqrt{\Delta\chi^2}$	1.40

IV. SUMMARY AND DISCUSSION

We studied a linear seesaw model with a field content as minimum as possible, introducing a modular S_4 using $U(1)_{B-L}$ symmetries. Owing to the rank two neutrino mass matrix, we obtained a vanishing neutrino mass eigenvalue. Furthermore, only the normal mass hierarchy of neutrinos was favored via the modular S_4 symmetry. In our numerical $\Delta\chi^2$ analysis, we determined a rather sharp prediction on the sum of neutrino masses to be approximately 60 meV. The imaginary part of τ was restricted at 1.35-1.5, while the real part ran through the entire range in the fundamental region. Other remarks are presented below:

1. α_{21} was within $\{0°-120°, 240°-310°\}$, while $\delta_{\rm CP}$ was within $\{70°-290°\}$.

2. $\langle m_{ee}\rangle$ was allowed by $\{0.001-0.004\}$ eV.

Therefore our model indicates several predictions in the neutrino sector that was triggered by the minimal structure with the S_4 modular symmetry.

ACKNOWLEDGMENTS

This research was supported by the Korean Local Governments - Gyeongsangbuk-do Province and Pohang City (H.O.). H. O. is sincerely grateful to the KIAS member.

APPENDIX A

Here, we review some properties of the modular S_4 symmetry. In general, the modular group $\bar{\Gamma}$ is the group of linear fractional transformation γ acting on the modulus τ, which belongs to the upper-half complex plane and transforms as

$$\tau \longrightarrow \gamma\tau = \frac{a\tau+b}{c\tau+d}, \tag{A1}$$

where $a,b,c,d \in \mathbb{Z}$ and $ad-bc=1$, $\mathrm{Im}[\tau]>0$.

This is isomorphic to $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{I,-I\}$ transformation. Then modular transformation is generated by two transformations S and T defined as:

$$S:\tau \longrightarrow -\frac{1}{\tau}\,, \qquad T:\tau \longrightarrow \tau+1\,, \tag{A2}$$

and they satisfy the following algebraic relations,

$$S^2 = \mathbb{I}\,, \qquad (ST)^3 = \mathbb{I}\,. \tag{A3}$$

Here, we introduce the series of groups $\Gamma(N)$ ($N = 1,2,3,\ldots$) which are defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod} N) \right\}, \tag{A4}$$

and we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I,-I\}$ for $N=2$. Because the element $-I$ does not belong to the $\Gamma(N)$ for $N>2$ case, we have $\bar{\Gamma}(N) = \Gamma(N)$, which are the infinite normal subgroup of $\bar{\Gamma}$ known as principal congruence subgroups. Hence, we obtain finite modular groups as the quotient groups defined by $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$. For these finite groups Γ_N, $T^N = \mathbb{I}$ is imposed, and the groups Γ_N with $N = 2,3,4,5$ are isomorphic to S_3, A_4, S_4 and A_5, respectively [11].

Modular forms of level N are holomorphic functions $f(\tau)$ transformed under the action of $\Gamma(N)$, which is expressed as:

$$f(\gamma\tau) = (c\tau+d)^k f(\tau)\,,\ \ \gamma \in \Gamma(N)\,, \tag{A5}$$

where k denotes the reputed modular weight.

Here, we discuss the modular symmetric theory framework without imposing supersymmetry explicitly, considering the S_4 ($N=4$) modular group. Under the modular transformation in Eq. (A1), a field $\phi^{(I)}$ is also transformed as

$$\phi^{(I)} \to (c\tau+d)^{-k_I} \rho^{(I)}(\gamma)\phi^{(I)}, \tag{A6}$$

where $-k_I$ represents the modular weight and $\rho^{(I)}(\gamma)$ denotes a unitary representation matrix of $\gamma \in \Gamma(4)$. Hence, a Lagrangian such as the Yukawa terms can be invariant if the sum of modular weight from fields and the modular form in the corresponding term is zero (also invariant under S_4 and gauge symmetry).

