Sonic velocity in holographic fluids and its applications[*]

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Abstract: Gravity/fluid correspondence acts as an important tool in investigating the strongly correlated fluids. We carefully investigate the holographic fluids at the finite cutoff surface by considering different boundary conditions in the scenario of gravity/fluid correspondence. We find that the sonic velocity of the boundary fluids at the finite cutoff surface is critical in clarifying the superficial similarity between the bulk viscosity and perturbation of the pressure for the holographic fluid, where we set a special boundary condition at the finite cutoff surface to explicitly express this superficial similarity. Moreover, we further take the sonic velocity into account to investigate a case with a more general boundary condition. In this more genaral case, although two parameters in the first order stress tensor of holographic fluid cannot be fixed, one can still extract the information about the transport coefficients by considering the sonic velocity seriously.

Keywords: gravity/fluid correspondence, boundary condition, bulk viscosity, sonic velocity

PACS: 04.70.Dy, 11.25.Tq, 04.65.+e   **DOI:** 10.1088/1674-1137/43/1/013107

1   Introduction

The AdS/CFT correspondence [1–4] is a significant progress in theoretical physics. This correspondence provides new insights and useful tools to investigate the strongly related field theory by using the weakly correlated gravity theory [5–10].

At long wave limit the AdS/CFT correspondence reduces to gravity/fluid correspondence [11]. In the gravity/fluid correspondence, the dual field theory usually resides on the infinite boundary (conformal boundary or UV boundary), and has conformal dynamics [11–21]. In fact, the gravity/fluid correspondence can be gener-

alized to study nonconformal dual systems. A simple way of achieving this is to break the conformal symmetry by introducing a finite cutoff on the radial coordinate in the bulk, which has implied a deep relation between the Navier–Stokes (NS) equations and the Einstein equations [22–29]. In addition, from the renormalization group (RG) viewpoint, the radial direction of the bulk space-time corresponds to the energy scale of the dual field theory [29–37]. Thus, investigations of the holographic fluids at a finite cutoff surface were started [24–26, 29, 38–43]. The holographic fluid on the cutoff surface is usually nonconformal [38–43].

In this study, we focus on the stress tensor of noncon-

formal fluids, whose transport coefficients are obtained via holography. It is found that the sonic velocity of the holographic fluids at the finite cutoff surface is critical in clarifying the superficial similarity between bulk viscosity and perturbation of the pressure for holographic fluids and further simplifies the first order stress tensor of holographic fluid at the finite cutoff surface. Under more general boundary conditions at the finite cutoff surface, we also investigate applications of sonic velocity of non-conformal holographic fluids in detail.

This article is organized as follows. In Sec. II, we focus on the first order perturbative solution of the Schwarzschild-AdS black brane solution. Since this part is simple and fundamental, which can be seen in the previous reports , here we just give a brief review as a warm-up to make the whole paper more readable. In Sec. III, several boundary conditions are carefully analyzed; it is classified into two cases under the choice of boundary condition $h(r_c)$. Besides the general expressions of perturbations of pressure and energy density in the holographic fluid are expressed, the superficial similarity has been also explicitly seen between the bulk viscosity and perturbation of pressure. A crucial outcome is a method proposed to distinguish this superficial similarity through studies of sonic velocity in the holographic fluid. Moreover, we further take the sonic velocity into account to investigate a case with more general boundary condition, which is $h(r_c) \neq 0$. Sec. IV is devoted to the conclusion and discussion. Note that, Latin index repeated is usually represented to take the summation in our whole paper. For example, $\partial_i \beta_i = \Sigma_i \partial_i \beta_i$. However, if it is not so, it will be pointed out in the manuscript.

2 Warm-up: The first order perturbative solution of the Schwarzschild-AdS black brane solution

We make a concise review of the Schwarzschild-AdS black brane. The action of five-dimensional Einstein gravity with a negative cosmological constant $\Lambda = -6/\ell^2$ is as follows:

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^5 x \sqrt{-g^{(5)}} (R - 2\Lambda). \quad (1)$$

The corresponding field equation is given below.

$$R_{AB} - \frac{1}{2} R g_{AB} + \Lambda g_{AB} = 0 . \quad (2)$$

Here, the AdS radius $\ell = 1$ and $16\pi G = 1$ have been set for later convenience. The Schwarzschild-AdS black brane solution is

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2 f(r)} + r^2 \left(\sum_{i=1}^{3} \mathrm{d}x_i^2 \right) - r^2 f(r) \mathrm{d}t^2, \quad (3)$$

Where,

$$f(r) = 1 - \frac{2M}{r^4}, \quad (4)$$

The Hawking temperature of the Schwarzschild-AdS black brane solution is given by the following equation.

$$T_+ = \left. \frac{(r^2 f(r))'}{4\pi} \right|_{r=r_+} = \frac{r_+}{\pi} , \quad (5)$$

where r_+ is the location of horizon and positive root of $f(r) = 0$.

In the Eddington–Finkelstin coordinates, the black brane solution takes the following form.

$$\mathrm{d}s^2 = -r^2 f(r) \mathrm{d}v^2 + 2 \mathrm{d}v \mathrm{d}r + r^2 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \quad (6)$$

Where, $v = t + r_*$ and r_* is the tortoise coordinate satisfying $\mathrm{d}r_* = \mathrm{d}r/(r^2 f)$. Note that, the holographic fluid is investigated to reside at some cutoff hypersurface with constant radial coordinate $r = r_c$ (r_c is a constant). It is helpful to make the following coordinate - transformation: $v \to v/\sqrt{r_c^2 f(r_c)}$ and $x_i \to x_i/r_c$ in the solution (6) , which makes the induced metric on the cutoff surface to be explicitly flat metric, i.e. the cutoff surface with metric $\mathrm{d}s^2 = -\mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. The Hawking temperature is expressed as $T = T_+/\sqrt{r_c^2 f(r_c)}$ with respect to the killing observer $(\partial/\partial v)^a$ in the new coordinate system; the Schwarzschild-AdS black brane solution then takes the following form.

$$\mathrm{d}s^2 = -\frac{r^2 f(r)}{r_c^2 f(r_c)} \mathrm{d}v^2 + \frac{2}{r_c \sqrt{f(r_c)}} \mathrm{d}v \mathrm{d}r + \frac{r^2}{r_c^2} (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \quad (7)$$

while the entropy density is $s = \frac{r_+^3}{4 G r_c^3}$. The boosted Schwarzschild-AdS black brane solution is

$$\mathrm{d}s^2 = -\frac{r^2 f(r)}{r_c^2 f(r_c)} (u_\mu \mathrm{d}x^\mu)^2 - \frac{2}{r_c \sqrt{f(r_c)}} u_\mu \mathrm{d}x^\mu \mathrm{d}r$$
$$+ \frac{r^2}{r_c^2} P_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \quad (8)$$

with

$$u^v = \frac{1}{\sqrt{1 - \beta_i^2}}, \quad u^i = \frac{\beta_i}{\sqrt{1 - \beta_i^2}}, \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu , \quad (9)$$

Where, $x^\mu = (v, x_i)$ represents the boundary coordinates at the cutoff surface, $P_{\mu\nu}$ is the projector onto spatial directions, velocities β^i are constants, and the boundary indices (μ, ν) are raised and lowered by using the Minkowski metric $\eta_{\mu\nu}$, while the bulk indices are distinguished by (A, B).

