Statistical method in quark combination model*

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Abstract: We present a new method for solving the probability distribution for baryons, antibaryons, and mesons at the hadronization of the constituent quark and antiquark system. The hadronization is governed by the quark combination rule in the quark combination model developed by the Shandong Group. We employ the method of the generating function to derive the outcome of the quark combination rule, which is significantly simpler and easier to generalize than the original method. Furthermore, we use the formula of the quark combination rule and its generalization to study the property of the multiplicity distribution of net-protons. Taking a naive case of quark number fluctuations and correlations at hadronization, we calculate ratios of multiplicity cumulants of final-state net-protons and discuss the potential applicability of the quark combination model by studying hadronic multiplicity fluctuations and the underlying phase transition property in relativistic heavy-ion collisions.

Keywords: hadronization, quark combination model, particle fluctuation

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1   Introduction

Most hadronization models, such as the Lund string model [1–4] or the coalescence/recombination models [5–21] in electron-positron and hadron-hadron collisions, assume that a hadron is formed by quarks and antiquarks in the neighborhood of phase space. Normally, the phase space can be decomposed into the longitudinal and transverse directions. The longitudinal phase space plays a particular role. For example, in the Lund model, a string is formed between a quark and an antiquark moving back to back, which is an object with only one spatial dimension, i.e., with only the longitudinal phase space. A quark-antiquark pair is produced as the result of the string breaking into two pieces. The process of string breaks continues until it is terminated at a lowest energy scale. The transverse momentum space of the excited quark-antiquark pair is rather limited and can be described by the exponentially suppressed function of the transverse mo-

mentum. In relativistic heavy-ion collisions, the longitudinal phase space remains dominant, although transverse expansion is significant.

The quark combination model developed by the Shandong group (SDQCM) [9–11, 22–35] is a kind of exclusive or statistical hadronization model, which differs from the Lund string model and the coalescence model. The model first takes the constituent quark degrees of freedom as an effective description for the strongly-interacted quark gluon system at hadronization. Subsequently, the model adopts a quark combination rule (QCR) to combine the quarks and antiquarks in the neighborhood of the longitudinal phase space into baryons and mesons. Since the longitudinal phase space is easily described by the momentum rapidity, the correlation in rapidity is the basis of the QCR. This type of QCR in SDQCM has successfully explained many data of hadronic production in e^+e^- and pp collisions [24, 25, 36, 37] and rapidity distributions of hadrons in heavy-ion collisions [26, 27, 31, 38].

The probability distributions of the particle numbers of baryons, antibaryons, and mesons is an important observable in high energy collisions and is closely related to hadronization dynamics. Especially, the particle number distributions are essential to the search for the critical end point of the QCD phase diagram in relativistic heavy-ion collisions [39–43]. In this study, we propose a new method to solve the baryon and meson number distribution in the context of the QCR in the SDQCM. The new method is significantly simpler and easier to generalize to more sophisticated cases than the original one [44]. We followingly employ these formula to calculate the multiplicity distribution and ratios of cumulants for net protons in relativistic heavy-ion collisions.

The paper is organized as follows. In Sec. 2, we briefly present the original QCR in the SDQCM. Then, we apply the generating function method to solve the particle number probability of baryons, antibaryons, and mesons for a given number of quarks and antiquarks. In Sec. 3, we generalize the original QCR and derive the corresponding particle number probability of baryons, antibaryons, and mesons using the generating function method. In Sec. 4, we compare the numerical difference between the original QCR and the generalized QCR in terms of moments of the antibaryon number. In Sec. 5, we formulate the fluctuation and correlation of identified hadrons. In Sec. 6 and 7, we study the effects of the quark number fluctuation and resonance decays. In Sec. 8, we provide an illustrative example of applying the QCR and generalized QCR to calculate the ratios of cumulants for net protons in heavy-ion collisions and compare it with the obtained data. Finally, we provide a summary and discussions in Sec. 9.

2   Baryon and meson formation in original quark combination rule

Because of the non-perturbative QCD feature, the transition from quarks and/or gluons to hadrons is not solved from the first principle, and it is presently only described by the phenomenological models. Inspired by the simple and effective description of the constituent quark model in explaining the static property of hadrons, SDQCM assumes the constituent quarks and antiquarks as effective degrees of freedom for the strongly-interacting quarks and gluons at hadronization ,and therefore builds a simple hadronization phenomenology by the combination of these constituent quarks and antiquarks into hadrons. We emphasize that there are explicit experimental signals for such constituent quark degrees of free-

dom in high energy collisions, such as the quark number scaling property of elliptic flows and the transverse momentum spectra for hadrons observed in recent years [45– 50]. A phenomenological quark combination rule was proposed in Ref. [22] to describe the combination of these constituent quarks and antiquarks in a one-dimensional phase space and can successfully describe the data of yields and momentum distributions of hadrons in high-energy reactions [24–27, 31, 36–38]. In this section, we use the generating function method to solve the probability distribution for the number of baryons, antibaryons, and mesons formed by QCR.

2.1   Original quark combination rule

In the QCM, N_q quarks and $N_{\bar{q}}$e antiquarks generated in an event are placed into a queue and then allowed to combine into hadrons one by one following the quark combination rule (QCR) [22]. QCR is based on the basic property of QCD. A $q\bar{q}$ pair may occur in a color octet with a repulsive interaction or a singlet with a attractive interaction. If $q\bar{q}$ is adjacent in phase space, $q\bar{q}$ will have sufficient time or opportunity to be in a color state and hadronize into a meson. For a qq pair, it may be in a sextet or an antitriplet. If its nearest neighbor is a q in phase space, they can hadronize into a baryon. If the neighbor of qq is a $\bar{q}$, $q\bar{q}$ will win the competition to form a meson and leave a q to combine with other quarks and/or antiquarks. This is because the attraction strength of the singlet for $q\bar{q}$ is two times that of the antitriplet for qq (via counting the color factor in the one-gluon exchange case). The original QCR proposed in Ref. [22] reads:

1. Check if there are partons in the queue. If there are no partons, the process ends. Otherwise, start from the first parton (q or $\bar{q}$) in the queue and proceed to the next step.

2. Look at the second parton. If there is no second parton, the process ends. Otherwise, if the baryon number of the second parton in the queue is different from the first one, i.e., the first two partons are either $q\bar{q}$ or $\bar{q}q$r, they combine into a meson and are removed from the queue, repeat step 1; Otherwise, if they are either qq or $\bar{q}\bar{q}$, proceed to the next step.

3. Look at the third parton. If there is no third parton, the process ends. Otherwise, if the third parton is different in the baryon number from the first one, the first and third parton form a meson and are removed from the queue, repeat step 1; Otherwise, if the first three partons combine into a baryon or an antibaryon and are removed from the queue, go to step 1.

The following example demonstrates the function of the QCR.

$$q_1\bar{q}_2q_3q_4q_5\bar{q}_6q_7q_8\bar{q}_9\bar{q}_{10}q_{11}\bar{q}_{12}\bar{q}_{13}\bar{q}_{14}\bar{q}_{15}q_{16}q_{17}q_{18}\bar{q}_{19}\bar{q}_{20} \rightarrow M(q_1\bar{q}_2)B(q_3q_4q_5)M(\bar{q}_6q_7)M(q_8\bar{q}_9)M(\bar{q}_{10}q_{11})$$
$$\bar{B}(\bar{q}_{12}\bar{q}_{13}\bar{q}_{14})M(\bar{q}_{15}q_{16})M(q_{17}\bar{q}_{19})M(q_{18}\bar{q}_{20}).$$

$$(1)$$

In relativistic heavy-ion collisions, the longitudinal rapidity space is predominant, and the rapidity density of quarks and antiquarks is quite large. Therefore, it is suitable and straightforward to apply such a QCR in one-dimensional longitudinal rapidity space. However, it is quite complicated in three dimensional phase space, because one can not easily define a particular hadronization order to apply QCR [51].

2.2 Recursive relation for $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$

We consider the system consisting of N_q quarks and $N_{\bar{q}}$ antiquarks stochastically populated in one-dimensional phase space. After combination by QCR, there are N_B baryons, $N_{\bar{B}}$ antibaryons, and N_M mesons formed, and N_r quarks and $N_{\bar{r}}$ antiquarks left. There are only five different configurations with $(N_r, N_{\bar{r}}) = (0,0), (0,1), (1,0), (2,0)$, and $(0,2)$. The quark number conservation gives

$$
\begin{aligned}
N_M + 3N_B + N_r &= N_q, \\
N_M + 3N_{\bar{B}} + N_{\bar{r}} &= N_{\bar{q}}.
\end{aligned}
\tag{2}
$$

The outcome of implementing the QCR to the queue of N_q quarks and $N_{\bar{q}}$ antiquarks yields a group of numbers $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$. The queue $(N_M, N_B, N_{\bar{B}}, 0, 0)$ can be reached by one of the following ways of adding one q or $\bar{q}$ to the end of the other four queues with a smaller number of quarks and antiquarks

$$
\begin{aligned}
(a) &(N_M - 1, N_B, N_{\bar{B}}, 1, 0) + \bar{q}, \\
(b) &(N_M - 1, N_B, N_{\bar{B}}, 0, 1) + q, \\
(c) &(N_M, N_B - 1, N_{\bar{B}}, 2, 0) + q, \\
(d) &(N_M, N_B, N_{\bar{B}} - 1, 0, 2) + \bar{q}.
\end{aligned}
\tag{3}
$$

We use $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ to denote the number of different queues for a given group $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$. We define $F(0,0,0,0,0) = 1$ and $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}}) = 0$ for the case that any N_M, N_B, $N_{\bar{B}}$, N_r, and $N_{\bar{r}}$ are negative. Under the constraint of Eq. (2), the sum of $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ over all different groups of $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ should be

$$
\begin{aligned}
S(N_q, N_{\bar{q}}) &= \sum_{\{N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}}\}} F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}}) \\
&\quad \times \delta_{N_M + 3N_B + N_r, N_q} \delta_{N_M + 3N_{\bar{B}} + N_{\bar{r}}, N_{\bar{q}}} \\
&= \binom{N_q + N_{\bar{q}}}{N_q} \equiv \frac{(N_q + N_{\bar{q}})!}{N_q! N_{\bar{q}}!},
\end{aligned}
\tag{4}
$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

For non-zero N_r and $N_{\bar{r}}$, $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ has a property

$$
F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}}) = \sum_{i=0}^{N_M} F(i, N_B, N_{\bar{B}}, 0, 0).
\tag{5}
$$

Proof of Property. We first take $(N_r, N_{\bar{r}}) = (1,0)$ as an example. We note that the queue yielding $(N_M, N_B, N_{\bar{B}}, 1, 0)$ can be achieved in two ways: (a) adding q to the end of the queue with $(N_M, N_B, N_{\bar{B}}, 0, 0)$; (b) adding $\bar{q}$ to the end

of the queue with $(N_M - 1, N_B, N_{\bar{B}}, 2, 0)$, which can further be obtained from the queue with $(N_M - 1, N_B, N_{\bar{B}}, 1, 0)$ by adding a q at the end. Thus, we obtain the recursive relation

$$
\begin{aligned}
F(N_M, N_B, N_{\bar{B}}, 1, 0) =& F(N_M, N_B, N_{\bar{B}}, 0, 0) \\
&+ F(N_M - 1, N_B, N_{\bar{B}}, 2, 0) \\
=& F(N_M, N_B, N_{\bar{B}}, 0, 0) \\
&+ F(N_M - 1, N_B, N_{\bar{B}}, 1, 0).
\end{aligned}
\tag{6}
$$

Solving the above equation recursively, we obtain Eq. (5), where we used $F(0, N_B, N_{\bar{B}}, 1, 0) = F(0, N_B, N_{\bar{B}}, 0, 0)$. The proof of the case $(N_r, N_{\bar{r}}) = (2, 0)$ is straightforward, because the queue yielding $(N_M, N_B, N_{\bar{B}}, 2, 0)$ can be only obtained from the queue with $(N_M, N_B, N_{\bar{B}}, 1, 0)$ by adding a q at the end. The proof for the cases $(N_r, N_{\bar{r}}) = (0, 1)$ and $(N_r, N_{\bar{r}}) = (0, 2)$ is similar.