The kinetic and quadratic terms of the scalar fields can be expressed as:

$$\sum_I \frac{|\partial_\mu \phi^{(I)}|^2}{(-i\tau+i\bar{\tau})^{k_I}}\,, \quad \sum_I \frac{|\phi^{(I)}|^2}{(-i\tau+i\bar{\tau})^{k_I}}\,, \tag{A7}$$

which is invariant under the modular transformation, and the overall factor is eventually absorbed by a field redefinition consistently. Therefore, the Lagrangian associated with these terms should be invariant under the modular symmetry.

The basis of modular forms with weight 2, $Y_2^{(2)} = (y_1, y_2)$ and $Y_{3'}^{(2)} = (y_3, y_4, y_5)$, which are transformed as a doublet and triplet under S_4, are determined in terms of the Dedekind eta-function $\eta(\tau)$ and its derivative [10]:

$$y_1(\tau) = \frac{\mathrm{i}}{8}\left(8\frac{\eta'\left(\tau+\frac{1}{2}\right)}{\eta\left(\tau+\frac{1}{2}\right)} + 32\frac{\eta'(4\tau)}{\eta(4\tau)} - \frac{\eta'\left(\frac{\tau}{4}\right)}{\eta\left(\frac{\tau}{4}\right)} - \frac{\eta'\left(\frac{\tau+1}{4}\right)}{\eta\left(\frac{\tau+1}{4}\right)} - \frac{\eta'\left(\frac{\tau+2}{4}\right)}{\eta\left(\frac{\tau+2}{4}\right)} - \frac{\eta'\left(\frac{\tau+3}{4}\right)}{\eta\left(\frac{\tau+3}{4}\right)} \right),$$

$$y_2(\tau) = \frac{\mathrm{i}\sqrt{3}}{8}\left(\frac{\eta'\left(\frac{\tau}{4}\right)}{\eta\left(\frac{\tau}{4}\right)} - \frac{\eta'\left(\frac{\tau+1}{4}\right)}{\eta\left(\frac{\tau+1}{4}\right)} + \frac{\eta'\left(\frac{\tau+2}{4}\right)}{\eta\left(\frac{\tau+2}{4}\right)} - \frac{\eta'\left(\frac{\tau+3}{4}\right)}{\eta\left(\frac{\tau+3}{4}\right)} \right),$$

$$y_3(\tau) = \mathrm{i}\left(\frac{\eta'\left(\tau+\frac{1}{2}\right)}{\eta\left(\tau+\frac{1}{2}\right)} - 4\frac{\eta'(4\tau)}{\eta(4\tau)} \right),$$

$$y_4(\tau) = \frac{\mathrm{i}}{4\sqrt{2}}\left(-\frac{\eta'\left(\frac{\tau}{4}\right)}{\eta\left(\frac{\tau}{4}\right)} + \mathrm{i}\frac{\eta'\left(\frac{\tau+1}{4}\right)}{\eta\left(\frac{\tau+1}{4}\right)} + \frac{\eta'\left(\frac{\tau+2}{4}\right)}{\eta\left(\frac{\tau+2}{4}\right)} - \mathrm{i}\frac{\eta'\left(\frac{\tau+3}{4}\right)}{\eta\left(\frac{\tau+3}{4}\right)} \right),$$

$$y_5(\tau)=\frac{\mathrm{i}}{4\sqrt{2}}\left[-\frac{\eta'\left(\frac{\tau}{4}\right)}{\eta\left(\frac{\tau}{4}\right)}-i\frac{\eta'\left(\frac{\tau+1}{4}\right)}{\eta\left(\frac{\tau+1}{4}\right)}+\frac{\eta'\left(\frac{\tau+2}{4}\right)}{\eta\left(\frac{\tau+2}{4}\right)}+\mathrm{i}\frac{\eta'\left(\frac{\tau+3}{4}\right)}{\eta\left(\frac{\tau+3}{4}\right)}\right]. \tag{A8}$$

y_i's can be expanded in terms of q as follows:

$$y_1=-3\pi\left(\frac{b_1}{8}+3b_5\right),\ \ y_2=3\sqrt{3}\pi b_3,\ \ y_3=-\pi\left(-\frac{b_1}{4}+2b_5\right),$$
$$y_4=-\pi\sqrt{2}b_2,\ \ y_5=-4\pi\sqrt{2}b_4, \tag{A9}$$

where b_i are given by

$$b_1\sim 1,\ \ b_2\sim q,\ \ b_3\sim q^2,\ \ b_4\sim 0,\ \ b_5\sim 0, \tag{A10}$$

with $q=\exp(2\pi i\tau)$ and $|q|\ll 1$ [96].