We define a useful tensor

$$W_{AB} = R_{AB} + 4 g_{AB}, \quad (10)$$

while solutions of equation motions are equivalent to $W_{AB} = 0$. Viewed from the gravity/fluid correspondence

scenario, one needs perturb the gravitational solutions in the bulk spacetime to obtain transport coefficients of holographic fluids like shear viscosity η. The general procedure is to promote the constant parameters β^i and M in (8) to functions of boundary coordinates x^μ, i.e. $\beta^i(x^\mu)$ and $M(x^\mu)$ [11, 12]. Therefore, (8) will be no longer the solution of the field equation represented by (2) since the parameters now depend on the boundary coordinates and hence extra correction terms need to be added to make (8) a self-consistent solution.

For the extra correction terms, we can just focus on the extra correction terms around the origin $x^\mu = 0$; the first order extra correction terms around $x^\mu = 0$ are [11] as given below.

$$
\begin{aligned}
\mathrm{d}s^2_{(1)} =\ & \frac{k(r)}{r^2}\mathrm{d}v^2 + 2\frac{h(r)}{r_c\sqrt{f(r_c)}}\mathrm{d}v\mathrm{d}r + 2\frac{j_i(r)}{r^2}\mathrm{d}v\mathrm{d}x^i \\
& + \frac{r^2}{r_c^2}\left(\alpha_{ij}(r) - \frac{2}{3}h(r)\delta_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j,
\end{aligned} \tag{11}
$$

where, an appropriate gauge has been chosen, i.e. the background field gauge in a previous report [11] (G_{AB} represents the full metric).

$$
G_{rr}=0,\ \ G_{r\mu}\propto u_\mu,\ \ Tr((G^{(0)})^{-1}G^{(1)})=0, \tag{12}
$$

where $G^{(0)}$, $G^{(1)}$ are the corresponding zero order and first order terms in G_{AB}; $\alpha_{ij}(r)$ is in fact traceless for this background field gauge since $Tr((G^{(0)})^{-1}G^{(1)}) = \sum_i \alpha_{ii}$. Note that, parameters around $x^\mu = 0$ expanded to the first order are

$$
\beta_i(x^\mu) = \ \partial_\mu\beta_i|_{x^\mu=0}x^\mu,\ M(x^\mu)=M(0)+\partial_\mu M|_{x^\mu=0}x^\mu, \tag{13}
$$

Where, $\beta_i(0) = 0$ are assumed at the origin $x^\mu = 0$. Thus, after inserting the metric (8) with nonconstant parameters and (13) into W_{AB} , the nonzero $-W_{AB}$ is usually considered as the first order source term $S^{(1)}_{AB}$, while the first order perturbation solution around $x^\mu = 0$ can be obtained from the vanishing $W_{AB} =$ (effect from correction)$-S^{(1)}_{AB}$, which are casted into the Appendix A.

Still, there are two constraint equations

$$
W_{vv}+\frac{r^2f(r)}{r_c\sqrt{f(r_c)}}W_{vr}=0 \Rightarrow S^{(1)}_{vv}+\frac{r^2f(r)}{r_c\sqrt{f(r_c)}}S^{(1)}_{vr}=0,
$$

$$
W_{vi}+\frac{r^2f(r)}{r_c\sqrt{f(r_c)}}W_{ri}=0 \Rightarrow S^{(1)}_{vi}+\frac{r^2f(r)}{r_c\sqrt{f(r_c)}}S^{(1)}_{ri}=0. \tag{14}
$$

From the Appendix A, one rewrites these constrain equations (14) as

$$
3\partial_v M+4M\partial_i\beta_i=0,
$$

$$
\partial_i M+4M\partial_v\beta_i=\frac{-4M\partial_i M}{r_c^4 f(r_c)}, \tag{15}
$$

which are nothing but the conservation equations of the zeroth order stress-energy tensor [11, 12, 18, 19]. Fur-

ther, one analytically obtains

$$
h(r) = C_{h2}+\frac{C_{h1}}{r^4},
$$

$$
k(r) = C_{k2}-\frac{2C_{h2}r^4}{r_c^2 f(r_c)}+\frac{4C_{h1}M}{3r^4 r_c^2 f(r_c)}+\frac{2r^3\partial_i\beta_i}{3r_c\sqrt{f(r_c)}},
$$

$$
\begin{aligned}
j_i(r) &= \frac{r^3}{r_c^5 f(r_c)^{\frac{3}{2}}}(\partial_i M+r_c^4 f(r_c)\partial_v\beta_i)+\frac{C_{i1}r^4}{4}+C_{i2} \\
&= \frac{r^3 r_c^3\sqrt{f(r_c)}}{2M+r_c^4}\partial_v\beta_i+\frac{C_{i1}r^4}{4}+C_{i2},
\end{aligned}
$$

$$
\alpha_{ij}(r) = \alpha(r)\left\{(\partial_i\beta_j+\partial_j\beta_i)-\frac{2}{3}\delta_{ij}\partial_k\beta^k\right\}, \tag{16}
$$

Where, $\alpha(r)$ is $\alpha(r) = r_c\sqrt{f(r_c)}\int_{r_c}^r \frac{s^3-r_+^3}{-s^5 f(s)}\mathrm{d}s$, and $C_{h1}, C_{h2}, C_{k2}, C_{i1}$, and C_{i2} are nine constants of integration.

3   The stress tensor of first order holographic fluid under different boundary conditions at the finite cutoff surface

Note that the previous studies usually investigated the holographic fluid just residing at the UV boundary or infinite cutoff surface (i.e., r_c to infinity) [11, 12]. Here, we will try to use the gravity/fluid correspondence to shed some insights on the holographic fluid at the finite cutoff surface, which can be considered as a simple generalization of the previous works. However, it should be emphasized that this generalization is nontrivial as the stress tensor of the holographic fluid at the finite cutoff surface is usually nonconformal and depends on the choice of boundary conditions. All these points can be seen more clearly in the following content.