Using properties in Eq. (5) and Eq. (3), we obtain

$$
\begin{aligned}
F(N_M, N_B, N_{\bar{B}}, 0, 0) =& F(N_M - 1, N_B, N_{\bar{B}}, 1, 0) \\
&+ F(N_M - 1, N_B, N_{\bar{B}}, 0, 1) \\
&+ F(N_M, N_B - 1, N_{\bar{B}}, 2, 0) \\
&+ F(N_M, N_B, N_{\bar{B}} - 1, 0, 2).
\end{aligned}
\tag{7}
$$

We make the replacement $N_M \to N_M - 1$ in the above equation and take the difference between the two. Finally, we use Eq. (5) to obtain the recursive equation for $F(N_M, N_B, N_{\bar{B}}, 0, 0)$

$$
\begin{aligned}
F(N_M, N_B, N_{\bar{B}}, 0, 0) =& 3F(N_M - 1, N_B, N_{\bar{B}}, 0, 0) \\
&+ F(N_M, N_B - 1, N_{\bar{B}}, 0, 0) \\
&+ F(N_M, N_B, N_{\bar{B}} - 1, 0, 0),
\end{aligned}
\tag{8}
$$

where $N_M \geqslant 0$, $N_B \geqslant 0$, $N_{\bar{B}} \geqslant 0$ excluding two cases $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}}) = (0,0,0,0,0), (1,0,0,0,0)$, which we have $F(0,0,0,0,0) = 1$ by definition and $F(1,0,0,0,0) = 2$ by simple counting.

2.3 Solution to recursive equation by generating function method

First, we consider a special simple case: $N_M > 1$, $N_B = N_{\bar{B}} = 0$, $N_r = N_{\bar{r}} = 0$. Eq. (8) is simplified to

$$
F(N_M, 0, 0, 0, 0) = 3F(N_M - 1, 0, 0, 0, 0),
\tag{9}
$$

which immediately leads to the solution

$$
F(N_M, 0, 0, 0, 0) = 2 \times 3^{N_M - 1},
\tag{10}
$$

for $N_M \geqslant 1$ with $F(1, 0, 0, 0, 0) = 2$.

Now, we consider the general case for $F(N_M, N_B, N_{\bar{B}}, 0, 0)$ with $N_M \geqslant 0$, $N_B \geqslant 0$, $N_{\bar{B}} \geqslant 0$. We define the generating function

$$
\begin{aligned}
A(x, y, z) = \sum_{N_M=0}^{\infty} \sum_{N_B=0}^{\infty} \sum_{N_{\bar{B}}=0}^{\infty} \\
\times F(N_M, N_B, N_{\bar{B}}, 0, 0) x^{N_M} y^{N_B} z^{N_{\bar{B}}},
\end{aligned}
\tag{11}
$$

and $F(N_M, N_B, N_{\bar{B}}, 0, 0)$ is the coefficient of $x^{N_M} y^{N_B} z^{N_{\bar{B}}}$ in the polynomial expansion of $A(x, y, z)$, once solved.

To solve $A(x, y, z)$, we define a partial generating function

$$A(x; N_B, N_{\bar{B}}) = \sum_{N_M=0}^{\infty} F(N_M, N_B, N_{\bar{B}}, 0, 0) x^{N_M}. \tag{12}$$

Inserting Eq. (8) for $N_M \geqslant 1$, we obtain

$$(1 - 3x) A(x; N_B, N_{\bar{B}}) - F(0, N_B, N_{\bar{B}}, 0, 0) = A(x; N_B - 1, N_{\bar{B}}) + A(x; N_B, N_{\bar{B}} - 1) - F(0, N_B - 1, N_{\bar{B}}, 0, 0) - F(0, N_B, N_{\bar{B}} - 1, 0, 0). \tag{13}$$

In a special case with $N_B = 0$, we obtain

$$A(x; 0, N_{\bar{B}}) = \frac{1}{(1 - 3x)^{N_{\bar{B}}}} A(x; 0, 0)$$
$$= \frac{2x}{(1 - 3x)^{N_{\bar{B}}+1}} + \frac{1}{(1 - 3x)^{N_{\bar{B}}}}, \tag{14}$$

where we used Eq. (10) and assumed that $3|x| < 1$. In the same way, we obtain

$$A(x; N_B, 0) = \frac{2x}{(1 - 3x)^{N_B+1}} + \frac{1}{(1 - 3x)^{N_B}}. \tag{15}$$

We also obtain two sum rules

$$\sum_{N_B=0}^{\infty} A(x; N_B, 0) y^{N_B} = \frac{1 - x}{1 - 3x - y}, \tag{16}$$

$$\sum_{N_{\bar{B}}=0}^{\infty} A(x; 0, N_{\bar{B}}) z^{N_{\bar{B}}} = \frac{1 - x}{1 - 3x - z}, \tag{17}$$

where the convergent region is $|y|, |z| < |1 - 3x|$.

We now multiply Eq. (13) by $y^{N_B} z^{N_{\bar{B}}}$ and take a sum over N_B from 1 to infinity, and $N_{\bar{B}}$ from 1 to infinity. By noticing $A(x, y, z) = \sum_{N_B=0}^{\infty} \sum_{N_{\bar{B}}=0}^{\infty} A(x; N_B, N_{\bar{B}}) y^{N_B} z^{N_{\bar{B}}}$ and employing Eqs. (16) and (17), we solve the generating function as

$$A(x, y, z) = \frac{1 - x}{1 - 3x - y - z}. \tag{18}$$

To obtain $F(N_M, N_B, N_{\bar{B}}, 0, 0)$, we first extract the coefficient of $z^{N_{\bar{B}}}$

$$C(z^{N_{\bar{B}}}) = \frac{1 - x}{(1 - 3x - y)^{N_{\bar{B}}+1}}. \tag{19}$$

Then, we extract the coefficient of y^{N_B} in $C(z^{N_{\bar{B}}})$ as

$$C(z^{N_{\bar{B}}} y^{N_B}) = \binom{N_B + N_{\bar{B}}}{N_B} \frac{1 - x}{(1 - 3x)^{N_B + N_{\bar{B}} + 1}}, \tag{20}$$

where we apply

$$(1 \pm w)^{-n} = \sum_{k=0}^{\infty} \binom{n - 1 + k}{k} (\mp 1)^k w^k, \tag{21}$$

for $n > 0$ and $|w| < 1$. Finally, we extract the coefficient of x^{N_M} in $C(z^{N_{\bar{B}}} y^{N_B})$, i.e.,

$$F(N_M, N_B, N_{\bar{B}}, 0, 0) = 3^{N_M} \binom{N_B + N_{\bar{B}}}{N_B} \left[\binom{N_M + N_B + N_{\bar{B}}}{N_M} - \frac{1}{3} \binom{N_M - 1 + N_B + N_{\bar{B}}}{N_M - 1} \right], \tag{22}$$

which yields the final result for

$$F(N_M, N_B, N_{\bar{B}}, 0, 0) = \begin{cases} \dfrac{(2N_M + 3N_B + 3N_{\bar{B}})(N_M + N_B + N_{\bar{B}} - 1)!}{N_M! N_B! N_{\bar{B}}!} 3^{N_M - 1} & \text{for } N_M > 0, \\ \binom{N_B + N_{\bar{B}}}{N_B} & \text{for } N_M = 0. \end{cases} \tag{23}$$

Eq. (23) is the result of QCR in Sect. 2.1 and was first given in Ref. [44] by mathematical induction and the traditional combination method. The current derivation is a simplified one, based on the recursive equation and the method of generating functions. The probability of forming N_M mesons, N_B baryons, and $N_{\bar{B}}$ antibaryons in a quark system with N_q quarks and $N_{\bar{q}}$ antiquarks is given by

$$P(N_M, N_B, N_{\bar{B}}) = \frac{F(N_M, N_B, N_{\bar{B}}, 0, 0)}{\binom{N_B + N_{\bar{B}}}{N_B}}. \tag{24}$$

Here, we have removed the label '0,0' from $P(N_M, N_B, N_{\bar{B}})$, since this is the real case in hadronization.

3　Baryon and meson formation in generalized quark combination rule

3.1　Generalized quark combination rule

The ratio of baryons to mesons (B/M) given by QCR in Sect. 2.1 is larger than the observation of heavy-ion collisions [52, 53]. To suppress the B/M ratio, we can generalize the QCR in Sect. 2.1 to decrease the formation probability of baryons relative to mesons. The generalized QCR (gQCR) reads:

1. Check if there are partons in the queue. If there are no partons, the process ends. Otherwise, start from the first parton and proceed to the next step.

2. Look at the second parton. If there is no second

parton, the process ends. Otherwise, if the baryon number of the second parton is different from the first one, i.e., the first two partons are either $q\bar{q}$ or $\bar{q}q$, they combine into a meson and are removed from the queue, repeat step 1; Otherwise if they are either qq or $\overline{qq}$, proceed to the next step.

3. Look at the third parton. If there is no third parton, the process ends. Otherwise, if the baryon number of the third parton is different from the first one, the first and third parton form a meson and are removed from the queue, repeat to step 1; Otherwise if the first three partons are either qqq or $\overline{qqq}$, proceed to next step.

4. Look at the fourth parton. If there is no fourth parton, the first three partons form a baryon or an antibaryon and the process ends. Otherwise, if the baryon number of the fourth parton is different from the first one, the first and fourth parton form a meson and are removed from the queue, repeat to step 1; Otherwise the first three partons combine into a baryon or an antibaryon and are removed from the queue, proceed to step 1.

The outcome of implementing gQCR to the queue of stochastically populated N_q quarks and $N_{\bar{q}}$ antiquarks yields a group of numbers $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$. Similarly as in the QCR case, the special queue with $(N_M, N_B, N_{\bar{B}}, 0, 0)$ can be reached by one of four ways in Eq. (3). Other queues with $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ where $N_r \neq 0$ or $N_{\bar{r}} \neq 0$ can be reached by adding q or $\bar{q}$ to the end of queues with smaller N_M, N_B, and $N_{\bar{B}}$. Queues marked as $(N_M, N_B, N_{\bar{B}}, 1, 0)$ can be achieved by

$$(a)[(N_M, N_B, N_{\bar{B}}, 0, 0) - (N_M, N_B, N_{\bar{B}} - 1, 0, 3)] + q,$$
$$(b)(N_M - 1, N_B, N_{\bar{B}}, 2, 0) + \bar{q}. \tag{25}$$

In approach (a), the special queue $(N_M, N_B, N_{\bar{B}} - 1, 0, 3)$ (included in $(N_M, N_B, N_{\bar{B}}, 0, 0)$) is excluded because of step 4 in gQCR. Queues marked as $(N_M, N_B, N_{\bar{B}}, 0, 1)$ can be achieved by

$$(a)[(N_M, N_B, N_{\bar{B}}, 0, 0) - (N_M, N_B - 1, N_{\bar{B}}, 3, 0)] + \bar{q},$$
$$(b)(N_M - 1, N_B, N_{\bar{B}}, 0, 2) + q. \tag{26}$$

Queues marked as $(N_M, N_B, N_{\bar{B}}, 2, 0)$ can be achieved by

$$(a)(N_M, N_B, N_{\bar{B}}, 1, 0) + q,$$
$$(b)(N_M - 1, N_B, N_{\bar{B}}, 3, 0) + \bar{q}. \tag{27}$$

Queues marked as $(N_M, N_B, N_{\bar{B}}, 0, 2)$ can be achieved by

$$(a)(N_M, N_B, N_{\bar{B}}, 0, 1) + \bar{q},$$
$$(b)(N_M - 1, N_B, N_{\bar{B}}, 0, 3) + q. \tag{28}$$