Subsequently, Yukawas with higher weights are constructed by the multiplication rules of S_4, and the following couplings can be obtained:

$$Y^{(4)}_{\mathbf{2}}=\begin{bmatrix} y_2^2-y_1^2 \\ 2y_1y_2\end{bmatrix},\ \ Y^{(4)}_{\mathbf{3}}=\begin{bmatrix} -2y_2y_3 \\ \sqrt{3}y_1y_5+y_2y_4 \\ \sqrt{3}y_1y_4+y_2y_5\end{bmatrix},$$
$$Y^{(4)}_{\mathbf{3'}}=\begin{bmatrix} 2y_1y_3 \\ \sqrt{3}y_2y_5-y_1y_4 \\ \sqrt{3}y_2y_4-y_1y_5\end{bmatrix}. \tag{A11}$$

References

[1] P. Minkowski, Phys. Lett. B **67**, 421 (1977)
[2] T. Yanagida, in *Proceedings of the Workshop on the Unified Theory and the Baryon Number in the Universe* (O. Sawada and A. Sugamoto, eds.), KEK, Tsukuba, Japan, 1979, p. 95
[3] M. Gell-Mann, P. Ramond, and R. Slansky, *Supergravity* (P. van Nieuwenhuizen et al. eds.), North Holland, Amsterdam, 1979, p. 315; S. L. Glashow, *The future of elementary particle physics*, in *Proceedings of the 1979 Cargèse Summer Institute on Quarks and Leptons* (M. Levy et al. eds.), Plenum Press, New York, 1980, p. 687
[4] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980)
[5] R. N. Mohapatra and J. W. F. ValleSenjanovic, Phys. Rev. D **34**, 1642 (1986)
[6] D. Wyler and L. Wolfenstein, Nucl. Phys. B **218**, 205 (1983)
[7] E. K. Akhmedov, M. Lindner, E. Schnapka *et al.*, Phys. Lett. B **368**, 270 (1996), arXiv:[hep-ph/9507275]
[8] E. K. Akhmedov, M. Lindner, E. Schnapka *et al.*, Phys. Rev. D **53**, 2752 (1996), arXiv:[hep-ph/9509255]
[9] M. Malinsky, J. C. Romao, and J. W. F. Valle, Phys. Rev. Lett. **95**, 161801 (2005), arXiv:hep-ph/0506296 [hep-ph]
[10] F. Feruglio, doi: 10.1142/9789813238053_0012 arXiv: 1706.08749[hep-ph]
[11] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Nucl. Phys. B **858**, 437-467 (2012), arXiv:1112.1340[hep-ph]
[12] J. C. Criado and F. Feruglio, SciPost Phys. **5**(5), 042 (2018), arXiv:1807.01125[hep-ph]
[13] T. Kobayashi, N. Omoto, Y. Shimizu *et al.*, JHEP **11**, 196 (2018), arXiv:1808.03012[hep-ph]
[14] H. Okada and M. Tanimoto, Phys. Lett. B **791**, 54-61 (2019), arXiv:1812.09677[hep-ph]
[15] T. Nomura and H. Okada, Phys. Lett. B **797**, 134799 (2019), arXiv:1904.03937[hep-ph]
[16] H. Okada and M. Tanimoto, Eur. Phys. J. C **81**(1), 52 (2021), arXiv:1905.13421[hep-ph]