According to the gravity/fluid correspondence, the stress tensor $T_{\mu\nu}$ of holographic fluid residing at the cutoff surface with the induced metric $\gamma_{\mu\nu}$ is given by [22, 29, 44–48]

$$
T_{\mu\nu}=2(K_{\mu\nu}-K\gamma_{\mu\nu}-C\gamma_{\mu\nu}), \tag{17}
$$

where, $\gamma_{\mu\nu}$ is the boundary metric obtained from the usual ADM decomposition

$$
\mathrm{d}s^2=\gamma_{\mu\nu}(\mathrm{d}x^\mu+V^\mu \mathrm{d}r)(\mathrm{d}x^\nu+V^\nu \mathrm{d}r)+N^2\mathrm{d}r^2, \tag{18}
$$

the extrinsic curvature is $K_{\mu\nu}=-\frac{1}{2}(\nabla_\mu n_\nu+\nabla_\nu n_\mu)$, and n^μ is the unit normal vector of the constant hypersurface $r=r_c$ pointing toward the increasing r direction. In addition, the term $C\gamma_{\mu\nu}$ is usually related to the boundary counterterm added to cancel the divergence of the stress tensor $T_{\mu\nu}$ when the boundary $r=r_c$ approaches infinity, for example, $C=3$ in the asymptotical AdS$_5$ case. However, there is no divergence of the stress tensor in our case with finite boundary. In the following, we still add the boundary counterterm with $C=3$ in the stress ten-

sor simply because we require our result to reduce to the previous result when r_c goes to infinity [11, 12, 42, 43]. Therefore, after obtaining the first order perturbative so-

lution in the bulk, we can obtain the general formula of stress tensor $T_{\mu\nu}$ of the holographic fluid at the cutoff surface, i.e. around the origin $x^\mu=0$

$$
\begin{aligned}
T_{vv}^{(0)} &= 2\left(C-3\sqrt{f(r_c)}\right), \\
T_{xx}^{(0)} &= T_{yy}^{(0)}=T_{zz}^{(0)}=\frac{-4M+2\left(3-C\sqrt{f(r_c)}\right)r_c^4}{\sqrt{f(r_c)}r_c^4}, \\
T_{vv}^{(1)} &= -2\partial_i\beta_i+6\sqrt{f(r_c)}h(r_c)+\frac{\left(-2C+9\sqrt{f(r_c)}\right)k(r_c)}{r_c^2}+2\sqrt{f(r_c)}r_ch'(r_c), \\
T_{vi}^{(1)} &= \frac{\partial_iM}{f(r_c)r_c^4}-\partial_v\beta_i+2\frac{\left(2-C\sqrt{f(r_c)}+3f(r_c)\right)j_i(r_c)}{\sqrt{f(r_c)}r_c^2}-\frac{\sqrt{f(r_c)}j_i'(r_c)}{r_c},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
T_{ij}^{(1)} &= 2(\delta_{ij}\partial_k\beta_k-\partial_{(i}\beta_{j)})+2\delta_{ij}\frac{\partial_vM}{f(r_c)r_c^4}+2\left(-C+\frac{-2M+3r_c^4}{\sqrt{f(r_c)}r_c^4}\right)a_{ij}(r_c)-\sqrt{f(r_c)}r_ca_{ij}'(r_c) \\
&\quad +2\delta_{ij}\left(\left(\frac{2C}{3}+\frac{5(2M-3r_c^4)}{3\sqrt{f(r_c)}r_c^4}\right)h(r_c)-\frac{2}{3}\sqrt{f(r_c)}r_ch'(r_c)\right) \\
&\quad +2\delta_{ij}\left(\frac{(-2M+(3-2f(r_c))r_c^4)k(r_c)}{2\sqrt{f(r_c)}r_c^6}-\frac{\sqrt{f(r_c)}k'(r_c)}{2r_c}\right).
\end{aligned}
\tag{20}
$$

Obviously, the further explicit results of the first order stress tensor depend on several conditions and hence extract the information of transport coefficients. In the following, we will carefully investigate the boundary conditions; particularly, the boundary condition related to $h(r_c)$ will be investigated as the cases under this boundary condition are complicated. Moreover, this boundary condition can be relaxed to arbitrary at the finite cutoff surface, which has not been investigated before.

3.1 Boundary condition with $h(r_c)=0$

It is clear that one can fix the nine parameters $C_{h1}, C_{h2}, C_{k2}, C_{i1}$, and C_{i2} in (16) to extract the exact transport coefficients of first order holographic fluid at the finite cutoff surface in (20). Therefore, several conditions can be assumed. In fact, the Dirichlet boundary condition is usually chosen in (11) like [24, 38, 42, 43]

$$
h(r_c)=0, \quad k(r_c)=0, \quad j_i(r_c)=0. \tag{21}
$$

In addition, the following condition can be chosen.

$$
T_{vi}^{(1)}=0, \tag{22}
$$

since $T_{vi}^{(1)}=0$ is a gauge choice usually considered in the Landau frame, i.e. $T_{vv}^{(1)}=T_{vi}^{(1)}=0$ which corresponds that the velocity u^μ is identified as the 4-velocity of energy of the relativistic fluid or a (normalized) time-like eigenvector of $T_{\mu\nu}$. Therefore, one final condition is needed to fix the nine parameters. Note that, obviously, the final condition can be chosen as $T_{vv}^{(1)}=0$, which is just the Landau frame case with (22), and the corresponding re-

sults have been explicitly obtained in the Appendix B. However, from (20), we find that $T_{vv}^{(1)}=0$ under (21) just corresponds to a special boundary condition related to $h'(r_c)$, while $T_{vv}^{(1)}$ will be nonzero for many other boundary condition cases, i.e. $h'(r_c)=0$. Therefore, it will be interesting to investigate another special boundary condition case, i.e., $h(r_c)=0$ and $h'(r_c)$ is kept as an arbitrary constant. Moreover, one will find that this special boundary condition will also be critical to explicitly see the superficial similarity between the bulk viscosity and perturbation of pressure in the stress tensor of the holographic fluid, while $T_{vv}^{(1)}=0$ case is a little more difficult to note this superficial similarity. Therefore, in the following, we will just focus on carefully investigating the stress tensor of holographic fluid under this case of special boundary conditions.