The special queues $(N_M, N_B, N_{\bar{B}}, 3, 0)$ and $(N_M, N_B, N_{\bar{B}}, 0, 3)$ can be build using normal ones

$$(N_M, N_B, N_{\bar{B}}, 3, 0) = (N_M, N_B, N_{\bar{B}}, 2, 0) + q, \tag{29}$$

$$(N_M, N_B, N_{\bar{B}}, 0, 3) = (N_M, N_B, N_{\bar{B}}, 0, 2) + \bar{q}. \tag{30}$$

Properties (25−30) lead to following recursive equations,

$$F(N_M, N_B, N_{\bar{B}}, 1, 0) = F(N_M, N_B, N_{\bar{B}}, 0, 0)$$
$$- F(N_M, N_B, N_{\bar{B}} - 1, 0, 2)$$
$$+ F(N_M - 1, N_B, N_{\bar{B}}, 2, 0),$$

$$F(N_M, N_B, N_{\bar{B}}, 0, 1) = F(N_M, N_B, N_{\bar{B}}, 0, 0)$$
$$- F(N_M, N_B - 1, N_{\bar{B}}, 2, 0)$$
$$+ F(N_M - 1, N_B, N_{\bar{B}}, 0, 2),$$

$$F(N_M, N_B, N_{\bar{B}}, 2, 0) = F(N_M, N_B, N_{\bar{B}}, 1, 0)$$
$$+ F(N_M - 1, N_B, N_{\bar{B}}, 2, 0),$$

$$F(N_M, N_B, N_{\bar{B}}, 0, 2) = F(N_M, N_B, N_{\bar{B}}, 0, 1)$$
$$+ F(N_M - 1, N_B, N_{\bar{B}}, 0, 2). \tag{31}$$

For non-zero N_r and $N_{\bar{r}}$, $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ can be obtained with the help of two properties in Appendix A. Using these, we can derive the recursive equation for $F(N_M, N_B, N_{\bar{B}}, 0, 0)$

$$F(N_M, N_B, N_{\bar{B}}, 0, 0) = F(N_M, N_B - 1, N_{\bar{B}}, 0, 0)$$
$$+ F(N_M, N_B, N_{\bar{B}} - 1, 0, 0)$$
$$- F(N_M, N_B - 1, N_{\bar{B}} - 1, 0, 0)$$
$$+ 6F(N_M - 1, N_B, N_{\bar{B}}, 0, 0)$$
$$- 3F(N_M - 1, N_B - 1, N_{\bar{B}}, 0, 0)$$
$$- 3F(N_M - 1, N_B, N_{\bar{B}} - 1, 0, 0)$$
$$- 10F(N_M - 2, N_B, N_{\bar{B}}, 0, 0)$$
$$+ F(N_M - 2, N_B - 1, N_{\bar{B}}, 0, 0)$$
$$+ F(N_M - 2, N_B, N_{\bar{B}} - 1, 0, 0)$$
$$+ 4F(N_M - 3, N_B, N_{\bar{B}}, 0, 0). \tag{32}$$

The detailed derivation of Eq. (32) is given in Appendix B.

3.2   Solution to recursive equation by generating functions for gQCR

We start with the most simple case $F(N_M, 0, 0, 0, 0)$ with $N_M \geqslant 1$, Eq. (32) is simplified as

$$F(N_M, 0, 0, 0, 0) = 6F(N_M - 1, 0, 0, 0, 0)$$
$$- 10F(N_M - 2, 0, 0, 0, 0)$$
$$+ 4F(N_M - 3, 0, 0, 0, 0).$$

With the initial values $F(0, 0, 0, 0, 0) = 1$, $F(1, 0, 0, 0, 0) = 2$, $F(2, 0, 0, 0, 0) = 6$, and $F(3, 0, 0, 0, 0) = 20$, we obtain the solution

$$F(N_M, 0, 0, 0, 0) = \frac{1}{2}\left[\left(2 + \sqrt{2}\right)^{N_M} + \left(2 - \sqrt{2}\right)^{N_M}\right]. \tag{33}$$

Now, we multiply Eq. (32) by x^{N_M} and sum over $N_M \geqslant 3$, obtaining the recursive equation Eq. (C3) for the partial generating function $A(x; N_B, N_{\bar{B}})$ defined in Eq. (12). In the special case with $N_B = 0$ in Eq. (C3), we obtain

$$A(x; 0, N_{\bar{B}}) = \frac{(1 - 3x + x^2)^{N_{\bar{B}}}}{(1 - 6x + 10x^2 - 4x^3)^{N_{\bar{B}}}} \cdot \frac{1 - 2x}{1 - 4x + 2x^2}, \tag{34}$$

where we have used

$$A(x; 0, 0) = \frac{1 - 2x}{1 - 4x + 2x^2}. \tag{35}$$

In the same way, we derive $A(x; N_B, 0)$, whose result is given by Eq. (34) by the replacement $N_{\bar{B}} \to N_B$. We also obtain two summation properties

$$\sum_{N_B=0}^{\infty} A(x; N_B, 0) y^{N_B} = \frac{1 - 2x}{1 - 4x + 2x^2}$$
$$\times \frac{1 - 6x + 10x^2 - 4x^3}{1 - 6x + 10x^2 - 4x^3 - y(1 - 3x + x^2)}, \tag{36}$$

$$\sum_{N_{\bar{B}}=0}^{\infty} A(x; 0, N_{\bar{B}}) z^{N_{\bar{B}}} = \frac{1 - 2x}{1 - 4x + 2x^2}$$
$$\times \frac{1 - 6x + 10x^2 - 4x^3}{1 - 6x + 10x^2 - 4x^3 - z(1 - 3x + x^2)}. \tag{37}$$

We multiply Eq. (C3) by $y^{N_B} z^{N_{\bar{B}}}$ and take sums over $N_B \geqslant 1$ and $N_{\bar{B}} \geqslant 1$, such that we can solve $A(x, y, z)$ as

$$A(x, y, z)$$
$$= \frac{1 - 4x + 4x^2 - yz}{1 - 6x + 10x^2 - 4x^3 - y + 3xy - x^2 y - z + 3xz - x^2 z + yz}. \tag{38}$$

The detailed derivation of Eq. (38) is shown in (C1–C6).

We extract $F(N_M, N_B, N_{\bar{B}}, 0, 0)$ as the coefficient of $x^{N_M} y^{N_B} z^{N_{\bar{B}}}$ in the polynomial expansion of $A(x, y, z)$. The result is a sum of four terms,

$$F(N_M, N_B, N_{\bar{B}}, 0, 0) = I_1 + I_2 + I_3 + I_4, \tag{39}$$

where these terms are given by Eq. (D8) in Appendix D.

4   Particle number distribution for baryons and mesons

In a system of N_q quark and $N_{\bar{q}}$ antiquark, we obtain in Eq. (24) the probability $P(N_M, N_B, N_{\bar{B}})$ to form N_M mesons, N_B baryons, and $N_{\bar{B}}$ antibaryons. A general form of the raw moments of meson and baryon numbers is

$$\overline{N_M^m N_B^n N_{\bar{B}}^k} = \sum_{N_M N_B N_{\bar{B}}} N_M^m N_B^n N_{\bar{B}}^k P(N_M, N_B, N_{\bar{B}})$$
$$\times \delta_{N_M + 3N_B, N_q} \delta_{N_M + 3N_{\bar{B}}, N_{\bar{q}}}, \tag{40}$$

where we use an *overline* to denote the average at fixed quark and antiquark numbers. The central moments for baryons and mesons are related by the quark number conservation

$$\overline{\delta N_B^n} = \overline{\delta N_{\bar{B}}^n}, \quad \overline{\delta N_M^n} = (-3)^n \overline{\delta N_B^n}, \tag{41}$$

where $\delta N_i \equiv N_i - \overline{N_i}$ with $i = M, B, \bar{B}$.

In Fig. 1, we show the ratios of the cumulants for antibaryons as functions of the quark-antiquark asymmetry
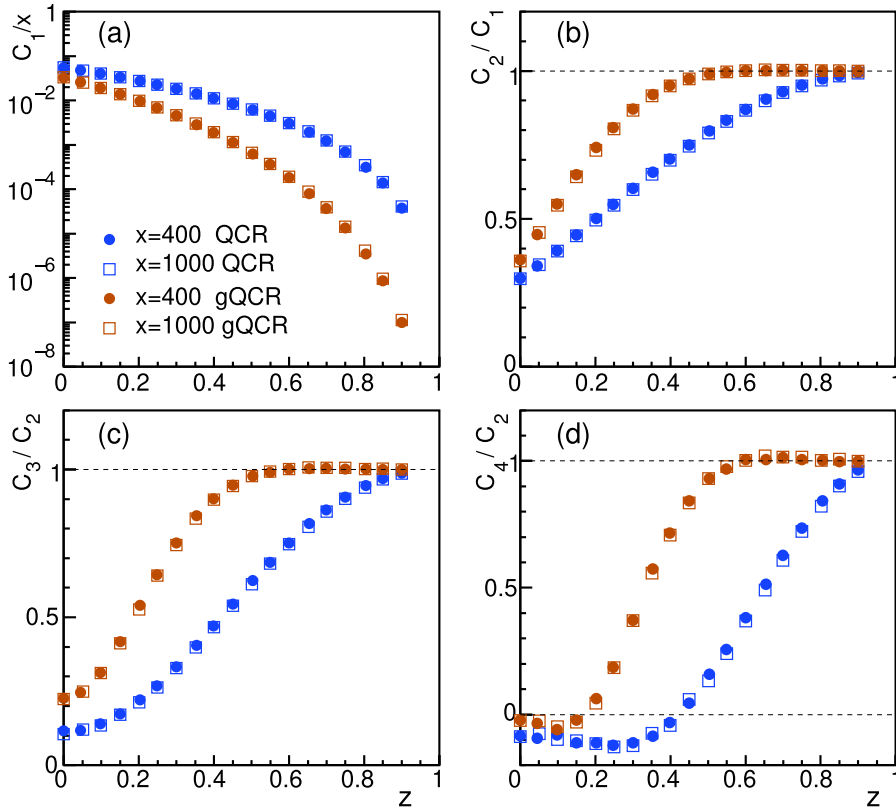

Fig. 1.   (color online) Ratios of cumulants for baryon or antibaryon number as functions of quark-antiquark asymmetry z with original QCR and gQCR at two different values of total quark number x: C_1/x, C_2/C_1, C_3/C_2, and C_4/C_2.

$$z = \frac{N_q - N_{\bar{q}}}{N_q + N_{\bar{q}}}, \qquad (42)$$

with the original QCR and gQCR at two values of the total quark number $x = N_q + N_{\bar{q}}$. The cumulants for antibaryons are given by

$$C_1 \equiv C_1^{\bar{B}} = \overline{N_{\bar{B}}}, \quad C_2 = \overline{\delta N_B^2} = \overline{\delta N_{\bar{B}}^2}, \quad C_3 = \overline{\delta N_B^3} = \overline{\delta N_{\bar{B}}^3},$$

$$C_4 = \overline{\delta N_B^4} - 3C_2^2 = \overline{\delta N_{\bar{B}}^4} - 3C_2^2. \qquad (43)$$

Note that the first order cumulant for baryons is related to that for antibaryons by $C_1^B = C_1 + zx/3$. As x is not small ($x \gtrsim 200$), C_1/x, C_2/C_1, C_3/C_2, and C_4/C_2 are almost independent of x (i.e., the system size) and are only mainly dependent on the quark-antiquark asymmetry z (proportional to the net-baryon density). Therefore, we plot their z dependence under the original and generalized QCR, respectively in Fig. 1. Panel (a) shows that fewer antibaryons are produced with gQCR than with the QCR. The antibaryon production with gQCR is more suppressed than the QCR for larger z. We can parameterize the antibaryon number [54] as $\overline{N_{\bar{B}}}/x = (z/3)(1-z)^a / [(1+z)^a - (1-z)^a]$ with $a = 3$ for QCR and $a = 5$ for gQCR. We see that the cumulant ratios C_2/C_1, C_3/C_2, and C_4/C_2 tend to unity at large z. This is because the antiquark number is small at large z, and the aggregation of quarks and antiquarks in phase space to form baryons and antibaryons is more stochastic and follows the Poisson distribution. Antibaryons are produced at a lower level in the gQCR, and thus the feature of the Poisson distribution is more obvious, which is reflected in the cumulant ratios in the gQCR approaching unity more quickly at large z than in the QCR. At small z, C_3/C_2 and C_4/C_2 are low and approach the Gaussian distribution.

