From geometry to non-geometry via T-duality *

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Abstract: Reconsideration of the T-duality of the open string allows us to introduce some geometric features in non-geometric theories. First, we have found what symmetry is T-dual to the local gauge transformations. It includes transformations of background fields but does not include transformations of the coordinates. According to this we have introduced a new, up to now missing term, with additional gauge field A_i^D (D denotes components with Dirichlet boundary conditions). It compensates non-fulfilment of the invariance under such transformations on the end-points of an open string, and the standard gauge field A_a^N (N denotes components with Neumann boundary conditions) compensates non-fulfilment of the gauge invariance. Using a generalized procedure we will perform T-duality of vector fields linear in coordinates. We show that gauge fields A_a^N and A_i^D are T-dual to ${}^\star A_D^a$ and ${}^\star A_N^i$ respectively. We introduce the field strength of T-dual non-geometric theories as derivatives of T-dual gauge fields along both T-dual variable y_μ and its double $\tilde{y}_\mu$. This definition allows us to obtain gauge transformation of non-geometric theories which leaves the T-dual field strength invariant. Therefore, we introduce some new features of non-geometric theories where field strength has both antisymmetric and symmetric parts. This allows us to define new kinds of truly non-geometric theories.

Keywords: T-duality, non-geometry, open string, neumann boundary conditions, dirichlet boundary conditions

PACS: 11.25.Tq, 11.25.Uv **DOI:** 10.1088/1674-1137/42/8/083106

1 Introduction

String theory has more symmetries than point particle theory. This is the source of an unusual situation, which is described by so-called non-geometry [1–3]. In fact, when going around a loop in space-time the field configuration is well defined only after applying some string symmetry (T-duality) as a transition function.

Geometric spaces appear when diffeomorphisms and gauge transformations have been used as transition functions to overlap coordinate patches. According to Ref. [3], there are two kinds of non-geometric backgrounds. In the first case (a benign form, which for a three-torus is usually referred to as a theory with Q-flux) the background is locally geometric but globally non-geometric. This is T-fold , when T-duality transformations can be used as transition functions [4]. In the second case (a severe form, which for a three-torus is usually referred to as a theory with R-flux) we lose the local geometric description of space-time points and the background is non-geometric even locally. This is a mysterious background, when T-duality is performed along some non-isometry directions.

In the great majority of papers, Abelian T-duality has been applied along the coordinates with global shift symmetry. A problem occurs when we try to perform T-duality along the coordinates on which background fields depend. Then we should apply the generalized Buscher's procedure, developed in Refs. [5, 6], where the metric and Kalb-Ramond fields are coordinate-dependent. In that case, the argument of T-dual background fields is not simply the T-dual variable y_a but it is the line integral of world-sheet gauge fields v_+^a and v_-^a. Explicitly, we have $V^a[v_+,v_-]\equiv\int_P \mathrm{d}\xi^\alpha v_\alpha^a=\int_P(\mathrm{d}\xi^+ v_+^a+\mathrm{d}\xi^- v_-^a)$, which on the solution for gauge fields turns to $V^a=-\kappa\theta^{ab}y_b+G_E^{-1ab}\tilde{y}_b$, where $\dot{\tilde{y}}_a=y_a'$ and $\tilde{y}_a'=\dot{y}_a$. We will claim that in such cases T-dual theories are locally non-geometric, although the initial theory is geometric. So far we have two reasons for that. Besides the fact that we perform T-dualization along non-isometry directions, we obtain a locally non-geometric background (the argument is the line integral). The additional two features: non-commutativity of the closed string coordinates and non-associativity, which have been shown in Refs. [7, 8].

Received 12 February 2018, Revised 24 May 2018, Published online 9 July 2018

∗ Supported by the Serbian Ministry of Education and Science (171031)

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In order to better understand non-commutativity, we prefer to use a canonical method. It can help us to reproduce nicely the main result of open string theory and non-commutative geometry, discovered in Ref. [9]. In fact, we can solve the Neumann boundary condition and obtain an effective theory with effective canonical variables q^i and p^i and effective background, named the open string background in Ref. [9]. Then, the initial coordinate x^i depends not only on effective coordinates but also on effective momenta, $x^i = q^i - 2\theta^{ij}\int^\sigma \mathrm{d}\sigma_1 p_j$. Since q^i and p_i, as independent variables, satisfy standard Poisson brackets, it is clear that initial coordinates non-commute. In Ref. [10] it is described how an antisymmetric field B_{ab}, regarded as a magnetic field on the Dp-brane, has an effect on non-commutativity.