From (16), it is easy to find that keeping $h'(r_c)$ as an arbitrary constant is equivalent to keeping the parameter C_{h1} as an arbitrary constant. Therefore, the other eight parameters C_{h2}, C_{k2}, C_{i1}, and C_{i2} can be solved from (21) and (22), which are all expressed in C_{h1}

$$
\begin{aligned}
C_{h2}&=-\frac{C_{h1}}{r_c^4}, \quad C_{k2}=-\frac{2\partial_i\beta_ir_c^2}{3\sqrt{f(r_c)}}-\frac{2C_{h1}(2M+3r_c^4)}{3r_c^6f(r_c)}, \\
C_{i1}&=-\frac{4r_c^2\partial_v\beta_i}{\sqrt{f(r_c)}(2M+r_c^4)}, \quad C_{i2}=\frac{2Mr_c^2\partial_v\beta_i}{\sqrt{f(r_c)}(2M+r_c^4)}.
\end{aligned}
\tag{23}
$$

After inserting (23) into (20), the non-zero components

of stress tensor $T_{\mu\nu}^{(1)}$ are

$$T_{vv}^{(1)}=-2\partial_i\beta_i+2r_c\sqrt{f(r_c)}h'(r_c)=-2\partial_i\beta_i-\frac{8\sqrt{f(r_c)}C_{h1}}{r_c^4},$$

$$T_{ij}^{(1)}=\frac{-2r_+^3\sigma_{ij}}{r_c^3}+\delta_{ij}\Big(\frac{-2(2M+r_c^4)}{3(-2M+r_c^4)}\partial_k\beta_k-\frac{8(2M+r_c^4)C_{h1}}{3r_c^8\sqrt{f(r_c)}}\Big). \tag{24}$$

Note that if the fluid is not considered under the Landau frame, usually the stress tensor of holographic fluid at the cutoff surface with the induced metric $\gamma_{\mu\nu}=\eta_{\mu\nu}$ can be written in the following general form [41].

$$T_{\mu\nu}=\rho u_\mu u_\nu+pP_{\mu\nu}-2\eta\sigma_{\mu\nu}-\zeta\theta P_{\mu\nu}-\zeta'\theta u_\mu u_\nu-\kappa a_{(\mu}u_{\nu)}, \tag{25}$$

where

$$P_{\mu\nu}=\eta_{\mu\nu}+u_\mu u_\nu,$$
$$\sigma^{\mu\nu}\equiv\frac{1}{2}P^{\mu\alpha}P^{\nu\beta}(\nabla_\alpha u_\beta+\nabla_\beta u_\alpha)-\frac{1}{3}P^{\mu\nu}\nabla_\alpha u^\alpha,$$
$$\theta=\nabla_\mu u^\mu,\qquad a^\nu=u^\mu\nabla_\mu u^\nu, \tag{26}$$

and ζ' is a shift of the local energy density by the expansion of the fluid, while κ is the heat conductivity. In our case, if we still consider the fluid with the velocity in (9), the above form of stress tensor can be represented as given below.

$$T_{\mu\nu}=\rho u_\mu u_\nu+pP_{\mu\nu}-2\eta\sigma_{\mu\nu}-\zeta\theta P_{\mu\nu}, \tag{27}$$

where $a^v=0$, $a^i=\partial_v\beta_i$ around $x^\mu=0$ has been used in our case and the $T_{vi}^{(1)}\neq0$ can be cancelled by the gauge choice in (22); in addition, it should be pointed out that here ρ and p can contain the first order terms with respect to the derivative of velocity although the stress tensor form looks like the form under the Landau frame.

A comparison between the results of (24) and (27) makes it easy to identify the energy density ρ and shear viscosity η. However, a superficial similarity between the pressure p and bulk viscosity ζ is explicitly seen in this case. Note that, from (19), the zero order pressure and energy density of dual fluid are $p_0=\frac{-4M+2\left(3-3\sqrt{f(r_c)}\right)r_c^4}{r_c^4\sqrt{f(r_c)}}$, $\rho_0=2\left(3-3\sqrt{f(r_c)}\right)$, and hence the entropy density s of dual fluid can be computed through the following equation

$$s=\frac{\partial p_0}{\partial T}=4\pi\frac{r_+^3}{r_c^3}, \tag{28}$$

which is consistent with the entropy density of the black brane solution (7) with $16\pi G=1$ recovered, and it is convenient to check this equation if we express p_0 and T in the functions of r_+. Furthermore, it can be easily checked that the familiar thermodynamic relation still holds on the arbitrary cutoff surface for the zero order

pressure and energy density.

$$\rho_0+p_0-Ts=0, \tag{29}$$

where T is the temperature of the dual fluid related to the Hawking temperature of the black brane solution by $T=T_+/\sqrt{r_c^2 f(r_c)}$. Therefore, the precise underlying superficial similarity, is in fact, between the perturbation of pressure p and the bulk viscosity ζ, i.e., the term proportional to $\partial_k\beta_k$ in $T_{ij}^{(1)}$ in (24) belongs to the perturbation of pressure or the bulk viscosity. For example, there can be two simple different choices, the first choice is

$$\rho = 2\left(3-3\sqrt{f(r_c)}\right)-2\theta-\frac{8\sqrt{f(r_c)}C_{h1}}{r_c^4},\quad \eta=\frac{r_+^3}{r_c^3},$$
$$p = \frac{-4M+2\left(3-3\sqrt{f(r_c)}\right)r_c^4}{r_c^4\sqrt{f(r_c)}}-\frac{8(2M+r_c^4)C_{h1}}{3r_c^8\sqrt{f(r_c)}},$$
$$\zeta = \frac{2(2M+r_c^4)}{3(-2M+r_c^4)}, \tag{30}$$

while the other is

$$\rho = 2\left(3-3\sqrt{f(r_c)}\right)-2\theta-\frac{8\sqrt{f(r_c)}C_{h1}}{r_c^4},\quad \eta=\frac{r_+^3}{r_c^3},$$
$$p = \frac{-4M+2\left(3-3\sqrt{f(r_c)}\right)r_c^4}{r_c^4\sqrt{f(r_c)}}-\frac{8(2M+r_c^4)C_{h1}}{3r_c^8\sqrt{f(r_c)}}$$
$$-\frac{2(2M+r_c^4)}{3(-2M+r_c^4)}\theta,\quad \zeta=0. \tag{31}$$

However, (30) and (31) cannot satisfy the thermodynamic relation between energy density and pressure at the same time. In addition, the bulk viscosity should be only one number in the same boundary condition case. Moreover, the bulk viscosity can increase the total entropy of fluid and hence, it is different from the other pressure term although sometimes it is also considered as the effective pressure. Therefore, we should use an underlying method to extract the physical information of the holographic fluids. In fact, after a careful consideration, we will find that there are two subtleties in the first choice or consideration (30). First, the $T_{vv}^{(1)}=0$ case as a special case contained in (24) has been explicitly shown in the Appendix B, and the bulk viscosity is zero, which will not be consistent with the results in the first choice with a nonzero bulk viscosity in (30). Second, the C_{h1} term in (30) can be also considered as the bulk viscosity term, particularly when it is also proportional to $\partial_k\beta_k$ in some boundary condition case and hence, there is an underlying ambiguity for the choice of bulk viscosity related to the term C_{h1} in (30). Therefore, for further obtaining the true transport coefficients particular the bulk viscosity, one needs find out a method.