[17] F. J. de Anda, S. F. King, and E. Perdomo, Phys. Rev. D **101**(1), 015028 (2020), arXiv:1812.05620[hep-ph]
[18] P. P. Novichkov, S. T. Petcov, and M. Tanimoto, Phys. Lett. B **793**, 247-258 (2019), arXiv:1812.11289[hep-ph]
[19] T. Nomura and H. Okada, Nucl. Phys. B **966**, 115372 (2021), arXiv:1906.03927[hep-ph]
[20] H. Okada and Y. Orikasa, arXiv: 1907.13520 [hep-ph]
[21] G. J. Ding, S. F. King, and X. G. Liu, JHEP **09**, 074 (2019), arXiv:1907.11714[hep-ph]
[22] T. Nomura, H. Okada, and O. Popov, Phys. Lett. B **803**, 135294 (2020), arXiv:1908.07457[hep-ph]
[23] T. Kobayashi, Y. Shimizu, K. Takagi *et al.*, Phys. Rev. D **100**(11), 115045 (2019) [Erratum: Phys. Rev. D **101**(3), 039904 (2020)] arXiv: 1909.05139[hep-ph]
[24] T. Asaka, Y. Heo, T. H. Tatsuishi a *et al.*, JHEP **01**, 144 (2020), arXiv:1909.06520[hep-ph]
[25] D. Zhang, Nucl. Phys. B **952**, 114935 (2020), arXiv:1910.07869[hep-ph]
[26] G. J. Ding, S. F. King, X. G. Liu *et al.*, JHEP **12**, 030 (2019), arXiv:1910.03460[hep-ph]
[27] T. Kobayashi, T. Nomura, and T. Shimomura, Phys. Rev. D **102**(3), 035019 (2020), arXiv:1912.00637[hep-ph]
[28] T. Nomura, H. Okada, and S. Patra, Nucl. Phys. B **967**, 115395 (2021), arXiv:1912.00379[hep-ph]
[29] X. Wang, Nucl. Phys. B **957**, 115105 (2020), arXiv:1912.13284[hep-ph]
[30] H. Okada and Y. Shoji, Nucl. Phys. B **961**, 115216 (2020), arXiv:2003.13219[hep-ph]
[31] H. Okada and M. Tanimoto, arXiv: 2005.00775 [hep-ph]
[32] M. K. Behera, S. Singirala, S. Mishra *et al.*, arXiv: 2009.01806 [hep-ph]
[33] M. K. Behera, S. Mishra, S. Singirala *et al.*, arXiv: 2007.00545 [hep-ph]
[34] T. Nomura and H. Okada, arXiv: 2007.04801 [hep-ph]
[35] T. Nomura and H. Okada, arXiv: 2007.15459 [hep-ph]
[36] T. Asaka, Y. Heo, and T. Yoshida, Phys. Lett. B **811**, 135956 (2020), arXiv:2009.12120[hep-ph]
[37] H. Okada and M. Tanimoto, Phys. Rev. D **103**(1), 015005 (2021), arXiv:2009.14242[hep-ph]
[38] K. I. Nagao and H. Okada, arXiv: 2010.03348 [hep-ph]
[39] H. Okada and M. Tanimoto, JHEP **03**, 010 (2021), arXiv:2012.01688[hep-ph]
[40] C. Y. Yao, J. N. Lu, and G. J. Ding, JHEP **05**, 102 (2021), arXiv:2012.13390 [hep-ph]
[41] P. Chen, G. J. Ding, and S. F. King, JHEP **04**, 239 (2021), arXiv:2101.12724 [hep-ph]