In Refs. [11, 12] it has been shown that the commutators of closed string coordinates are proportional to the flux and winding number. For the relation between space-time symmetry and non-commutativity see Ref. [13] and references therein. We can expect that in the canonical formulation closed string coordinates, in order to be non-commutative, should have similar expressions to the open string ones. However, closed string coordinates do not have end points and so there are no boundary conditions and so no effective coordinates.

In Ref. [7] this problem has been solved by T-duality performed along non-isometry directions, so that T-dual background fields depend on V^μ. To understand the essence it is useful to perform the Buscher procedure on the Lagrangian in canonical form $S = \int \mathrm{d}^2\xi[\pi_\mu \dot{x}^\mu - \mathcal{H}(x,x',\pi)]$. The x-dependence comes from the arguments of background fields. Then the corresponding auxiliary action (after gauging shift symmetry, putting the corresponding field strength to zero and fixing the gauge, see Section 3.3.1) takes the form $S_{aux} = \int \mathrm{d}^2\xi[\pi_\mu v_0^\mu - \mathcal{H}(V,v_1,\pi) - \kappa(v_0^\mu y'_\mu - v_1^\mu \dot{y}_\mu)]$. Formally, we substitute $\dot{x}^\mu \to v_0^\mu$, $x^\mu \to V^\mu$ and add the last term. Now, we can consider two cases. First, when we work only with isometry directions then we have $\mathcal{H}(v_1,\pi)$ (the Hamiltonian $\mathcal{H}$ does not depend on x^μ and consequently on V^μ). Varying with respect to v_0^μ we obtain the relation $\kappa y'_\mu = \pi_\mu$. (After integration over σ it produces a well known relation between momenta and winding numbers). This relation cannot help us, because T-dual variables y_μ commute, as they depend only on initial momenta and not on initial coordinates. The second case, when we work with non-isometry directions, is more interesting. Then the argument of T-dual background fields V^μ is a non-local expression because it is a line integral of both v_0^μ and v_1^μ. So, variation with respect to v_0^μ produces a new term and we obtain $\kappa y'_\mu = \pi_\mu - \frac{\kappa}{3} B_{\mu\nu\rho} x'^\nu x^\rho$ (see relation (2.22) of Ref. [7]). Now, the T-dual variables y_μ depend on both initial variable x^μ and its canonically conjugated momentum π_μ, which provides non-commutativity between T-dual variables y_μ. The fact that this expression is quadratic in coordinates provides non-associativity between T-dual variables y_μ, see Ref.[8].

In the present article we are going to perform T-dualization for a background with constant metric and Kalb-Ramond fields but where the vector gauge fields are linear in coordinates. We will do this in two ways: in terms of vector fields and in terms of corresponding field strengths. In the first case, T-dual gauge fields will depend on the same expression V^μ, introduced above. So in this case, due to the presence of vector background fields, the T-dual theory will be non-local and hence locally non-geometric. On the other hand, this theory can be described as a theory with constant field strength, but with both antisymmetric and symmetric parts of field strength. So, in the second case, we can perform standard Buscher T-duality and obtain explicitly T-dual field strengths. The main contribution of the paper is the relation between T-dual gauge fields and T-dual field strengths. It is non-standard for two reasons. First, because we must use derivatives of vector fields with respect to two variables: the T-dual variable y_μ and its double $\tilde{y}_\mu$. Second, because the T-dual field strength contains both antisymmetric and symmetric parts. So, using T-duality we are able to introduce some geometry (the field strength in terms of gauge fields) for non-geometric theories.

To prepare this, we will first reconsider the T-duality of vector background fields. There is a standard way to introduce vector fields at the end of an open string (see for example Ref. [14]). In fact, gauge invariance of the Kalb-Ramond field $B_{\mu\nu}$, which is valid for the closed string, has failed on open string ends with Neumann boundary conditions. To restore it we should add the corresponding vector fields A_a at the string end-points. Then neither the Kalb-Ramond field B_{ab} nor the field strength $F_{ab} = \partial_a A_b - \partial_b A_a$ are gauge invariant. There is a new invariant quantity $\mathcal{B}_{ab} = B_{ab} + F_{ab}$.