In the following, we will propose a method by checking the underlying consistency in (30) or (31) with the thermodynamic relation between energy density and

pressure, i.e. through the studies of sonic velocity c_s between the perturbations of energy density and pressure. As we know, the first order term in ρ can also be considered as the perturbation of energy density $\delta\rho$, while this perturbation of energy density usually deduces the perturbation of pressure of fluid δp. In our case, using the above explicit expressions of zero order pressure p_0 and energy density ρ_0 of holographic fluid, we can easily further obtain $p_0 = -\frac{\rho_0(6+\rho_0)}{3(-6+\rho_0)}$. Therefore, the perturbations of energy density and pressure should satisfy the underlying thermodynamic relation through the sonic velocity c_s, i.e. $\delta p = c_s^2\delta\rho$, while the square of sonic velocity can be easily obtained from the following expression.

$$c_s^2 = \left(\frac{\partial p_0}{\partial \rho_0}\right)_s = -\frac{\rho_0^2 - 12\rho_0 - 36}{3(\rho_0 - 6)^2} = \frac{(2M + r_c^4)}{3(-2M + r_c^4)}, \quad (32)$$

where the zero order energy density ρ_0 and pressure p_0 have been used and the derivative is usually taken for an adiabatic process, i.e. the constant entropy density $s = \frac{r_+^3}{4Gr_c^3}$ situation. In our case, we check that the perturbations of energy density $\delta\rho$ and pressure δp should be

$$\delta\rho = -2\theta - \frac{8\sqrt{f(r_c)}C_{h1}}{r_c^4},$$
$$\delta p = -\frac{8(2M + r_c^4)C_{h1}}{3r_c^8\sqrt{f(r_c)}} - \frac{2(2M + r_c^4)}{3(-2M + r_c^4)}\theta. \quad (33)$$

Therefore, it is obvious and interesting to find that the second choice (31) will be the right choice as it satisfies the underlying thermodynamic relation between the perturbations of energy density and pressure through the sonic velocity, i.e. $\delta p = \frac{(2M + r_c^4)}{3(-2M + r_c^4)}\delta\rho = c_s^2\delta\rho$. In addition, this choice is also consistent with the $T_{vv}^{(1)} = 0$ case with zero bulk viscosity in the Appendix B. Note that our proposal of taking the sonic velocity into account also implicates that the true bulk viscosity ζ_T should not be

ζ but $\zeta_T = \zeta - \zeta'(\frac{\partial p}{\partial\rho}) = \zeta - c_s^2\zeta'$ in (25), which is consistent with the discussion in [49], where a frame invariant scalar related to the bulk viscosity has been defined in (2.10) and later explicitly obtained in (2.24).

3.2   Boundary with $h(r_c) \neq 0$

In the above subsection, we have proposed a method to clarify the superficial similarity between the bulk viscosity and perturbation of the pressure. Note that while using the Dirichlet boundary condition (21), the main underlying simple reason is to keep a well-defined boosted transformation at the finite cutoff surface, $r = r_c$, i.e. $\gamma_{\mu\nu} = \eta_{\mu\nu}$. However, after a careful observation at the corrected metric (11), we find that the condition $h(r_c) = 0$ in (21) can be relaxed as $h(r_c) \neq 0$, which also keeps a well-defined boosted transformation at the finite cutoff surface $r = r_c$. The cost is that the traceless condition in (12) $Tr((G^{(0)})^{-1}G^{(1)}) = 0$ has been broken as

$$Tr((G^{(0)})^{-1}G^{(1)}) = 2h(r_c), \quad (34)$$

where we have used the deduced condition $\alpha_{xx}(r_c) = \alpha_{yy}(r_c) = \alpha_{zz}(r_c) = \frac{2}{3}h(r_c)$ from the order $\gamma_{\mu\nu} = \eta_{\mu\nu}$. In addition, for the corrected metric in (11) with a non-traceless $\alpha_{ij}(r)$, i.e. $\sum_i \alpha_{ii}(r) \neq 0$, the new components of tensor $W_{AB} =$ (effect from correction) $- S_{AB}$ become more complicated, which have also been expressed in Appendix C.

However, from these new components W_{AB}, we find that the solutions $h(r)$, $k(r)$, and $j_i(r)$ are the same as those from (16), while $\alpha_{ij}(r)$ can be instead as

$$\alpha_{ij}(r) = \alpha(r)\left\{(\partial_i\beta_j + \partial_j\beta_i) - \frac{2}{3}\delta_{ij}\partial_k\beta^k\right\} + b\delta_{ij}, \quad (35)$$

where b is a constant. In addition, the first order of stress tensors in (20) also have been changed and become more complicate

$$T_{vv}^{(1)} = -2\partial_i\beta_i + 6\sqrt{f(r_c)}h(r_c) + \frac{\left(-2C + 9\sqrt{f(r_c)}\right)k(r_c)}{r_c^2} + 2\sqrt{f(r_c)}r_c h'(r_c) - r_c\sqrt{f(r_c)}B(r_c),$$

$$T_{vi}^{(1)} = \frac{\partial_i M}{f(r_c)r_c^4} - \partial_v\beta_i + 2\frac{\left(2 - C\sqrt{f(r_c)} + 3f(r_c)\right)j_i(r_c)}{\sqrt{f(r_c)}r_c^2} - \frac{\sqrt{f(r_c)}j_i'(r_c)}{r_c},$$

$$T_{ij}^{(1)} = 2(\delta_{ij}\partial_k\beta_k - \partial_{(i}\beta_{j)}) + 2\delta_{ij}\frac{\partial_v M}{f(r_c)r_c^4} + 2\left(-C + \frac{-2M + 3r_c^4}{\sqrt{f(r_c)}r_c^4}\right)a_{ij}(r_c) - \sqrt{f(r_c)}r_c a_{ij}'(r_c)$$