5   Multiplicity property of identified hadrons

Following the method of Refs. [35, 55], we obtain some multiplicity properties of identified hadrons by taking advantage of the stochastic combination rule. In this study, we only consider the production of octet baryons with $J^P = (1/2)^+$ and decuplet baryons with $J^P = (3/2)^+$, pseudo-scalar mesons with $J^P = 0^-$, and vector mesons $J^P = 1^-$ in the flavor $SU(3)$ ground state. The mean values of the multiplicities for identified baryons and mesons are given by

$$\overline{N}_{B_i} = g_{B_i} \frac{N_{B_i}^{(q)}}{N_q(N_q - 1)(N_q - 2)} \overline{N}_B,$$

$$\overline{N}_{M_i} = g_{M_i} \frac{N_{M_i}^{(q)}}{N_q N_{\bar{q}}} \overline{N}_M, \qquad (44)$$

with

$$N_{B_i}^{(q)} = S_{B_i} \prod_f \prod_{j=1}^{n_{f,B_i}} (N_f - j + 1),$$

$$N_{M_i}^{(q)} = \prod_f \prod_{j=1}^{n_{f,M_i}} (N_f - j + 1), \qquad (45)$$

where f denotes the quark or antiquark flavors in the baryon B_i, and the meson M_i, n_{f,B_i}, and n_{f,M_i} are the number of flavor f quarks in B_i and M_i respectively. S_{B_i} counts the number of different permutations in quarks and antiquarks, g_{B_i} and g_{M_i} are spin selection factors for B_i and M_i, respectively. We introduce a parameter, $R_{V/P}$, to denote the relative weight of vector mesons to pseudo-scalar mesons with the same quark content, and introduce parameter $R_{D/O}$ to denote the relative weight of decuplet baryons to octet baryons with the same quark content. We assume $R_{V/P} = 0.45$ and $R_{D/O} = 0.5$ from studying hadronic yields in relativistic high-energy collisions [56, 57]. Because the values of the two parameters are extracted from the experimental data of hadronic yields, we emphasize that the two parameters can absorb the effects/contribution of various excited states and higher-mass resonances to a certain extent.

Taking the proton as an example, we have $N_p^{(q)} = 3N_u(N_u - 1)N_d$ for all possible combinations of uud. Then the ratio $N_p^{(q)}/N_q(N_q - 1)(N_q - 2)$ stands for the formation probability of the proton with the quark flavor uud. The spin selection factor for the proton is $g_p = 1/(1 + R_{D/O})$.

We introduce the configuration of multiple hadrons as $\{k_{h_1}h_1, k_{h_2}h_2, ...\} \equiv \{k_h h\}$, where h_i denotes one hadron, and k_{h_i} its number in the configuration. We define the joint moment of the multiplicity for such a configuration as $N_{\{k_h h\}} = \prod_{h_i} N_{h_i}^{k_{h_i}}$, where we have used the falling factorial of the hadron multiplicity $N_h^{k_h} = \prod_{n=1}^{k_h} (N_h - n + 1)$. The average value of the joint moment of the multiplicity reads

$$\overline{N_{\{k_h h\}}} = \left(\prod_{h_i} g_{h_i}^{k_{h_i}} \right) \frac{N_{\{h\}}^{(q)}}{N_q^{k_q} N_{\bar{q}}^{k_{\bar{q}}}} \overline{N_B^{k_B} N_{\bar{B}}^{k_{\bar{B}}} N_M^{k_M}}, \qquad (46)$$

where $k_B = \sum_{h_i} k_{h_i} Q_{B,h_i}$ counts the number of baryons in the multi-hadron configuration with $Q_{B,h_i} = 1$ with h_i as a baryon and $Q_{B,h_i} = 0$ for h_i as a meson and an antibaryon. Similarly, $k_{\bar{B}}$ counts the number of antibaryons, and k_M counts the number of mesons in the multi-hadron configuration. $k_q = \sum_{h_i} k_{h_i} Q_{q,h_i}$ counts the number of constituent quarks in the multi-hadron configuration with $Q_{q,h_i} = 3$ for h_i as a baryon, $Q_{q,h_i} = 0$ for h_i as an antibaryon, and $Q_{q,h_i} = 1$ for h_i as a meson. Similarly, $k_{\bar{q}}$ counts the number of antiquarks. The numerator in Eq. (46) is defined as

$$N_{\{h\}}^{(q)} = \left(\prod_{h_i} S_{h_i} \right) \prod_f \prod_{j=1}^{n_f} (N_f - j + 1), \qquad (47)$$

which denotes the number of all possible combinations

out of all quarks with specific flavors in the hadrons in the multi-hadron configuration, where f spans $u, d, s, \bar{u}, \bar{d}, \bar{s}$, and $n_f = \sum_{h_i} k_{h_i} n_{f,h_i}$ counts the number of f flavor quarks or antiquarks in the multi-hadron configuration.

We take a few examples of multi-hadron configurations to illustrate Eq. (46). The first example is the configuration with two protons, where we have $k_p = 2$ and $N_p^{\underline{2}} \equiv N_p(N_p - 1)$, thus we obtain

$$\overline{N_p^{\underline{2}}} = \overline{N_p(N_p - 1)} = g_p^2 \frac{N_{pp}^{(q)}}{N_q^6} \overline{N_B(N_B - 1)}, \qquad (48)$$

where $\overline{N_B(N_B - 1)}$ denotes the average for all possible baryon pairs, $N_{pp}^{(q)} = 3^2 N_u^{\underline{4}} N_d^{\underline{2}}$ is the number of all possible combinations out of six quarks with specific flavors in two protons. The ratio $N_{pp}^{(q)}/N_q^6$ provides the probability of the six-quark combination with a particular flavor structure $(uud)(uud)$. The second example is the configuration $pp\bar{n}$, we have

$$
\begin{aligned}
\overline{N_p^{\underline{2}} N_{\bar{n}}} &= \overline{N_p(N_p - 1)N_{\bar{n}}} \\
&= \left(g_p^2 \frac{N_{pp}^{(q)}}{N_q^6} \right) \left(g_{\bar{n}} \frac{N_{\bar{n}}^{(q)}}{N_{\bar{q}}^3} \right) \overline{N_B(N_B - 1)N_{\bar{B}}}, \qquad (49)
\end{aligned}
$$

where the first parentheses in the second line yield the probability of two baryons with the flavor and spin structure of two protons, whereas the second parentheses yield the probability of an antibaryon with the flavor and spin structure of the antineutron.

All orders of moments and correlation functions of hadron multiplicity can be built from Eq. (46). The two-body correlation function reads

$$
\begin{aligned}
C_{\alpha\beta} &\equiv \overline{\delta N_\alpha \delta N_\beta} = \overline{N_\alpha N_\beta} - \overline{N_\alpha}\,\overline{N_\beta} \\
&= \overline{N_{\alpha\beta}} + \delta_{\alpha,\beta}\overline{N_\alpha} - \overline{N_\alpha}\,\overline{N_\beta}, \qquad (50)
\end{aligned}
$$

where α, β denote two hadrons, and we have used $\overline{N_{\alpha\beta}} \equiv \overline{N_\alpha N_\beta}$ for $\alpha \neq \beta$ and $\overline{N_{\alpha\beta}} \equiv \overline{N_\alpha^{\underline{2}}}$ for $\alpha = \beta$. The three-body correlation function reads

$$
\begin{aligned}
C_{\alpha\beta\gamma} &\equiv \overline{\delta N_\alpha \delta N_\beta \delta N_\gamma} = \overline{N_\alpha N_\beta N_\gamma} - \overline{N_\alpha} C_{\beta\gamma} \\
&\quad - \overline{N_\beta} C_{\alpha\gamma} - \overline{N_\gamma} C_{\alpha\beta} - \overline{N_\alpha}\,\overline{N_\beta}\,\overline{N_\gamma}, \qquad (51)
\end{aligned}
$$

where $\overline{N_\alpha N_\beta N_\gamma}$ can be expressed by falling factorials,

$$
\begin{aligned}
\overline{N_\alpha N_\beta N_\gamma} &= \overline{N_{\alpha\beta\gamma}} + \delta_{\alpha,\beta}\overline{N_{\alpha\gamma}} + \delta_{\alpha,\gamma}\overline{N_{\alpha\beta}} + \delta_{\beta,\gamma}\overline{N_{\alpha\beta}} \\
&\quad + \delta_{\alpha,\beta}\delta_{\alpha,\gamma}\overline{N_\alpha}. \qquad (52)
\end{aligned}
$$

Here we used $\overline{N_{\alpha\beta\gamma}} \equiv \overline{N_\alpha N_\beta N_\gamma}$ for $\alpha \neq \beta \neq \gamma$, $\overline{N_{\alpha\beta\gamma}} \equiv \overline{N_\alpha^{\underline{2}} N_\gamma}$ for $\alpha = \beta \neq \gamma$, and $\overline{N_{\alpha\beta\gamma}} \equiv \overline{N_\alpha^{\underline{3}}}$ for $\alpha = \beta = \gamma$. Notably, $\overline{N_{\alpha\beta\gamma}}$ is symmetric for any permutation of α, β, and γ. The four-body correlation function can be written as

$$
\begin{aligned}
C_{\alpha\beta\gamma\epsilon} &= \overline{\delta N_\alpha \delta N_\beta \delta N_\gamma \delta N_\epsilon} \\
&= \overline{N_\alpha N_\beta N_\gamma N_\epsilon} - \left(\overline{N_\alpha} C_{\beta\gamma\epsilon} + \text{permutation} \right) \\
&\quad - \left(\overline{N_\alpha}\,\overline{N_\beta} C_{\gamma\epsilon} + \text{permutation} \right) - \overline{N_\alpha}\,\overline{N_\beta}\,\overline{N_\gamma}\,\overline{N_\epsilon}, \qquad (53)
\end{aligned}
$$

where $\overline{N_\alpha N_\beta N_\gamma N_\epsilon}$ can be expressed by falling factorials

$$
\begin{aligned}
\overline{N_\alpha N_\beta N_\gamma N_\epsilon} &= \overline{N_{\alpha\beta\gamma\epsilon}} + \left(\delta_{\alpha,\beta}\overline{N_{\alpha\gamma\epsilon}} + \text{permutation} \right) \\
&\quad + \left(\delta_{\alpha,\gamma}\delta_{\beta,\epsilon}\overline{N_{\alpha\beta}} + \text{permutation} \right) \\
&\quad + \delta_{\alpha,\beta}\delta_{\alpha,\gamma}\delta_{\alpha,\epsilon}\overline{N_\alpha}. \qquad (54)
\end{aligned}
$$

Here, we have used $\overline{N_{\alpha\beta\gamma\epsilon}} \equiv \overline{N_\alpha N_\beta N_\gamma N_\epsilon}$ for $\alpha \neq \beta \neq \gamma \neq \epsilon$, $\overline{N_{\alpha\beta\gamma\epsilon}} \equiv \overline{N_\alpha^{\underline{2}} N_\gamma N_\epsilon}$ for $\alpha = \beta \neq \gamma \neq \epsilon$, $\overline{N_{\alpha\beta\gamma\epsilon}} \equiv \overline{N_\alpha^{\underline{3}} N_\epsilon}$ for $\alpha = \beta = \gamma \neq \epsilon$, and $\overline{N_{\alpha\beta\gamma\epsilon}} \equiv \overline{N_\alpha^{\underline{4}}}$ for $\alpha = \beta = \gamma = \epsilon$. $\overline{N_{\alpha\beta\gamma\epsilon}}$ is symmetric for any permutation of α, β, γ and ϵ.