We will show that there exists a procedure T-dual to that explained in the previous paragraph. The main point is to understand what is T-dual to the local gauge invariance of the Kalb-Ramond field $B_{\mu\nu}$. In the space-time formulation it is known that this is general coordinate transformations, but the world-sheet action is invariant under general coordinate transformations. It does not fail on open string ends with Dirichlet boundary conditions, as we need. We will show that a transformation which includes transformation of background fields but not transformation of coordinates, is really T-dual to the local gauge invariance. The closed string is invariant under such symmetry on the equations of motion. In this article we are going to show that this symmetry fails on open string ends with Dirichlet boundary conditions. In analogy with previous case we will intro-

duce a corresponding additional vector field A_i, which restores general coordinate transformations at the string end-points.

In accordance with the boundary conditions, we will rename the vector fields A_a to A_a^N and A_i to A_i^D, where A_a^N are fields corresponding to the Neumann boundary conditions and A_i^D are fields corresponding to the Dirichlet boundary conditions.

The Dirichlet vector field is not coupled with $\dot{x}^\mu$ but with expression $\gamma_\mu^{(0)}(x)$, which depends on both $\dot{x}^\mu$ and x'^μ. We will introduce $\gamma_\mu^{(0)}(x)$ in Section 2 and we will call it σ-momentum, because standard momentum π_μ and $\gamma_\mu^{(0)}(x)$ are components of the same world-sheet vector. Consequently, we introduce a pair of *effective vector fields* $\mathcal{A}_{\alpha\mu}(\alpha=0,1)$ as a world-sheet and space-time vector. Its world-sheet components: the standard one $\mathcal{A}_{0\mu}$ is a coefficient in front of $\dot{x}^\mu$ and a new one $\mathcal{A}_{1\mu}$ is a coefficient in front of x'^μ. We will show that the field strength corresponding to $\mathcal{A}_{0\mu}$ is antisymmetric while the non-standard one, corresponding to $\mathcal{A}_{1\mu}$, is symmetric.

The space-time equations of motion in the lowest order in slope parameter α' are a consequence of the requirement of world-sheet conformal invariance on the quantum level. We will consider the simplest solutions for the closed string background fields (metric and Kalb-Ramond field) which satisfy the space-time equations of motion $G_{\mu\nu}=\text{const}, \text{B}_{\mu\nu}=\text{const}$. For gauge fields we will choose non-trivial solutions of the space-time equations of motion [15]: we will take them linear in coordinates with infinitesimal coefficients, so that the field strength is infinitesimal and constant. This is a non-trivial generalization of the standard consideration in the literature. As is well known [16], the constant part of the Dirichlet vector field A_i^D carries out uniform translation of the Dp-brane. In the present article the vector field additionally contains an infinitesimal coordinate dependent part. According to Ref. [16] such a term can produce coordinate-dependent translations. In other words it can curve the Dp-brane. We will assume that A_i^D depends only on coordinates x^i orthogonal to Dp-brane. So, in this paper we will work with a plane Dp-brane.

Let us now consider T-duality. In the present article we will work only with Abelian T-duality. When we use the formulation with field strength, according to Buscher's procedure [17], we will gauge global shift symmetry. In the formulation with vector gauge fields we should apply the generalization of such procedure Ref. [6]. Canonical momenta are T-dual to the σ-derivative of the coordinates. After integration over σ it turns to T-duality between momenta and winding numbers. On the other hand, canonical momenta are generators of the general coordinate transformations, while the σ-derivative of the coordinates are generators of the gauge symmetry [18–20]. It follows that general coordinate transformations are T-dual to the gauge symmetry, which is a fact used in double field theories. In the open string case, after T-dualization additional vector fields with Neumann boundary conditions turn into vector fields with Dirichlet boundary conditions, $A_a^N \to {}^\star A_D^b$, and vice versa, $A_i^D \to {}^\star A_N^j$.