$$+ r_c\sqrt{f(r_c)}B(r_c)\delta_{ij} + 2\delta_{ij}\left(\left(\frac{2C}{3} + \frac{5(2M - 3r_c^4)}{3\sqrt{f(r_c)}r_c^4}\right)h(r_c) - \frac{2}{3}\sqrt{f(r_c)}r_c h'(r_c)\right)$$

$$+ 2\delta_{ij}\left(\frac{(-2M + (3 - 2f(r_c))r_c^4)k(r_c)}{2\sqrt{f(r_c)}r_c^6} - \frac{\sqrt{f(r_c)}k'(r_c)}{2r_c}\right). \quad (36)$$

where $B(r) = \sum_i \alpha'_{ii}(r)$. Therefore, from the Dirichlet boundary condition, $k(r_c) = 0$, $j_i(r_c) = 0$ and $T_{vx}^{(1)} = 0$ in (21) and (22), we can obtain the parameters C_{k2}, C_{i1}, and C_{i2}

$$C_{k2} = \frac{2C_{h2}r_c^2}{f(r_c)} - \frac{4C_{h1}M}{3r_c^6 f(r_c)} - \frac{2r_c^2 \partial_i \beta_i}{3\sqrt{f(r_c)}},$$

$$C_{i1} = -\frac{4r_c^2 \partial_v \beta_i}{\sqrt{f(r_c)}(2M + r_c^4)}, \quad C_{i2} = \frac{2Mr_c^2 \partial_v \beta_i}{\sqrt{f(r_c)}(2M + r_c^4)},$$

(37)

where C_{h1} and C_{h2} are arbitrary parameters related to the unfixed $h(r_c)$ and $B(r)$ will be found to be zero in this case. Substituting (37) into (36), i.e. the nonzero first order stress tensor of holographic fluid at finite cutoff surface, one can obtain

$$T_{vv}^{(1)} = -2\partial_i \beta_i + 6\sqrt{f(r_c)}C_{h2} - \frac{2\sqrt{f(r_c)}C_{h1}}{r_c^4},$$

$$T_{ij}^{(1)} = \frac{-2r_+^3 \sigma_{ij}}{r_c^3} + \delta_{ij}\left(\frac{-2(2M + r_c^4)}{3(-2M + r_c^4)}\partial_k \beta_k\right.$$
$$- \frac{2f(r_c) + 12(1 - \sqrt{f(r_c)})}{3r_c^4 \sqrt{f(r_c)}}C_{h1}$$
$$+ \frac{4(1 + 3\sqrt{f(r_c)}) - 10f(r_c)}{3\sqrt{f(r_c)}}C_{h2}$$
$$\left. - 2b\left(3 + \frac{2M - 3r_c^4}{r_c^4 \sqrt{f(r_c)}}\right)\right).$$

(38)

Note that after making some tedious calculations, one can finally obtain a simple result

$$T_{ij}^{(1)} = \frac{-2r_+^3 \sigma_{ij}}{r_c^3} + \delta_{ij}\left(\frac{-2(2M + r_c^4)}{3(-2M + r_c^4)}\partial_k \beta_k\right.$$
$$\left. - \frac{2(2M + r_c^4)}{3r_c^8 \sqrt{f(r_c)}}C_{h1} + \frac{2(2M + r_c^4)}{r_c^4 \sqrt{f(r_c)}}C_{h2}\right),$$
$$= \frac{-2r_+^3 \sigma_{ij}}{r_c^3} + \delta_{ij}(c_s^2 T_{vv}^{(1)}),$$

(39)

where the condition $b = \frac{2}{3}h(r_c)$ has been used to keep $\alpha_{xx}(r_c) = \alpha_{yy}(r_c) = \alpha_{zz}(r_c) = \frac{2}{3}h(r_c)$. From these results and taking the method into account, one will be surprised that precise transport coefficients can still be extracted although some parameters have not been fixed, i.e. C_{h1} and C_{h2}. The bulk viscosity is still zero in this more general boundary condition case with $h(r_c) \neq 0$.

4 Conclusion and discussion

In this study, after constructing the first order perturbative solution of the Schwarzschild-AdS black brane spacetime, we used the gravity/fluid correspondence to carefully investigate the stress tensor of first order holographic fluid at a finite cutoff surface by considering different boundary conditions. In general, some frame to discuss the fluid, such as Landau frame or Eckart frame in fluid mechanics is selected. However, recent studies show that the physical results may be different in different frames [50], especially in the studies of stability problem. Therefore, it seemed better to relax the constraints of the Landau frame, i.e. admitting the perturbation of energy density in our case. However, an important question is that how we can eliminate the ambiguity freedom in $T_{xx}^{(1)}$ if we relax the constraint. As the first key point, we obtained that this ambiguity freedom is related to the perturbation of the pressure and bulk viscosity terms in $T_{xx}^{(1)}$, which are very similar. Furthermore, we find a method by taking the sonic velocity in (32) into account to clarify this superficial similarity between bulk viscosity and perturbation of the pressure to obtain the physical transport coefficients. The second key point of our paper is that we have explicitly expressed this similarity between bulk viscosity and perturbation of the pressure terms in $T_{xx}^{(1)}$ by investigating another special boundary condition case related to the scalar mode $h(r)$ of metric perturbation, i.e. $h(r_c) = 0$; however, $h'(r_c)$ is arbitrary, which has not been investigated and seen before; we found that this condition, $h'(r_c) \neq 0$, is crucial to explicitly yield the perturbation of pressure and see the superficial similarity between pressure perturbation and bulk viscosity. However, by using this method, we can easily obtain the physical transport coefficients in this case. The third key point of our study is the investigation of a more general boundary condition case, i.e. $h(r_c)$ is not zero, which has not been reported yet. This case was more complicated than the cases considered before, since some results have been changed, i.e. the traceless condition $Tr((G^{(0)})^{-1}G^{(1)})$ has been broken and the formula of stress tensor in (36) becomes more complicated. Moreover, the two parameters C_{h1} and C_{h2} cannot be fixed now due to the nonzero $h(r_c)$. However, it is surprising that one can still extract exact information of transport coefficients from the complicated formula $T_{xx}^{(1)}$ by using the method, and we obtain that bulk viscosity is still zero in this more general boundary condition case.