The cumulants of the net proton number $N_p - N_{\bar{p}}$ can be calculated by combinations of the above multi-body correlation functions (except C_1, which is simply the mean value of the net proton number)

$$
\begin{aligned}
C_1 &= \overline{N_p} - \overline{N_{\bar{p}}}, \\
C_2 &= C_{pp} - 2C_{p\bar{p}} + C_{\bar{p}\bar{p}}, \\
C_3 &= C_{ppp} - 3C_{pp\bar{p}} + 3C_{p\bar{p}\bar{p}} - C_{\bar{p}\bar{p}\bar{p}}, \\
C_4 &= C_{pppp} - 4C_{ppp\bar{p}} + 6C_{pp\bar{p}\bar{p}} - 4C_{p\bar{p}\bar{p}\bar{p}} + C_{\bar{p}\bar{p}\bar{p}\bar{p}} - 3C_2^2. \\
\end{aligned}
\qquad (55)
$$

In Fig. 2, we show the cumulant ratios for the net proton number with QCR and the gQCR at a given total quark number $x = 2000$ as functions of the quark-antiquark asymmetry z. We checked that the cumulant ratios of the net proton number are independent of x for large x. Here, we assume that the number of strange quarks is $N_s = N_{\bar{s}} = 0.45N_{\bar{u}}$, where the strangeness conservation is satisfied, and the strangeness suppression factor 0.45 is consistent with the observation in relativistic heavy-ion collisions [27]. Because the net-baryon number is fixed at given numbers of quarks and antiquarks, Fig. 2 only shows the fluctuations of the net proton number brought by the quark combination process. C_1/C_2 and C_3/C_2 of the net proton number increase with z and approach to 1.5 and 1/3 as $z \to 1$, respectively. C_4/C_2 decreases with z and approaches $-1/3$ as $z \to 1$. These results differ from those of the statistical model for hadron resonance gas with thermal equilibrium [58], in which C_1/C_2 and C_3/C_2 increase to one, and C_4/C_2 almost keeps a constant of one at large baryon number chemical potential.

6   Effects of quark number fluctuations at hadronization

The numbers of quarks and antiquarks may fluctuate in high-energy collisions event by event, such that hadronic observables should be influenced by fluctuations at the quark level. We denote the distribution of quark num-
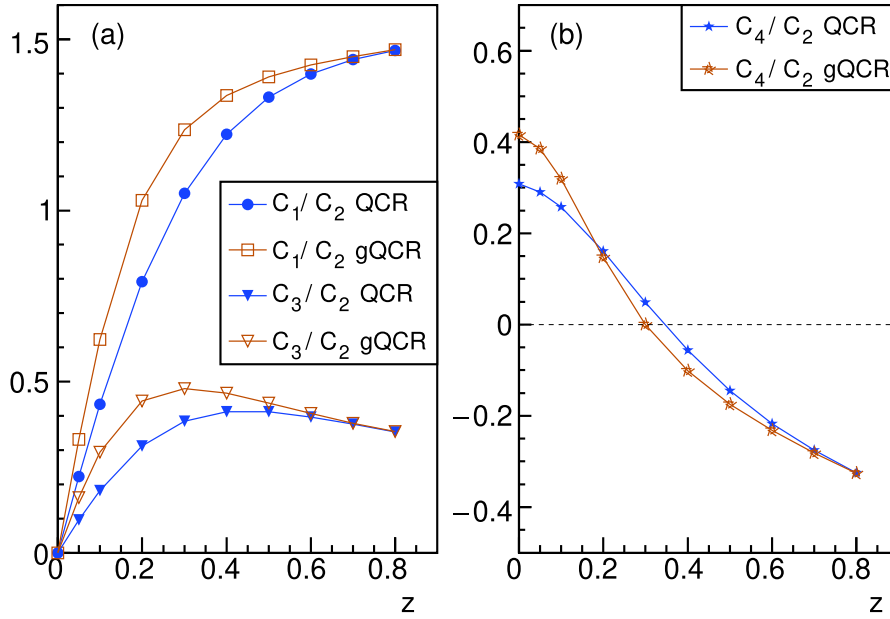
Fig. 2.   (color online) Cumulant ratios of net proton number directly produced through QCR and gQCR as functions of quark-anti-quark asymmetry z at $x = 2000$.

bers at hadronization as

$$P\left(\{N_f\}\right) \equiv P(N_u, N_{\bar{u}}, N_d, N_{\bar{d}}, N_s, N_{\bar{s}}), \qquad (56)$$

and the event average of a hadronic observable A_h is given by

$$\langle A_h \rangle = \sum_{\{N_f\}} P\left(\{N_f\}\right) \overline{A}_h\left(\{N_f\}\right), \qquad (57)$$

where $\overline{A}_h\left(\{N_f\}\right)$ denote the average at fixed quark and antiquark numbers, obtained in the previous section.

In the practical evaluation, it is more convenient to take the expansion of $\overline{A}_h\left(\{N_f\}\right)$ in N_f around the event-average numbers of quarks and antiquarks $\{\langle N_f \rangle\}$ as

$$\overline{A}_h = \overline{A}_{h,0} + \sum_f \frac{\partial \overline{A}_h}{\partial N_f}\bigg|_0 \delta N_f$$
$$+ \frac{1}{2} \sum_{f_1 f_2} \frac{\partial^2 \overline{A}_h}{\partial N_{f_1} \partial N_{f_2}}\bigg|_0 \delta N_{f_1} \delta N_{f_2} + \cdots, \qquad (58)$$

where $\delta N_f = N_f - \langle N_f \rangle$ and the subscript '0' denotes the values at $\langle N_f \rangle$. Substituting into Eq. (57), we obtain

$$\langle A_h \rangle = \overline{A}_{h,0} + \frac{1}{2} \sum_{f_1 f_2} \frac{\partial^2 \overline{A}_h}{\partial N_{f_1} \partial N_{f_2}}\bigg|_0 \left\langle \delta N_{f_1} \delta N_{f_2} \right\rangle$$
$$+ \frac{1}{3!} \sum_{f_1 f_2 f_3} \frac{\partial^3 \overline{A}_h}{\partial N_{f_1} \partial N_{f_2} \partial N_{f_3}}\bigg|_0 \left\langle \delta N_{f_1} \delta N_{f_2} \delta N_{f_3} \right\rangle$$
$$+ \dots, \qquad (59)$$

which involves multi-body correlations for the quark and antiquark number with the quark number distribution

$$P\left(\{N_f\}\right)$$

$$C_{f_1 f_2} \equiv \left\langle \delta N_{f_1} \delta N_{f_2} \right\rangle,$$
$$C_{f_1 f_2 f_3} \equiv \left\langle \delta N_{f_1} \delta N_{f_2} \delta N_{f_3} \right\rangle,$$
$$C_{f_1 f_2 f_3 f_4} \equiv \left\langle \delta N_{f_1} \delta N_{f_2} \delta N_{f_3} \delta N_{f_4} \right\rangle,$$
$$\dots\dots \qquad (60)$$

where we used the same symbols C as in Sec. 5 to denote the quark number correlation functions, but with quark flavor indices.

Using the above moment expansion method, we can study, for the selected phase-space window such as the midrapidity region $|y| < 0.5$, the influence of different quark and antiquark distributions in the window on the production of hadrons at hadronization. This extends the applicability of QCR described in Sec. 2 and 3, where only the stochastic quark-antiquark population is considered.

7   Influence of resonance decays

The long life (stable) hadrons measured in experiments include contributions from resonance decays. In this section, we consider the influence of resonance decays on multiplicity correlations of long-life hadrons. For a resonance h_i, the hardons' stable daughters are denoted as α, β, γ, etc., and the corresponding decay branching ratios are $D_{i\alpha}, D_{i\beta}, D_{i\gamma}$, etc., respectively. The distribution function of $\{N_\alpha, N_\beta, N_\gamma, \cdots\}$ from decays of the resonance h_i with the particle number N_{h_i} is denoted as $f\left(N_{h_i}, \{N_\alpha^i\}\right)$,

where N_α^i denotes number of the stable hadron α from the resonance h_i. We take the multi-nominal distribution for the selection of decay channels. Convoluting with the joint distribution of directly produced hadrons $P(\{N_{h_i}\}, \{\langle N_f \rangle\})$ at hadronization, we obtain joint distributions of stable hadrons $F(\{N_\alpha, N_\beta, N_\gamma, \cdots\})$ [35].

From the joint distribution functions of stable hadrons, we obtain the average yield of a stable hadron α

$$\langle N_\alpha \rangle = \sum_i \langle N_i \rangle D_{i\alpha}, \tag{61}$$

where the index i runs over all directly produced hadron species, including stable hadrons (we define $D_{\alpha\alpha} = 1$), and we use the shorthand notation $N_i \equiv N_{h_i}$. The two-body multiplicity correlation function is

$$C_{\alpha\beta} = \sum_{ij} C_{ij} D_{i\alpha} D_{j\beta} + \sum_i \langle N_i \rangle D_{i\alpha} (\delta_{\alpha\beta} - D_{i\beta}). \tag{62}$$

The three-body multiplicity correlation function of stable hadrons is

$$\begin{aligned}
C_{\alpha\beta\gamma} = &\sum_{ijk} C_{ijk} D_{i\alpha} D_{j\beta} D_{k\gamma} + \sum_{ij} C_{ij} D_{i\alpha} [\delta_{\alpha\beta} - D_{i\beta}] D_{j\gamma} \\
&+ \sum_{ij} C_{ij} D_{i\alpha} D_{j\beta} \{[\delta_{\alpha\gamma} - D_{i\gamma}] + [\delta_{\beta\gamma} - D_{j\gamma}]\} \\
&+ \sum_i \langle N_i \rangle D_{i\alpha} [(\delta_{\alpha\beta} - D_{i\beta})(\delta_{\alpha\gamma} - D_{i\gamma}) \\
&+ D_{i\beta}(D_{i\gamma} - \delta_{\beta\gamma})].
\end{aligned} \tag{63}$$