[42] M. Kashav and S. Verma, arXiv: 2103.07207 [hep-ph]
[43] H. Okada, Y. Shimizu, M. Tanimoto *et al.*, arXiv: 2105.14292 [hep-ph]
[44] I. de Medeiros Varzielas, and J. Lourenço, arXiv: 2107.04042 [hep-ph]
[45] T. Nomura, H. Okada, and Y. Orikasa, arXiv: 2106.12375 [hep-ph]
[46] P. T. P. Hutauruk, D. W. Kang, J. Kim *et al.*, arXiv: 2012.11156 [hep-ph]
[47] G. J. Ding, S. F. King, and J. N. Lu, arXiv: 2108.09655 [hep-ph]
[48] K. I. Nagao and H. Okada, arXiv: 2108.09984 [hep-ph]
[49] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, Phys. Rev. D **98**(1), 016004 (2018), arXiv:1803.10391[hep-ph]
[50] T. Kobayashi, Y. Shimizu, K. Takagi *et al.*, Phys. Lett. B **794**, 114-121 (2019), arXiv:1812.11072[hep-ph]
[51] T. Kobayashi, Y. Shimizu, K. Takagi *et al.*, PTEP **2020**(5), 053B05 (2020), arXiv:1906.10341[hep-ph]
[52] H. Okada and Y. Orikasa, Phys. Rev. D **100**(11), 115037 (2019), arXiv:1907.04716[hep-ph]
[53] S. Mishra, arXiv: 2008.02095 [hep-ph]
[54] X. Du and F. Wang, JHEP **02**, 221 (2021), arXiv:2012.01397[hep-ph]
[55] J. T. Penedo and S. T. Petcov, Nucl. Phys. B **939**, 292-307 (2019), arXiv:1806.11040[hep-ph]
[56] P. P. Novichkov, J. T. Penedo, S. T. Petcov *et al.*, JHEP **04**, 005 (2019), arXiv:1811.04933[hep-ph]
[57] T. Kobayashi, Y. Shimizu, K. Takagi *et al.*, JHEP **02**, 097 (2020), arXiv:1907.09141[hep-ph]
[58] S. F. King and Y. L. Zhou, Phys. Rev. D **101**(1), 015001 (2020), arXiv:1908.02770[hep-ph]
[59] H. Okada and Y. Orikasa, arXiv: 1908.08409 [hep-ph]
[60] J. C. Criado, F. Feruglio, and S. J. D. King, JHEP **02**, 001 (2020), arXiv:1908.11867[hep-ph]
[61] X. Wang and S. Zhou, JHEP **05**, 017 (2020), arXiv:1910.09473[hep-ph]
[62] Y. Zhao and H. H. Zhang, JHEP **03**, 002 (2021), arXiv:2101.02266 [hep-ph]
[63] S. F. King and Y. L. Zhou, JHEP **04**, 291 (2021), arXiv:2103.02633 [hep-ph]
[64] G. J. Ding, S. F. King, and C. Y. Yao, arXiv: 2103.16311 [hep-ph]
[65] X. Zhang and S. Zhou, arXiv: 2106.03433 [hep-ph]
[66] Bu-Yao Qu, Xiang-Gan Liu, Ping-Tao Chen *et al.*, arXiv: 2106.11659 [hep-ph]
[67] P. P. Novichkov, J. T. Penedo, S. T. Petcov *et al.*, JHEP **04**, 174 (2019), arXiv:1812.02158[hep-ph]
[68] G. J. Ding, S. F. King, and X. G. Liu, Phys. Rev. D **100**(11), 115005 (2019), arXiv:1903.12588[hep-ph]
[69] X. Wang, B. Yu, and S. Zhou, Phys. Rev. D **103**(7), 076005 (2021), arXiv:2010.10159[hep-ph]
[70] C. Y. Yao, X. G. Liu, and G. J. Ding, Phys. Rev. D **103**(9), 095013 (2021), arXiv:2011.03501[hep-ph]
[71] X. Wang and S. Zhou, arXiv: 2102.04358 [hep-ph]
[72] M. K. Behera and R. Mohanta, arXiv: 2108.01059 [hep-ph]
[73] A. Baur, H. P. Nilles, A. Trautner *et al.*, Phys. Lett. B **795**, 7-14 (2019), arXiv:1901.03251[hep-th]
[74] I. de Medeiros Varzielas, S. F. King *et al.*, Phys. Rev. D **101**(5), 055033 (2020), arXiv:1906.02208[hep-ph]
[75] X. G. Liu and G. J. Ding, JHEP **08**, 134 (2019), arXiv:1907.01488[hep-ph]
[76] P. Chen, G. J. Ding, J. N. Lu *et al.*, Phys. Rev. D **102**(9), 095014 (2020), arXiv:2003.02734[hep-ph]
[77] C. C. Li, X. G. Liu, and G. J. Ding, arXiv: 2108.02181 [hep-ph]