Note that our results of sound velocity in holographic fluids via gravity/fluid correspondence are nontrivial. First, our results are the original ones among the references because almost all the previous studies via gravity/fluid correspondence are just considered under the Landau frame, i.e. $T_{vv}^{(1)} = 0$. It should be pointed out that the corresponding transport coefficients may also be finally obtained under the Landau frame if some boundary condition is lost just like the case with the boundary condition $h(r_c) \neq 0$ in the present study; however, the calculations will be more complicated. Moreover, under the Landau frame, the explicit coefficient c_s^2 in front of $T_{vv}^{(1)}$ shall not be obtained in the expression of $T_{xx}^{(1)}$ in (39). In fact, (39) is an important equation to extract

some underlying relationship between $T_{xx}^{(1)}$ and $T_{vv}^{(1)}$. Second, based on the result in the case of boundary condition $h(r_c) \neq 0$, it implicates that the sonic velocity further simplified the complicated expression of $T_{xx}^{(1)}$ in (20). Indeed, our subsequent work in [51] has shown this point. Using this simplification, we have further deduced an underlying universality in the expression of $T_{xx}^{(1)}$, which has shed some insights on the clue of obtaining the nonzero bulk viscosity for the holographic fluid at the finite cutoff surface. More details and some other discussions related to our work are:

(1) Usually the superficial similarity between bulk viscosity and perturbation of pressure is hardly observed and difficult to distinguish. We proposed an approach to extract the physical transport coefficients of the holographic fluids in this study. In addition, our proposal also implicates that the true bulk viscosity ζ_T should be not ζ but $\zeta_T = \zeta - \zeta'(\frac{\partial p}{\partial \rho})$ through (25) and is consistent with the discussion in [49], where a frame invariant scalar related to the bulk viscosity has been defined in (2.10) and explicitly obtained in (2.24).

(2) Our approach is useful to deduce the true bulk viscosity term in the scenario of gravity/fluid correspondence. Further studies of holographic fluid with other different boundary conditions at the finite cutoff surface are in process, where we simplify the $T_{xx}^{(1)}$ by taking the sonic velocity into account in [51]. Moreover, we have found some underlying universality in the $T_{xx}^{(1)}$ after taking the sonic velocity into account. In addition, note that here we chose these boundary conditions just simply from the mathematical point of view, i.e. these bound-

ary conditions are mathematically permitted. However, the underlying physical meaning of these boundary conditions is lost; therefore, it will be interesting and important to find out the underlying physical meaning of these different boundary conditions in the future work. In addition, there have been other methods and studies investigating the bulk viscosity [38–41, 52–56].It will be interesting to make the comparisons between these methods and the method based on gravity/fluid correspondence, which may give some insight into the underlying physical meaning of these different boundary conditions.

(3) All our discussions are considered in the so-called background gauge in (11). In fact, as discussed in [41], there is an ambiguity in the extra correction term $g^{(1)}$ in (11). This ambiguity can affect our choices of the boundary conditions, and hence, may affect the stress tensor with transport coefficients. Several studies showing that bulk viscosity can also appear in other gauge have been reported [40, 41]. In addition, there are gauge invariant quantities for the metric and energy momentum tensor under perturbation [57], and whether bulk viscosity depends on these gauge invariant quantities is still an open issue. Therefore, the underlying relations between gauge, boundary conditions, gauge invariant quantities, and stress tensors for holographic fluids with transport coefficients are of interest for further studies.

Appendix A: The tensor components of W_{AB} and S_{AB}

The tensor components of $W_{AB} =$ (effect from correction)$-S_{AB}$ are

$$W_{vv} = -\frac{8r^2 f(r) h(r)}{r_c^2 f(r_c)} - \frac{2(2M + r^4) f(r) h'(r)}{r r_c^2 f(r_c)}$$
$$+ \frac{f(r) k'(r)}{2r} - \frac{1}{2} f(r) k''(r) - S_{vv}^{(1)},$$

$$W_{vi} = \frac{3 f(r) j_i'(r)}{2r} - \frac{1}{2} f(r) j_i''(r) - S_{vi}^{(1)}(r), \quad \text{(A1)}$$

$$W_{vr} = \frac{8 h(r)}{r_c \sqrt{f(r_c)}} + \frac{2(2M + r^4) h'(r)}{r^3 r_c \sqrt{f(r_c)}} - \frac{r_c \sqrt{f(r_c)} k'(r)}{2r^3}$$
$$+ \frac{r_c \sqrt{f(r_c)} k''(r)}{2r^2} - S_{vr}^{(1)}, \quad \text{(A2)}$$

$$W_{ri} = -\frac{3 r_c \sqrt{f(r_c)} j_i'(r)}{2r^3} + \frac{r_c \sqrt{f(r_c)} j_i''(r)}{2r^2} - S_{ri}^{(1)}, \quad \text{(A3)}$$

$$W_{rr} = \frac{5 h'(r)}{r} + h''(r) - S_{rr}^{(1)}, \quad \text{(A4)}$$

$$W_{ii} = \frac{8r^2}{r_c^2} h(r) + \frac{(-14M + 11r^4) h'(r)}{3 r r_c^2} + \frac{1}{3 r_c^2} r^4 f(r) h''(r)$$
$$+ \frac{f(r_c) k'(r)}{r} + \frac{(2M - 5r^4) \alpha_{ii}'(r)}{2 r r_c^2}$$
$$- \frac{1}{2 r_c^2} r^4 f(r) \alpha_{ii}''(r) - S_{ii}^{(1)},$$

(here $ii = xx, yy, zz$ with no summation) (A5)

$$W_{ij} = \frac{(2M - 5r^4) \alpha_{ij}'(r)}{2 r r_c^2}$$
$$- \frac{1}{2 r_c^2} r^4 f(r) \alpha_{ij}''(r) - S_{ij}^{(1)}, \ (i \neq j), \quad \text{(A6)}$$

$$cW_{ij} - \frac{1}{3} \delta_{ij} \left(\sum_k W_{kk} \right) = \frac{(2M - 5r^4) \alpha_{ij}'(r)}{2 r r_c^2}$$
$$- \frac{1}{2 r_c^2} r^4 f(r) \alpha_{ij}''(r) - S_{ij}^{(1)} + \frac{1}{3} \delta_{ij} (\delta^{kl} S_{kl}^{(1)}), \quad \text{(A7)}$$

where the first order source terms are

$$S_{vv}^{(1)}(r) = -\frac{3\partial_v M}{r^3 r_c \sqrt{f(r_c)}} - \frac{(2M+r^4)\partial_i\beta_i}{r^3 r_c \sqrt{f(r_c)}}, \quad \text{(A8)}$$

$$S_{vi}^{(1)}(r) = \frac{(-2M+3r^4+2r_c^4)\partial_i M}{2r^3 r_c^5 f(r_c)^{3/2}} + \frac{(2M+3r^4)\partial_v\beta_i}{2r^3 r_c \sqrt{f(r_c)}}, \quad \text{(A9)}$$

$$S_{vr}^{(1)}(r) = \frac{\partial_i\beta_i}{r}, \quad \text{(A10)}$$

$$S_{ri}^{(1)}(r) = -\frac{3\partial_v\beta_i}{2r} - \frac{3\partial_i M}{2rr_c^4 f(r_c)}, \quad \text{(A11)}$$

$$S_{rr}^{(1)}(r) = 0, \quad \text{(A12)}$$

$$S_{ij}^{(1)}(r) = \left(\delta_{ij}\partial_k\beta_k + 3\partial_{(i}\beta_{j)}\right)\frac{r\sqrt{f(r_c)}}{r_c}. \quad \text{(A13)}$$