We are going to carry through T-dualization in two ways: in terms of vector field and in terms of its field strength. The first way is more challenging, because in that case the vector field is not constant and Buscher's procedure cannot be applied. The part with vector field which corresponds to the Dirichlet boundary conditions does not possess even global symmetry. So, we will use the T-dualization procedure of Ref. [6], which works in absence of global symmetry. We explicitly find T-dual vector fields in the form ${}^\star A_D^a(V) = G_E^{-1ab} A_b^N(V)$ and ${}^\star A_N^i(V) = G^{-1ij} A_j^D(V)$. It shows that, as we expect, T-dualization changes boundary conditions and exchanges Neumann with Dirichlet vector fields. Additionally we prove that T-dual vector fields do not depend only on the dual coordinates y_μ but on V^μ, which besides y_μ depends also on its double $\tilde{y}_\mu$.

The second way of T-dualization is simpler, because the field strength of the initial theory is constant. The antisymmetric part $\mathcal{F}_{\mu\nu}^{(a)}$ can be considered an extension of the Kalb-Ramond field while the symmetric part $\mathcal{F}_{\mu\nu}^{(s)}$ can be considered an extension of the metric tensor. So, it is easy to find complete T-dual background fields and T-dual field strength.

The particular form of $V^\mu = -\kappa\theta^{\mu\nu} y_\nu + G_E^{-1\mu\nu}\tilde{y}_\nu$ implies several features connected with non-geometric theories. For example, in Ref. [8] it was shown that it produces non-associativity of the coordinates, derived previously in the other way in Refs. [11, 12, 21–23]. In geometric theories the field strength for an Abelian vector field is simply $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Because in non-geometric theories the vector field depends on V^μ, we expect that T-dual field strength will contain derivatives with respect to both variables y_μ and $\tilde{y}_\mu$.

The case with T-dual vector fields includes additional problems. The source of non-geometry is not only the argument V^μ of the vector background field but also the T-dual σ-momentum ${}^\star\gamma_{(0)}^\mu(y)$, which depends on both $\dot{y}_\mu$ and y'_μ. In that case we can analogously introduce *T-dual effective vector fields* ${}^\star\mathcal{A}_0^\mu(V)$ and ${}^\star\mathcal{A}_1^\mu(V)$ in front of $\dot{y}_\mu$ and y'_μ respectively for both Neumann and Dirichlet sectors. T-duality allows us to find their dependence on the original vector fields A_μ: ${}^\star\mathcal{A}_\alpha^\mu(A_\mu), (\alpha=0,1)$, which is equivalent to dependence on the original field strength ${}^\star\mathcal{A}_\alpha^\mu(F_{\mu\nu}), (\alpha=0,1)$.

In the present article we will introduce the field strengths of non-geometric theories. In geometric theories, the term in the action with vector field (defined as integration over τ) multiplied by $\dot{x}^\mu$ can be trans-

formed to the term in the action with corresponding field strength (with integration over $\mathrm{d}^2\xi=\mathrm{d}\tau\mathrm{d}\sigma$). We can take this as a new definition for field strength. It agree with the standard one for geometric theories and provides us with new opportunities for non-geometric theories.

We can generalize such an approach to the case of non-geometric theories. We will define the effective T-dual field strength ${}^\star\mathcal{F}^{\mu\nu}$ as the term in action with integration over $\mathrm{d}^2\xi=\mathrm{d}\tau\mathrm{d}\sigma$ which is equivalent to the term with effective vector fields ${}^\star\mathcal{A}_0^\mu(V)$ and ${}^\star\mathcal{A}_1^\mu(V)$ multiplied by $\dot{y}_\mu$ and y'_μ respectively. As well as in the initial theory, besides the standard term antisymmetric in μ,ν indices ${}^\star\mathcal{F}^{\mu\nu}_{(a)}=-{}^\star\mathcal{F}^{\nu\mu}_{(a)}$, it appears the new one is symmetric in μ,ν indices ${}^\star\mathcal{F}^{\mu\nu}_{(s)}={}^\star\mathcal{F}^{\nu\mu}_{(s)}$. The T-dual effective field strength depends on the initial one ${}^\star\mathcal{F}^{\mu\nu}(F_{\mu\nu})$. The expressions ${}^\star\mathcal{A}^\mu_\alpha(F_{\mu\nu})$ and ${}^\star\mathcal{F}^{\mu\nu}(F_{\mu\nu})$ allow us to eliminate the initial field strength $F_{\mu\nu}$ and find expression ${}^\star\mathcal{F}^{\mu\nu}$ in terms of ${}^\star\mathcal{A}^\mu_\alpha$. In fact, first we can find all antisymmetric and symmetric derivatives of ${}^\star\mathcal{A}^\mu_\alpha(F_{\mu\nu})$ with respect to both y_μ and $\tilde{y}_\mu$. Comparing these results with the known expression for ${}^\star\mathcal{F}^{\mu\nu}(F_{\mu\nu})$ we obtain the desired result.