[78] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, Nucl. Phys. B **963**, 115301 (2021), arXiv:2006.03058[hep-ph]
[79] X. G. Liu, C. Y. Yao, and G. J. Ding, Phys. Rev. D **103**(5), 056013 (2021), arXiv:2006.10722[hep-ph]
[80] S. Kikuchi, T. Kobayashi, H. Otsuka *et al.*, JHEP **11**, 101 (2020), arXiv:2007.06188[hep-th]
[81] Y. Almumin, M. C. Chen, V. Knapp-Pérez *et al.*, JHEP **05**, 078 (2021), arXiv:2102.11286 [hep-th]
[82] G. J. Ding, F. Feruglio, and X. G. Liu, SciPost Phys. **10**, 133 (2021), arXiv:2102.06716 [hep-ph]
[83] F. Feruglio, V. Gherardi, A. Romanino *et al.*, JHEP **05**, 242 (2021), arXiv:2101.08718 [hep-ph]
[84] S. Kikuchi, T. Kobayashi, and H. Uchida, arXiv: 2101.00826 [hep-th]
[85] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, JHEP **04**, 206 (2021), arXiv:2102.07488 [hep-ph]
[86] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701-2729 (2010), arXiv:1002.0211[hep-ph]
[87] H. Ishimori, T. Kobayashi, H. Ohki *et al.*, Prog. Theor. Phys. Suppl. **183**, 1-163 (2010), arXiv:1003.3552[hep-th]
[88] H. Ishimori, T. Kobayashi, H. Ohki *et al.*, Notes Phys. **858**, 1-227 (2012)
[89] D. Hernandez and A. Y. Smirnov, Phys. Rev. D **86**, 053014 (2012), arXiv:1204.0445[hep-ph]
[90] S. F. King and C. Luhn, Rept. Prog. Phys. **76**, 056201 (2013), arXiv:1301.1340[hep-ph]
[91] S. F. King, A. Merle, S. Morisi *et al.*, New J. Phys. **16**, 045018 (2014), arXiv:1402.4271[hep-ph]
[92] S. F. King, Prog. Part. Nucl. Phys. **94**, 217-256 (2017), arXiv:1701.04413[hep-ph]
[93] S. T. Petcov, Eur. Phys. J. C **78**(9), 709 (2018), arXiv:1711.10806[hep-ph]
[94] A. Baur, H. P. Nilles, A. Trautner *et al.*, Nucl. Phys. B **947**, 114737 (2019), arXiv:1908.00805[hep-th]
[95] T. Kobayashi, Y. Shimizu, K. Takagi *et al.*, Phys. Rev. D **101**(5), 055046 (2020), arXiv:1910.11553[hep-ph]
[96] P. P. Novichkov, J. T. Penedo, S. T. Petcov *et al.*, JHEP **07**, 165 (2019), arXiv:1905.11970[hep-ph]
[97] M. Tanimoto and K. Yamamoto, arXiv: 2106.10919 [hep-ph].
[98] M. C. Chen, S. Ramos-Sánchez, and M. Ratz, Phys. Lett. B **801**, 135153 (2020), arXiv:1909.06910[hep-ph]
[99] I. de Medeiros Varzielas, M. Levy, and Y. L. Zhou, JHEP **11**, 085 (2020), arXiv:2008.05329[hep-ph]
[100] A. Das, T. Nomura, H. Okada *et al.*, Phys. Rev. D **96**(7), 075001 (2017), arXiv:1704.02078[hep-ph]
[101] W. Wang and Z. L. Han, Phys. Rev. D **92**, 095001 (2015), arXiv:1508.00706 [hep-ph]
[102] A. Gando *et al.* (KamLAND-Zen Collaboration), Phys. Rev. Lett. **117**(8), 082503 (2016) [Addendum: Phys. Rev. Lett. **117**(10), 109903 (2016)] arXiv: 1605.02889 [hep-ex]
[103] I. Esteban, M. C. Gonzalez-Garcia, A. Hernandez-Cabezudo *et al.*, JHEP **1901**, 106 (2019), arXiv:1811.05487[hep-ph]
[104] I. Esteban, M. C. Gonzalez-Garcia, A. Hernandez-Cabezudo *et al.*, *NuFIT 4.1 (2019),* www.nu-fit.org, (2019)
[105] E. Fernandez-Martinez, J. Hernandez-Garcia, and J. Lopez-Pavon, JHEP **08**, 033 (2016), arXiv:1605.08774[hep-ph]
[106] N. R. Agostinho, G. C. Branco, P. M. F. Pereira *et al.*, Eur. Phys. J. C **78**(11), 895 (2018), arXiv:1711.06229[hep-ph]