Appendix B: The case $T_{vv}^{(1)}=0$

In this case, the nine parameters can be fixed using $T_{vv}^{(1)}=0$, (21) and (22)

$$C_{h1} = -\frac{\partial_i\beta_i r_c^4}{4\sqrt{f(r_c)}}, \quad C_{h2} = \frac{\partial_i\beta_i}{4\sqrt{f(r_c)}},$$

$$C_{k2} = -\frac{\partial_i\beta_i\left(-10M+r_c^4\right)}{6f(r_c)^{3/2}r_c^2},$$

$$C_{i1} = -\frac{4r_c^2\partial_v\beta_i}{\sqrt{f(r_c)}(2M+r_c^4)}, \quad C_{i2} = \frac{2Mr_c^2\partial_v\beta_i}{\sqrt{f(r_c)}(2M+r_c^4)}. \quad \text{(B1)}$$

Consequently, the nonzero components of $T_{\mu\nu}^{(1)}$ are

$$T_{ij}^{(1)} = -2r_+^3\sigma_{ij}/r_c^3, \quad \sigma_{ij} = \partial_{(i}\beta_{j)} - \frac{1}{3}\delta_{ij}\partial_k\beta^k. \quad \text{(B2)}$$

From (25), one can simply read out

$$\rho = 6\left(1-\sqrt{f(r_c)}\right), \quad p = \frac{-4M+6\left(1-\sqrt{f(r_c)}\right)r_c^4}{r_c^4\sqrt{f(r_c)}},$$

$$\eta = r_+^3/r_c^3, \quad \zeta = 0. \quad \text{(B3)}$$

Thus, the dual fluid obtained at the finite cutoff surface is not conformal because the trace of $T_{\mu\nu}$ is nonzero, i.e. $\rho=3p$ has been broken. This result is consistent with that in Ref. [26], and expected from the fact that the conformal symmetry has been broken with a finite radial coordinate in the bulk. In addition, as $r_c\to\infty$, the results in (B3) can relate to those in the infinite boundary by just a conformal factor. Since the conformal symmetry is recovered in this case, these results can be related to each other by conformal transformation. Moreover, the entropy density from (7) is $s=\frac{r_+^3}{4Gr_c^3}$ and after substituting the coefficient $16\pi G$ in η, we can easily find that $\eta/s=1/(4\pi)$, which is consistent with the well-known η/s result for the dual fluid at the infinite boundary in the Einstein gravity [11, 12, 18, 19, 42].

Appendix C: New tensor components of W_{AB} and S_{AB}

For the corrected metric in (11) with a nontraceless $\alpha_{ij}(r)$ i.e., $\sum_i\alpha_{ii}(r)\neq0$, we can obtain the new tensor components of $W_{AB}=$(effect from correction)$-S_{AB}$ as

$$W_{vv} = -\frac{8r^2 f(r)h(r)}{r_c^2 f(r_c)} - \frac{2(2M+r^4)f(r)h'(r)}{rr_c^2 f(r_c)} + \frac{f(r)k'(r)}{2r}$$
$$-\frac{1}{2}f(r)k''(r) + \frac{(2M+r^4)r_c^2(2M-r^4)}{2r^5(2M-r_c^4)}B(r) - S_{vv}^{(1)},$$

$$W_{vi} = \frac{3f(r)j_i'(r)}{2r} - \frac{1}{2}f(r)j_i''(r) - S_{vi}^{(1)}(r),$$

$$W_{vr} = \frac{8h(r)}{r_c\sqrt{f(r_c)}} + \frac{2(2M+r^4)h'(r)}{r^3 r_c\sqrt{f(r_c)}} - \frac{r_c\sqrt{f(r_c)}k'(r)}{2r^3}$$
$$+\frac{r_c\sqrt{f(r_c)}k''(r)}{2r^2} - \frac{2M+r^4}{2r^3 r_c\sqrt{f(r_c)}}B(r) - S_{vr}^{(1)},$$

$$W_{ri} = -\frac{3r_c\sqrt{f(r_c)}j_i'(r)}{2r^3} + \frac{r_c\sqrt{f(r_c)}j_i''(r)}{2r^2} - S_{ri}^{(1)},$$

$$W_{rr} = \frac{5h'(r)}{r} + h''(r) - \frac{B(r)}{r} - \frac{B'(r)}{2} - S_{rr}^{(1)},$$

$$W_{ii} = \frac{8r^2}{r_c^2}h(r) + \frac{(-14M+11r^4)h'(r)}{3rr_c^2} + \frac{1}{3r_c^2}r^4 f(r)h''(r)$$
$$+\frac{f(r_c)k'(r)}{r} + \frac{(2M-5r^4)\alpha_{ii}'(r)}{2rr_c^2}$$
$$-\frac{1}{2r_c^2}r^4 f(r)\alpha_{ii}''(r) - \frac{r^3 f(r)}{2r_c^2}B(r) - S_{ii}^{(1)},$$
(here $ii=xx,yy,zz$ with no summation)

$$W_{ij} = \frac{(2M-5r^4)\alpha_{ij}'(r)}{2rr_c^2} - \frac{1}{2r_c^2}r^4 f(r)\alpha_{ij}''(r) - S_{ij}^{(1)}, \ (i\neq j),$$

$$W_{ij} - \frac{1}{3}\delta_{ij}\left(\sum_k W_{kk}\right) = \frac{(2M-5r^4)\left(\alpha_{ij}'(r) - \delta_{ij}\frac{1}{3}B(r)\right)}{2rr_c^2}$$
$$-\frac{1}{2r_c^2}r^4 f(r)\left(\alpha_{ij}'(r) + \delta_{ij}\frac{1}{3}B(r)\right)'$$
$$-S_{ij}^{(1)} + \frac{1}{3}\delta_{ij}(\delta^{kl}S_{kl}^{(1)}),$$

where $B(r)=\sum_i\alpha_{ii}'(r)$, and the first order source terms are the same as those in Appendix .

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