The four-body multiplicity correlation function of stable hadrons is

$$\begin{aligned}
C_{\alpha\beta\gamma\epsilon} = &\sum_{ijkl} C_{ijkl} D_{i\alpha} D_{j\beta} D_{k\gamma} D_{l\epsilon} + \sum_{ijk} [C_{ijk} + \langle N_i \rangle C_{jk}] \{(\delta_{\alpha\beta} - D_{i\beta}) D_{i\alpha} D_{j\gamma} D_{k\epsilon} + (\delta_{\alpha\gamma} - D_{i\gamma}) D_{i\alpha} D_{j\beta} D_{k\epsilon} \\
&+ (\delta_{\alpha\epsilon} - D_{i\epsilon}) D_{i\alpha} D_{j\gamma} D_{k\beta} + (\delta_{\beta\gamma} - D_{i\gamma}) D_{i\beta} D_{j\alpha} D_{k\epsilon} + (\delta_{\beta\epsilon} - D_{i\epsilon}) D_{i\beta} D_{j\alpha} D_{k\gamma} + (\delta_{\gamma\epsilon} - D_{i\epsilon}) D_{i\gamma} D_{j\alpha} D_{k\beta} \} \\
&+ \sum_{ij} [C_{ij} + \langle N_i \rangle \langle N_j \rangle] \{D_{i\alpha}(\delta_{\alpha\beta} - D_{i\beta}) D_{j\gamma}(\delta_{\gamma\epsilon} - D_{j\epsilon}) \\
&+ D_{i\alpha}(\delta_{\alpha\gamma} - D_{i\gamma}) D_{j\beta}(\delta_{\beta\epsilon} - D_{j\epsilon}) + D_{i\alpha}(\delta_{\alpha\epsilon} - D_{i\epsilon}) D_{j\beta}(\delta_{\beta\gamma} - D_{j\gamma})\} \\
&+ \sum_{ij} C_{ij} D_{i\alpha} \{(\delta_{\alpha\beta} - D_{i\beta})(\delta_{\alpha\gamma} - D_{i\gamma}) + D_{i\beta}(D_{i\gamma} - \delta_{\beta\gamma})\} D_{j\epsilon} + \sum_{ij} C_{ij} D_{i\alpha} \{(\delta_{\alpha\beta} - D_{i\beta})(\delta_{\alpha\epsilon} - D_{i\epsilon}) + D_{i\beta}(D_{i\epsilon} - \delta_{\beta\epsilon})\} D_{j\gamma} \\
&+ \sum_{ij} C_{ij} D_{i\alpha} \{(\delta_{\alpha\gamma} - D_{i\gamma})(\delta_{\alpha\epsilon} - D_{i\epsilon}) + D_{i\gamma}(D_{i\epsilon} - \delta_{\gamma\epsilon})\} D_{j\beta} + \sum_{ij} C_{ij} D_{i\beta} \{(\delta_{\beta\gamma} - D_{i\gamma})(\delta_{\beta\epsilon} - D_{i\epsilon}) + D_{i\gamma}(D_{i\epsilon} - \delta_{\gamma\epsilon})\} D_{j\alpha} \\
&+ \sum_i \langle N_i \rangle D_{i\alpha} \{-6 D_{i\beta} D_{i\gamma} D_{i\epsilon} + 2 [\delta_{\alpha\beta} D_{i\gamma} D_{i\epsilon} + (\delta_{\alpha\gamma} + \delta_{\beta\gamma}) D_{i\beta} D_{i\epsilon} + (\delta_{\alpha\epsilon} + \delta_{\beta\epsilon} + \delta_{\gamma\epsilon}) D_{i\beta} D_{i\gamma}] \\
&- [(\delta_{\alpha\gamma}\delta_{\beta\epsilon} + \delta_{\alpha\epsilon}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\alpha\epsilon} + \delta_{\beta\gamma}\delta_{\beta\epsilon}) D_{i\beta} + (\delta_{\alpha\beta}\delta_{\gamma\epsilon} + \delta_{\alpha\beta}\delta_{\alpha\epsilon}) D_{i\gamma} + \delta_{\alpha\beta}\delta_{\alpha\gamma} D_{i\epsilon}] + \delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\epsilon}\}.
\end{aligned} \tag{64}$$

The higher order multiplicity correlation functions can similarly be derived from the joint distribution functions of stable hadrons.

8   Application: cumulants for net protons in heavy ion collisions

In this section, we take the simple example of a quark system and calculate the cumulant ratios for net protons in the final state with gQCR. We discuss our results in the context of the experimental observation in Au+Au collisions at RHIC.

We consider a quark system with the property $S\sigma \equiv C_3/C_2 = z$ and $\kappa\sigma^2 \equiv C_4/C_2 = 1$ for the baryon quantum number, which is similar to the base line of the grand canonical ensemble in the statistical model [59]. A simple case for the quark number correlation functions

satisfying the above property is

$$\begin{aligned}
C_{ff} &= \langle N_f \rangle, \\
C_{fff} &= 3\langle N_f \rangle, \\
C_{ffff} &= 9\langle N_f \rangle + 3\langle N_f \rangle^2,
\end{aligned} \tag{65}$$

for diagonal elements and non-vanishing off-diagonal elements ($f_1 \neq f_2$)

$$C_{f_1 f_1 f_2 f_2} = \langle N_{f_1} \rangle \langle N_{f_2} \rangle. \tag{66}$$

We assume the following properties for correlation functions of the strangeness as a result of the strangeness conservation

$$\begin{aligned}
C_{s\bar{s}} &= C_{ss}, \\
C_{ss\bar{s}} &= C_{s\bar{s}\bar{s}} = C_{sss}, \\
C_{ss\bar{s}\bar{s}} &= C_{ssss}.
\end{aligned} \tag{67}$$

Because the total quark number x in the mid-rapidity region $|y| < 0.5$ in central heavy-ion collisions at RHIC energy is usually large ($x \gtrsim 500$), the cumulant ratios for net

protons in the final state in our model is not sensitive to x (the system size). Here, we just take $x = 5000$ in the calculation. We use Eq. (61) to fit the data of the $\bar{p}/p$ yield ratio in Au+Au collisions and determine z in the collision energy range $\sqrt{s_{NN}} \in [7, 200]$ GeV.

In Fig. 3, we show the results for the cumulant ratios for net protons in the final state at different collisional energies. Our results only incorporate the contributions of quark number fluctuations and correlations up to the fourth order, as in Eq. (59). The auxiliary horizontal axis shows the corresponding quark-antiquark asymmetry parameter z. Results including the contributions from strong and electromagnetic (S&EM) decays of resonances are depicted by dashed lines, while results with full decay contributions including weak decays are depicted by solid lines. The STAR data are shown in solid circles with error bars.

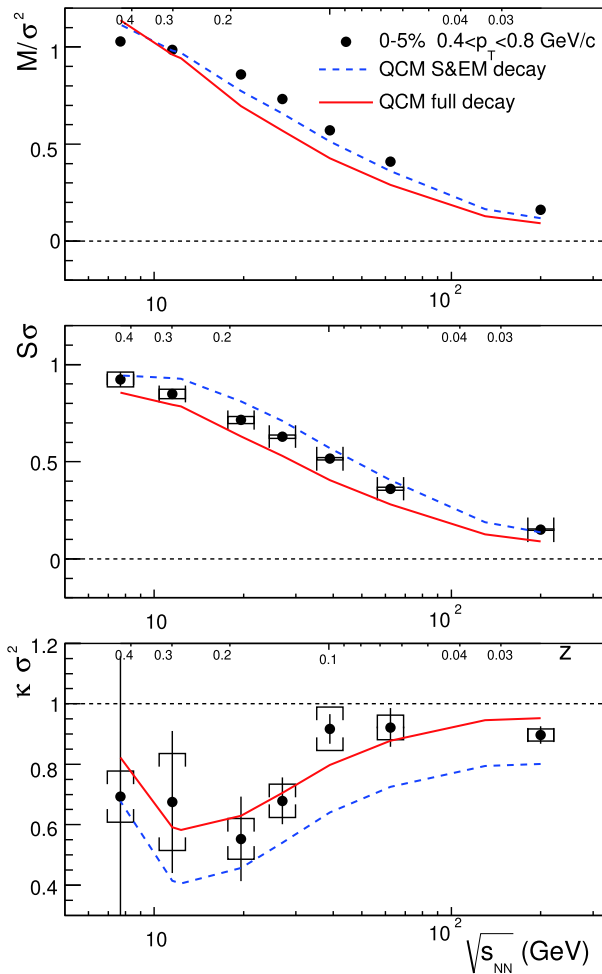In the figure, we see that the cumulant ratios for net

Fig. 3.   (color online) Cumulant ratios for net protons at different collision energies. Solid circles with error bars depict experimental data [43, 60]. Solid and dashed lines represent theoretical results of SDQCM.

protons as functions of collisional energies in our model describe the experimental data: M/σ^2 and $S\sigma$ increase with z or equivalently decrease with collisional energies, while $\kappa\sigma^2$ decreases with collisional energies until it reaches a minimum at $\sqrt{s_{NN}} = 20$ GeV and then increases toward unity at high energies. Our results for $\kappa\sigma^2$ are the consequence of the competition between two effects: (1) The cumulant ratio C_4/C_2 from directly produced net protons by quark combinations, as shown in Fig. 2, always decreases with increasing z; (2) The cumulant ratio C_4/C_2 for baryons or antibaryons, as shown in Fig. 1, rapidly increases as $z \gtrsim 0.2$ (corresponding to $\sqrt{s_{NN}} \lesssim 20$ GeV).

We now provide some remarks regarding the results for $\kappa\sigma^2$ of net protons. Although our results for net protons can reproduce the nontrivial dependence on collisional energies, we always have $\kappa\sigma^2 = 1$ for net baryons at all collisional energies because of the flavor/charge conservation in the quark combination. Therefore, our results indicate that cumulant ratios of the net proton number in this simple case do not exactly follow those of the net baryon number. This is different from the statistical model [57] for hadron resonance gas with thermal equilibrium, which predicts similar cumulant ratios for net-protons and net-baryons.

We emphasize that the current results are preliminary and are mainly used to show the potential application of our model in hadronic fluctuations, and the related phase transition in relativistic heavy-ion collisions. Some limitations in current calculations should be clarified. In this study, we only consider the production of ground-state hadrons. Effects of higher-mass resonances are only partially absorbed by parameters $R_{V/P}$ and $R_{D/O}$. According to our previous study [35], the production of hadrons with small yields tends to follow the Poisson distribution. Including those higher-mass resonances with small yields may enlarge the fourth moments of protons to a certain extent. Our model is a static one and does not consider the diffusion of hadrons/quarks during the finite hadronization time, which will have some influence on the calculation of net-proton fluctuations in a specific window. Further, to make final comparison with the data of net protons, we should consider more realistic quark number fluctuations and correlations in the studied rapidity window, which may be obtained by grand-canonical statistics of the thermal quark system or by canonical statistics with a Bernoulli trial selection of the specific window. We should also consider other effects related to the finite acceptance window, such as the diffusion/blur of charges during the hadronic scatterings stage as well as that caused by resonance decay [55, 58, 61–73]. We will study these effects in this framework in the future.

9   Summary

We developed a new statistical method to solve the probability distribution for the number of baryons, antibaryons, and mesons formed in the hadronization of a constituent quark and antiquark system governed by the quark combination rule (QCR) in the quark combination model developed by the Shandong Group. We use a set of numbers $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ to classify the outcome of implementing the QCR for a queue of N_q quarks and $N_{\bar{q}}$ antiquarks, where there are N_M mesons, N_B baryons, and $N_{\bar{B}}$ antibaryons formed by combination but with N_r quarks and $N_{\bar{r}}$ antiquarks left without forming hadrons. The number of different ways of a given configuration $(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ is denoted as $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$. We build the recursive relation for $F(N_M, N_B, N_{\bar{B}}, 0, 0)$. We adopt the generating function method to solve the recursive relation and provide the analytical expression of $F(N_M, N_B, N_{\bar{B}}, 0, 0)$. This method is far simpler than the previous one and is easy to generalize. To accommodate the baryon yield in experimental data in relativistic heavy-ion collisions, we consider a generalized combination rule (gQCR). We provide the solution of the baryon and meson probability distribution function under the new rule. Because the (anti)baryon production is more suppressed in the gQCR, we find that the cumulant ratios of the antibaryon number achieve Poisson statistics more rapidly in the gQCR than in the QCR with increasing baryon number density.

We studied the multiplicity fluctuation and correlation functions for identified baryons directly produced in collisions. We also studied correlation functions of final state hadrons including contributions from resonance decays. As an illustrative example, we consider a quark-antiquark system with the property $S\sigma = z$ and $\kappa\sigma^2 = 1$ for the baryon quantum number, which is similar to the base line in the grand-canonical ensemble in the statistical model. We calculate the cumulant ratios in the quark combination model and find that $S\sigma$ for net protons in the model increases with decreasing collisional energies, which is consistent with the experimental observation. More interesting is that $\kappa\sigma^2$ for net protons has a nontrivial energy behavior consistent with the data.

We dedicate this work to Qu-bing Xie (1935-2013) who was the teacher, mentor and friend of ZTL, FLS, and QW.

Appendix A: Derivation of two properties for gQCR

We present two properties for $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ in this appendix.