Using the above results we will introduce genuinely non-geometric theories. We will also discuss local gauge symmetries of T-dual non-geometric theories as transformation of T-dual effective vector fields ${}^\star\mathcal{A}^\mu_\alpha(V)$ which does not change the T-dual field strength ${}^\star\mathcal{F}^{\mu\nu}$. We will briefly discuss non-geometric matter fields.

2 T-duality of closed string

In this section we introduce some known features of the bosonic closed string, which we are going to generalize to the case of the open string in the next sections. In particular, we consider T-dual background fields and T-duality transformations in canonical form.

2.1 Closed bosonic string

Let us consider the closed bosonic string which propagates in D-dimensional space-time described by the action [24]

$$S[x]=\kappa\int_\Sigma \mathrm{d}^2\xi\sqrt{-g}\Big[\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}[x]+\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}[x]\Big]\partial_\alpha x^\mu\partial_\beta x^\nu,\quad(\varepsilon^{01}=-1).\tag{1}$$

The string, with coordinates $x^\mu(\xi)$, $\mu=0,1,...,D-1$ is moving in a non-trivial background, defined by the space metric $G_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$. Here $g_{\alpha\beta}$ is the intrinsic world-sheet metric and the integration goes over a two-dimensional world-sheet Σ with coordinates ξ^α ($\xi^0=\tau$, $\xi^1=\sigma$).

Choosing the conformal gauge $g_{\alpha\beta}=e^{2F}\eta_{\alpha\beta}$, and introducing light-cone coordinates $\xi^\pm=\frac{1}{2}(\tau\pm\sigma)$, $\partial_\pm=\partial_\tau\pm\partial_\sigma$, the action (1) can be rewritten in the form

$$S=\kappa\int_\Sigma \mathrm{d}^2\xi\ \partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu,\tag{2}$$

where we introduce a useful combination of background fields

$$\Pi_{\pm\mu\nu}=B_{\mu\nu}\pm\frac{1}{2}G_{\mu\nu}.\tag{3}$$

According to the action principle, variation of the action (2) with respect to x^μ produces an equation of motion

$$\partial_+\partial_-x^\mu+\Big(\Gamma^\mu_{\nu\rho}-B^\mu{}_{\nu\rho}\Big)\partial_+x^\nu\partial_-x^\rho=0,\tag{4}$$

and boundary conditions

$$\gamma^{(0)}_\mu(x)\delta x^\mu/_{\sigma=\pi}\ -\ \gamma^{(0)}_\mu(x)\delta x^\mu/_{\sigma=0}=0,\tag{5}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol and we introduce

$$\begin{aligned}\gamma^{(0)}_\mu(x)\ &\equiv\ \frac{\delta S}{\delta x'^\mu}=\kappa\Big[2B_{\mu\nu}\dot{x}^\nu-G_{\mu\nu}x'^\nu\Big]\\&=\ \kappa\Big(\Pi_{+\mu\nu}\partial_-x^\nu+\Pi_{-\mu\nu}\partial_+x^\nu\Big).\end{aligned}\tag{6}$$

The requirement of world-sheet conformal invariance on the quantum level leads to the space-time equations of motion, which at the lowest order in slope parameter α', for the constant dilaton field $\Phi=$const, are

$$R_{\mu\nu}-\frac{1}{4}B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma}=0,\qquad D_\rho B^\rho{}_{\mu\nu}=0.\tag{7}$$

Here $B_{\mu\nu\rho}=\partial_\mu B_{\nu\rho}+\partial_\nu B_{\rho\mu}+\partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and covariant derivative with respect to the space-time metric. We will consider the simplest solutions of Eq. (7),

$$G_{\mu\nu}=\text{const},\quad B_{\mu\nu}=\text{const},\tag{8}$$

which satisfy the space-time equations of motion.