Property 1. For $(N_r, N_{\bar{r}}) = (1,0), (0,1)$, $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ can be expressed in terms of $F(N'_M, N'_B, N'_{\bar{B}}, 0, 0)$ with $N'_M \leqslant N_M$, $N'_B \leqslant N_B$ and $N'_{\bar{B}} \leqslant N_{\bar{B}}$,

$$F(N_M, N_B, N_{\bar{B}}, 1, 0) = \sum_{n_M=0}^{N_M} \sum_{n_B=0}^{N_B} \sum_{n_{\bar{B}}=0}^{N_{\bar{B}}} C^{10}_{n_B, n_{\bar{B}}}(n_M) 2^{N_M - 1 - n_M + \delta_{n_M, N_M}} F(n_M, N_B - n_B, N_{\bar{B}} - n_{\bar{B}}, 0, 0),$$

$$F(N_M, N_B, N_{\bar{B}}, 0, 1) = \sum_{n_M=0}^{N_M} \sum_{n_B=0}^{N_B} \sum_{n_{\bar{B}}=0}^{N_{\bar{B}}} C^{01}_{n_B, n_{\bar{B}}}(n_M) 2^{N_M - 1 - n_M + \delta_{n_M, N_M}} F(n_M, N_B - n_B, N_{\bar{B}} - n_{\bar{B}}, 0, 0), \tag{A1}$$

where the coefficients $C^{10}_{n_B, n_{\bar{B}}}(n_M)$ are given by $C^{10}_{0,0}(n_M) = 1$ and

$$C^{10}_{a,b}(n_M) = (-1)^{a+b} \sum_{j_1=n_M}^{N_M} \cdots \sum_{j_{a+b}=j_{a+b-1}}^{N_M} 1, \quad \text{for} \quad b = a, a+1,$$

$$C^{10}_{a,b}(n_M) = 0, \quad \text{for} \quad b \neq a, a+1, \tag{A2}$$

for $a + b > 0$. The coefficients $C^{01}_{n_B, n_{\bar{B}}}(n_M)$ are given by $C^{01}_{0,0}(n_M) = 1$ and

$$C^{01}_{a,b}(n_M) = (-1)^{a+b} \sum_{j_1=n_M}^{N_M} \cdots \sum_{j_{a+b}=j_{a+b-1}}^{N_M} 1, \quad \text{for} \quad b = a, a-1,$$

$$C^{01}_{a,b}(n_M) = 0, \quad \text{for} \quad b \neq a, a-1, \tag{A3}$$

for $a + b > 0$. The sums over n_B, $n_{\bar{B}}$, and n_M in Eq. (A1) continue until any of the baryon, antibaryon, or meson number become negative, since $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}}) = 0$ for the case where any of N_M, N_B, $N_{\bar{B}}$, N_r, or $N_{\bar{r}}$ are negative.

Property 2. For $(N_r, N_{\bar{r}}) = (2,0), (0,2)$, $F(N_M, N_B, N_{\bar{B}}, N_r, N_{\bar{r}})$ can be expressed in terms of $F(N'_M, N'_B, N'_{\bar{B}}, 0, 0)$ with $N'_M \leqslant N_M$, $N'_B \leqslant N_B$ and $N'_{\bar{B}} \leqslant N_{\bar{B}}$,

$$F(N_M, N_B, N_{\bar{B}}, 2, 0) = \sum_{n_M=0}^{N_M} \sum_{n_B=0}^{N_B} \sum_{n_{\bar{B}}=0}^{N_B} C^{20}_{n_B, n_{\bar{B}}}(n_M) 2^{N_M - n_M} F(n_M, N_B - n_B, N_{\bar{B}} - n_{\bar{B}}, 0, 0),$$

$$F(N_M, N_B, N_{\bar{B}}, 0, 2) = \sum_{n_M=0}^{N_M} \sum_{n_B=0}^{N_B} \sum_{n_{\bar{B}}=0}^{N_B} C^{02}_{n_B, n_{\bar{B}}}(n_M) 2^{N_M - n_M} F(n_M, N_B - n_B, N_{\bar{B}} - n_{\bar{B}}, 0, 0), \tag{A4}$$

where the coefficients are identical to Eqs. (A2, A3): $C^{20}_{n_B, n_{\bar{B}}}(n_M) = C^{10}_{n_B, n_{\bar{B}}}(n_M)$ and $C^{02}_{n_B, n_{\bar{B}}}(n_M) = C^{01}_{n_B, n_{\bar{B}}}(n_M)$.

To demonstrate the two properties in Eq. (A1) and Eq. (A4), we can solve $F(N_M,N_B,N_{\bar{B}},2,0)$ by substituting the first equation into the third one in Eq. (31),

$$F(N_M,N_B,N_{\bar{B}},2,0) = \sum_{n_M=0}^{N_M} [F(n_M,N_B,N_{\bar{B}},0,0) - F(n_M,N_B,N_{\bar{B}}-1,0,2)]$$
$$\times 2^{N_M-n_M}.$$

(A5)

In the same manner, we can solve $F(N_M,N_B,N_{\bar{B}},0,2)$ by substituting

the second equation into the fourth one in Eq. (31)

$$F(N_M,N_B,N_{\bar{B}},0,2) = \sum_{n_M=0}^{N_M} [F(n_M,N_B,N_{\bar{B}},0,0) - F(n_M,N_B-1,N_{\bar{B}},2,0)]$$
$$\times 2^{N_M-n_M}.$$

(A6)

From Eqs. (A5, A6) we obtain Eq. (A4). Then from Eq. (A4) and the last two equations of Eq. (31) we obtain Eq. (A1).

Appendix B: Derivation of recursive equation Eq. (32)

We derive the recursive equation for $F(N_M,N_B,N_{\bar{B}},0,0)$ with the help of Eq. (31). We start from Eq. (7), which also holds for gQCR. Using Eq. (31), Eq. (7) can be rewritten as

$$F(N_M,N_B,N_{\bar{B}},0,0) = G(N_M,N_B,N_{\bar{B}}) - 3G(N_M-1,N_B,N_{\bar{B}})$$
$$+ G(N_M-2,N_B,N_{\bar{B}}),$$

(B1)

where G is defined by

$$G(N_M,N_B,N_{\bar{B}}) = F(N_M,N_B,N_{\bar{B}},2,0) + F(N_M,N_B,N_{\bar{B}},0,2).$$

(B2)

We need to define another auxiliary function

$$H(N_M,N_B,N_{\bar{B}}) = F(N_M,N_B-1,N_{\bar{B}},2,0) + F(N_M,N_B,N_{\bar{B}}-1,0,2)$$
$$= 2F(N_M,N_B,N_{\bar{B}},0,0) + 2G(N_M-1,N_B,N_{\bar{B}})$$
$$- G(N_M,N_B,N_{\bar{B}}) = G(N_M,N_B,N_{\bar{B}})$$
$$- 4G(N_M-1,N_B,N_{\bar{B}}) + 2G(N_M-2,N_B,N_{\bar{B}}),$$

(B3)

where we used the first two equalities of Eq. (31) to obtain second line and used Eq. (B1) to obtain the last line. In contrast, we can also derive another form of $H(N_M,N_B,N_{\bar{B}})$ by applying the last two

equalities and subsequently the first two equalities of Eq. (B3) and finally applying Eq. (31). The result is as follows:

$$H(N_M,N_B,N_{\bar{B}}) = F(N_M,N_B-1,N_{\bar{B}},1,0) + F(N_M-1,N_B-1,N_{\bar{B}},2,0)$$
$$+ F(N_M,N_B,N_{\bar{B}}-1,0,1) + F(N_M-1,N_B,N_{\bar{B}}-1,0,2)$$
$$= 2H(N_M-1,N_B,N_{\bar{B}}) - G(N_M,N_B-1,N_{\bar{B}}-1)$$
$$+ G(N_M,N_B-1,N_{\bar{B}}) - 3G(N_M-1,N_B-1,N_{\bar{B}})$$
$$+ G(N_M-2,N_B-1,N_{\bar{B}}) + G(N_M,N_B,N_{\bar{B}}-1)$$
$$- 3G(N_M-1,N_B,N_{\bar{B}}-1) + G(N_M-2,N_B,N_{\bar{B}}-1).$$

(B4)

From Eq. (B3), we also obtain

$$H(N_M,N_B,N_{\bar{B}}) - 2H(N_M-1,N_B,N_{\bar{B}}) = G(N_M,N_B,N_{\bar{B}}) - 6G(N_M-1,N_B,N_{\bar{B}})$$
$$+ 10G(N_M-2,N_B,N_{\bar{B}})$$
$$- 4G(N_M-3,N_B,N_{\bar{B}}).$$

(B5)

By setting Eq. (B4) equal to Eq. (B5), we derive the recursive equation for G and then for $F(N_M,N_B,N_{\bar{B}},0,0)$ through Eq. (B1), as in Eq. (32).

Appendix C: Derivation of generating functions for gQCR

We multiply Eq. (32) by x^{N_M} and sum over $N_M \geqslant 3$ (the equation is well defined for $N_M \geqslant 3$). Hence, we obtain

$$A(x;N_B,N_{\bar{B}}) - x^2 F(2,N_B,N_{\bar{B}},0,0) - xF(1,N_B,N_{\bar{B}},0,0) - F(0,N_B,N_{\bar{B}},0,0) = A(x;N_B-1,N_{\bar{B}}) - x^2 F(2,N_B-1,N_{\bar{B}},0,0) - xF(1,N_B-1,N_{\bar{B}},0,0)$$
$$- F(0,N_B-1,N_{\bar{B}},0,0) + A(x;N_B,N_{\bar{B}}-1) - x^2 F(2,N_B,N_{\bar{B}}-1,0,0) - xF(1,N_B,N_{\bar{B}}-1,0,0) - F(0,N_B,N_{\bar{B}}-1,0,0)$$
$$- A(x;N_B-1,N_{\bar{B}}-1) + x^2 F(2,N_B-1,N_{\bar{B}}-1,0,0) + xF(1,N_B-1,N_{\bar{B}}-1,0,0) + F(0,N_B-1,N_{\bar{B}}-1,0,0) + 6x[A(x;N_B,N_{\bar{B}})$$
$$- xF(1,N_B,N_{\bar{B}},0,0) - F(0,N_B,N_{\bar{B}},0,0)] - 3x[A(x;N_B-1,N_{\bar{B}}) - xF(1,N_B-1,N_{\bar{B}},0,0) - F(0,N_B-1,N_{\bar{B}},0,0)]$$
$$- 3x[A(x;N_B,N_{\bar{B}}-1) - xF(1,N_B,N_{\bar{B}}-1,0,0) - F(0,N_B,N_{\bar{B}}-1,0,0)] - 10x^2[A(x;N_B,N_{\bar{B}}) - F(0,N_B,N_{\bar{B}},0,0)] + x^2[A(x;N_B-1,N_{\bar{B}})$$
$$- F(0,N_B-1,N_{\bar{B}},0,0)] + x^2[A(x;N_B,N_{\bar{B}}-1) - F(0,N_B,N_{\bar{B}}-1,0,0)] + 4x^3 A(x;N_B,N_{\bar{B}}).$$

(C1)

One can verify that a complete cancellation occurs for terms proportional to x^2 and those proportional to x after applying Eq. (32) for $N_M = 2$ and $N_M = 1$, respectively. The constant terms in F read

$$I = F(0,N_B,N_{\bar{B}},0,0) - F(0,N_B-1,N_{\bar{B}},0,0) - F(0,N_B,N_{\bar{B}}-1,0,0)$$
$$+ F(0,N_B-1,N_{\bar{B}}-1,0,0)$$
$$= \delta_{N_B N_{\bar{B}},0} - \delta_{(N_B-1)N_{\bar{B}},0} - \delta_{N_B(N_{\bar{B}}-1),0} + \delta_{(N_B-1)(N_{\bar{B}}-1),0},$$

(C2)

where we used the initial value $F(0,N_B,N_{\bar{B}},0,0) = \delta_{N_B N_{\bar{B}},0}$. This means that if there are no mesons, baryons and antibaryons cannot

co-exist, and if there are only quarks or antiquarks in the system, the number of different queues is 1. Then, Eq. (C1) is simplified as

$$A(x;N_B,N_{\bar{B}}) = (6x+4x^3-10x^2)A(x;N_B,N_{\bar{B}}) + (1-3x+x^2)$$
$$\times [A(x;N_B-1,N_{\bar{B}}) + A(x;N_B,N_{\bar{B}}-1)]$$
$$- A(x;N_B-1,N_{\bar{B}}-1) + \delta_{N_B N_{\bar{B}},0} - \delta_{(N_B-1)N_{\bar{B}},0}$$
$$- \delta_{N_B(N_{\bar{B}}-1),0} + \delta_{(N_B-1)(N_{\bar{B}}-1),0}.$$

(C3)

We now multiply Eq. (C3) by $y^{N_B}z^{N_{\bar{B}}}$ and take a sum over $N_B \geqslant 1$ and $N_{\bar{B}} \geqslant 1$ to obtain

$$(1-6x+10x^2-4x^3)A(x,y,z) = (1-3x+x^2)\sum_{N_{\bar{B}}=1}^{\infty}\sum_{N_B=1}^{\infty} y^{N_B}z^{N_{\bar{B}}}[A(x;N_B-1,N_{\bar{B}}) + A(x;N_B,N_{\bar{B}}-1)] - \sum_{N_{\bar{B}}=1}^{\infty}\sum_{N_B=1}^{\infty} A(x;N_B-1,N_{\bar{B}}-1)y^{N_B}z^{N_{\bar{B}}}$$

$$+ \sum_{N_{\bar{B}}=1}^{\infty} \sum_{N_B=1}^{\infty} \left[\delta_{N_B N_{\bar{B}},0} - \delta_{(N_B-1)N_{\bar{B}},0} - \delta_{N_B(N_{\bar{B}}-1),0} + \delta_{(N_B-1)(N_{\bar{B}}-1),0} \right] y^{N_B} z^{N_{\bar{B}}}. \tag{C4}$$

To simplify the above equation, we use

$$\sum_{N_{\bar{B}}=1}^{\infty} \sum_{N_B=1}^{\infty} A(x; N_B - a, N_{\bar{B}} - b) y^{N_B} z^{N_{\bar{B}}} = \delta_{a,1} \delta_{b,1} yz A(x,y,z) + \delta_{a,0} \delta_{b,1} z \left[A(x,y,z) - \sum_{N_{\bar{B}}=0}^{\infty} A(x; 0, N_{\bar{B}}) z^{N_{\bar{B}}} \right] + \delta_{a,1} \delta_{b,0} y \left[A(x,y,z) - \sum_{N_B=0}^{\infty} A(x; N_B, 0) y^{N_B} \right]$$

$$+ \delta_{a,0} \delta_{b,0} \left[A(x,y,z) - \sum_{N_B=0}^{\infty} A(x; N_B, 0) y^{N_B} - \sum_{N_{\bar{B}}=0}^{\infty} A(x; 0, N_{\bar{B}}) z^{N_{\bar{B}}} + A(x; 0,0) \right], \tag{C5}$$

and

$$-yz = \sum_{N_{\bar{B}}=1}^{\infty} \sum_{N_B=1}^{\infty} \left[\delta_{N_B N_{\bar{B}},0} - \delta_{(N_B-1)N_{\bar{B}},0} - \delta_{N_B(N_{\bar{B}}-1),0} + \delta_{(N_B-1)(N_{\bar{B}}-1),0} \right] y^{N_B} z^{N_{\bar{B}}}, \tag{C6}$$

as well as Eqs. (36) and (37), we finally obtain Eq. (38).

Appendix D: Derivation of Eq. (39) in gQCR

From Eq. (38) we can extract $F(N_M, N_B, N_{\bar{B}}, 0, 0)$, the coefficient of $x^{N_M} y^{N_B} z^{N_{\bar{B}}}$ in the polynomial expansion of $A(x,y,z)$. To this end, we rewrite $A(x,y,z)$ as

$$A(x,y,z) = \frac{1 - 4x + 4x^2 - yz}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \left[\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} (y+z) - \frac{1}{1 - 6x + 10x^2 - 4x^3} yz \right]^i$$

$$= \frac{1 - 4x + 4x^2 - yz}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \sum_{j+k+l=i} \binom{N_q + N_{\bar{q}}}{N_q} y^{j+l} z^{k+l} \times \left(\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} \right)^{j+k} \left(\frac{-1}{1 - 6x + 10x^2 - 4x^3} \right)^l$$

$$= \frac{1 - 4x + 4x^2 - yz}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{i} \binom{i}{j,\, k,\, l} y^{i-k} z^{i-j} \times \left(\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} \right)^{j+k} \left(\frac{-1}{1 - 6x + 10x^2 - 4x^3} \right)^{i-j-k}, \tag{D1}$$

where we used the multinomial theorem

$$(w_1 + w_2 + \cdots + w_m)^n = \sum_{k_1 + k_2 + \cdots + k_m = n} \binom{n}{k_1,\, k_2,\, \ldots,\, k_m} \prod_{t=1}^{m} w_t^{k_t}, \tag{D2}$$

with multinomial coefficients

$$\binom{n}{k_1,\, k_2,\, \ldots,\, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}. \tag{D3}$$

Then, we can extract the coefficient of $y^{N_B} z^{N_{\bar{B}}}$ as

$$C(y^{N_B} z^{N_{\bar{B}}}) = C_1 + C_2 + C_3 + C_4, \tag{D4}$$

where $C_{1,2,3,4}$ are given by

$$C_1 = \frac{1}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \binom{i}{j,\, k,\, i-j-k} \left(\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} \right)^{2i - N_B - N_{\bar{B}}} \left(\frac{-1}{1 - 6x + 10x^2 - 4x^3} \right)^{N_B + N_{\bar{B}} - i},$$

$$C_2 = \frac{-4x}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \binom{i}{i - N_B,\, i - N_{\bar{B}},\, N_B + N_{\bar{B}} - i} \left(\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} \right)^{2i - N_B - N_{\bar{B}}} \left(\frac{-1}{1 - 6x + 10x^2 - 4x^3} \right)^{N_B + N_{\bar{B}} - i},$$

$$C_3 = \frac{4x^2}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \binom{i}{i - N_B,\, i - N_{\bar{B}},\, N_B + N_{\bar{B}} - i} \left(\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} \right)^{2i - N_B - N_{\bar{B}}} \left(\frac{-1}{1 - 6x + 10x^2 - 4x^3} \right)^{N_B + N_{\bar{B}} - i},$$

$$C_4 = - \frac{1}{1 - 6x + 10x^2 - 4x^3} \sum_{i=0}^{\infty} \binom{i}{i - N_B,\, i - N_{\bar{B}},\, N_B + N_{\bar{B}} - i} \times \left(\frac{1 - 3x + x^2}{1 - 6x + 10x^2 - 4x^3} \right)^{2i - N_B - N_{\bar{B}} + 2} \left(\frac{-1}{1 - 6x + 10x^2 - 4x^3} \right)^{N_B + N_{\bar{B}} - 2 - i}. \tag{D5}$$

We have expressed the summation indices j and k in terms of N_B and $N_{\bar{B}}$ for a given summation index i. The sum over i involves a finite number of terms. For example, in C_1, we have $N_B + N_{\bar{B}} \geq i \geq \mathrm{Max}(N_B, N_{\bar{B}})$, since any term in the summation with the factorial of a negative integer in the denominator is vanishing.

We can factorize the following two polynomials as

$$1 - 3x + x^2 = \left(1 - \frac{3 + \sqrt{5}}{2} x \right) \left(1 - \frac{3 - \sqrt{5}}{2} x \right), \tag{D6}$$

$$6x + 10x^2 - 4x^3 = 6x(1-x) \left(1 - \frac{2}{3} x \right), \tag{D7}$$

and expand C_1, C_2, C_3, C_4 into a power series of x with the help of the binomial theorem. Then, we can extract the coefficient of x^{N_M} in $C(y^{N_B} z^{N_{\bar{B}}})$. This yields the coefficient of $x^{N_M} y^{N_B} z^{N_{\bar{B}}}$ in $A(x,y,z)$ or

$F(N_M, N_B, N_{\bar{B}}, 0, 0)$ as the sum of the following four terms

$$
\begin{aligned}
I_1 =& \sum_{i=0}^{\infty} (-1)^{N_B + N_{\bar{B}} - i} \begin{pmatrix} i \\ i - N_B + 1, \ i - N_{\bar{B}} + 1, \ N_B + N_{\bar{B}} - 2 - i \end{pmatrix} \sum_{j+k+l+m+n=N_M} \begin{pmatrix} i \\ i - N_B, \ i - N_{\bar{B}}, \ N_B + N_{\bar{B}} - i \end{pmatrix} \begin{pmatrix} 2i - N_B - N_{\bar{B}} \\ j \end{pmatrix} \\
&\times \left(-\frac{3+\sqrt{5}}{2} \right)^j \left(-\frac{3-\sqrt{5}}{2} \right)^k \begin{pmatrix} 2i - N_B - N_{\bar{B}} \\ k \end{pmatrix} 6^l \begin{pmatrix} i+l \\ l \end{pmatrix} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \left(-\frac{2}{3} \right)^n, \\
I_2 =& -4 \sum_{i=0}^{\infty} (-1)^{N_B + N_{\bar{B}} - i} \begin{pmatrix} l \\ n \end{pmatrix} \sum_{j+k+l+m+n=N_M-1} \begin{pmatrix} i \\ i - N_B, \ i - N_{\bar{B}}, \ N_B + N_{\bar{B}} - i \end{pmatrix} \begin{pmatrix} 2i - N_B - N_{\bar{B}} \\ j \end{pmatrix} \\
&\times \left(-\frac{3+\sqrt{5}}{2} \right)^j \left(-\frac{3-\sqrt{5}}{2} \right)^k \begin{pmatrix} 2i - N_B - N_{\bar{B}} \\ k \end{pmatrix} 6^l \begin{pmatrix} i+l \\ l \end{pmatrix} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \left(-\frac{2}{3} \right)^n, \\
I_3 =& 4 \sum_{i=0}^{\infty} (-1)^{N_B + N_{\bar{B}} - i} \begin{pmatrix} l \\ n \end{pmatrix} \sum_{j+k+l+m+n=N_M-2} \begin{pmatrix} i \\ i - N_B, \ i - N_{\bar{B}}, \ N_B + N_{\bar{B}} - i \end{pmatrix} \begin{pmatrix} 2i - N_B - N_{\bar{B}} \\ j \end{pmatrix} \\
&\times \left(-\frac{3+\sqrt{5}}{2} \right)^j \left(-\frac{3-\sqrt{5}}{2} \right)^k \begin{pmatrix} 2i - N_B - N_{\bar{B}} \\ k \end{pmatrix} 6^l \begin{pmatrix} i+l \\ l \end{pmatrix} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \left(-\frac{2}{3} \right)^n, \\
I_4 =& -\sum_{i=0}^{\infty} (-1)^{N_B + N_{\bar{B}} - i} \begin{pmatrix} l \\ n \end{pmatrix} \sum_{j+k+l+m+n=N_M} \begin{pmatrix} i \\ i - N_B + 1, \ i - N_{\bar{B}} + 1, \ N_B + N_{\bar{B}} - 2 - i \end{pmatrix} \begin{pmatrix} 2i - N_B - N_{\bar{B}} + 2 \\ j \end{pmatrix} \\
&\times \left(-\frac{3+\sqrt{5}}{2} \right)^j \left(-\frac{3-\sqrt{5}}{2} \right)^k \begin{pmatrix} 2i - N_B - N_{\bar{B}} + 2 \\ k \end{pmatrix} 6^l \begin{pmatrix} i+l \\ l \end{pmatrix} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \left(-\frac{2}{3} \right)^n,
\end{aligned}
\tag{D8}
$$

which yield the final result of gQCR in Sec. 3.

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