2.2 Sigma-model T-duality for closed string

Applying the Buscher T-dualization procedure along all coordinates [17], we obtain the T-dual action

$${}^\star S[y]=\kappa\int\mathrm{d}^2\xi\ \partial_+y_\mu{}^\star\Pi^{\mu\nu}_+\partial_-y_\nu=\frac{\kappa^2}{2}\int\mathrm{d}^2\xi\ \partial_+y_\mu\theta^{\mu\nu}_-\partial_-y_\nu,\tag{9}$$

where

$$\theta^{\mu\nu}_\pm\ \equiv\ -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu}=\theta^{\mu\nu}\mp\frac{1}{\kappa}(G_E^{-1})^{\mu\nu}.\tag{10}$$

The symmetric and antisymmetric parts of $\theta^{\mu\nu}_\pm$ are the inverse of the effective metric $G^E_{\mu\nu}$ and the non-commutativity parameter $\theta^{\mu\nu}$

$$G^E_{\mu\nu}\ \equiv G_{\mu\nu}-4(BG^{-1}B)_{\mu\nu},\ \theta^{\mu\nu}\ \equiv-\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}.\tag{11}$$

Consequently, the T-dual background fields are

$${}^\star G^{\mu\nu}=(G_E^{-1})^{\mu\nu},\quad {}^\star B^{\mu\nu}=\frac{\kappa}{2}\theta^{\mu\nu}.\tag{12}$$

Note that the dual effective metric is just the inverse of the initial metric

$$ {}^\star G_E^{\mu\nu} \equiv {}^\star G^{\mu\nu} - 4({}^\star B {}^\star G^{-1} {}^\star B)^{\mu\nu} = (G^{-1})^{\mu\nu}, \tag{13} $$

and we will need the following relations

$$ ({}^\star B {}^\star G^{-1})^\mu{}_\nu = -(G^{-1}B)^\mu{}_\nu, \quad ({}^\star G^{-1} {}^\star B)_\mu{}^\nu = -(BG^{-1})_\mu{}^\nu. \tag{14} $$

2.3 T-duality transformations of closed string

The T-duality transformation, connecting the variables x^μ of the initial closed string theory with its corresponding T-dual ones y_μ, takes the form [5]

$$ \partial_\pm x^\mu \cong -\kappa \theta^{\mu\nu}_\pm \partial_\pm y_\nu, \tag{15} $$

where the symbol $\cong$ denotes the T-duality relation.

From this equation we can find the T-dual transformation laws for $\dot{x}^\mu$ and x'^μ,

$$ \dot{x}^\mu \cong -\kappa\theta^{\mu\nu}\dot{y}_\nu + (G_E^{-1})^{\mu\nu} y'_\nu, \tag{16} $$

$$ x'^\mu \cong (G_E^{-1})^{\mu\nu}\dot{y}_\nu - \kappa\theta^{\mu\nu} y'_\nu. \tag{17} $$

It has been shown in Ref. [5] that the T-dual of the T-dual action is the original one. The corresponding T-dual transformation is the inverse of Eq. (15),

$$ \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu, \tag{18} $$

and consequently the transformation laws for $\dot{y}_\mu$ and y'_μ are equal to

$$ \dot{y}_\mu \cong -2B_{\mu\nu}\dot{x}^\nu + G_{\mu\nu}x'^\nu, \tag{19} $$

$$ y'_\mu \cong G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu. \tag{20} $$

Using the expression for the canonical momentum of the original theory,

$$ \pi_\mu \equiv \frac{\delta S}{\delta \dot{x}^\mu} = \kappa\Big[G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu\Big], \tag{21} $$

and of the T-dual theory,

$$ {}^\star\pi^\mu \equiv \frac{\delta {}^\star S}{\delta \dot{y}_\mu} = \kappa\Big[(G_E^{-1})^{\mu\nu}\dot{y}_\nu - \kappa\theta^{\mu\nu}y'_\nu\Big], \tag{22} $$

we can rewrite the transformations (17) and (20) in the canonical form,

$$ \kappa x'^\mu \cong {}^\star\pi^\mu, \quad \pi_\mu \cong \kappa y'_\mu. \tag{23} $$

This relation connect momenta and winding numbers.

It was shown in Refs. [18–20] that π_μ is the generator of general coordinate transformations while x'^μ is the generator of gauge symmetry. Then, Eq. (23) shows that these symmetries are T-dual to each other.

Since $\partial_\alpha x^\mu = \{\dot{x}^\mu, x'^\mu\}$ is a world-sheet vector, variation with respect to $\partial_\alpha x^\mu$,

$$ \pi^\alpha_\mu \equiv \frac{\delta S}{\delta \partial_\alpha x^\mu} = \{\pi_\mu, \gamma^{(0)}_\mu(x)\}, \tag{24} $$

is also a world-sheet vector. So, the momentum π_μ and variable $\gamma^{(0)}_\mu(x)$, which will play important roles in the analysis of boundary conditions, are components of the same world-sheet vector. From now on we will call $\gamma^{(0)}_\mu(x)$ σ-momentum.

3 T-duality of open string

In this section we will consider boundary conditions on the open string end-points and adapt T-duality for such restrictions. Essentially, all changes will happen on the string end points, although it is useful to rewrite some expressions formally as if they are on the world-sheet.

We will consider vector gauge field A^N_a with Neumann boundary conditions, which appears regularly in the literature. It is a p+1 dimensional vector on the Dp-brane. It compensates the not-implemented gauge symmetry of the Kalb-Ramond field at the open string end-points. In this article we additionally introduce the $D-p-1$ dimensional vector field A^D_i with Dirichlet boundary conditions, orthogonal to the Dp-brane, which with previous ones completes a D-dimensional vector. It compensates the not-implemented general coordinate transformations at the open string end-points. We will show that field A^D_i is T-dual to the A^N_a one, as well as that the general coordinate transformations are T-dual to the gauge symmetry of the Kalb-Ramond field.

3.1 T-duality between Dirichlet and Neumann boundary conditions

Unlike the closed string, the open string must satisfy boundary conditions at the string end-points. For an initial string they take forms (5) and (6) while for a T-dual string we have

$$ {}^\star\gamma^\mu_{(0)}(y)\delta y_\mu/_{\sigma=\pi} - {}^\star\gamma^\mu_{(0)}(y)\delta y_\mu/_{\sigma=0} = 0, \tag{25} $$

where according to Eq. (9) the T-dual σ-momentum is

$$ \begin{aligned} {}^\star\gamma^\mu_{(0)}(y) &\equiv \frac{\delta {}^\star S}{\delta y'_\mu} = \kappa\Big[\kappa\theta^{\mu\nu}\dot{y}_\nu - (G_E^{-1})^{\mu\nu}y'_\nu\Big] \\ &= \kappa\Big[2{}^\star B^{\mu\nu}\dot{y}_\nu - {}^\star G^{\mu\nu}y'_\nu\Big] \\ &= \frac{\kappa^2}{2}\Big(\theta^{\mu\nu}_-\partial_- y_\nu + \theta^{\mu\nu}_+\partial_+ y_\nu\Big). \end{aligned} \tag{26} $$

We can rewrite the T-dual transformations (16) and (19) in the form

$$ -\kappa\dot{x}^\mu \cong {}^\star\gamma^\mu_{(0)}(y), \qquad \gamma^{(0)}_\mu(x) \cong -\kappa\dot{y}_\mu. \tag{27} $$

Note that we can put Eqs. (23) and (27) in compact, world-sheet covariant forms,

$$ \begin{aligned} \kappa\partial_\alpha x^\mu &\cong \frac{\delta {}^\star S}{\delta(\varepsilon^{\alpha\beta}\partial_\beta y_\mu)} = -\varepsilon_{\alpha\beta}{}^\star\pi^{\beta\mu}, \\ \kappa\partial_\alpha y_\mu &\cong \frac{\delta S}{\delta(\varepsilon^{\alpha\beta}\partial_\beta x^\mu)} = -\varepsilon_{\alpha\beta}\pi^\beta_\mu. \end{aligned} \tag{28} $$

Let us show that the above equations connect the Dirichlet and Neumann boundary conditions.

1) If the end points (we will denote them with $\partial\Sigma$ as a boundary of the world-sheet Σ) of the initial string satisfy Neumann boundary conditions (which means that variation of some string end points $\delta x^a/_{\partial\Sigma}$ with $a=0,1,\cdots,p$ is arbitrary) then $\gamma^{(0)}_a(x)/_{\partial\Sigma}=0$. Together with Eq. (27) it produces $\dot{y}_a/_{\partial\Sigma}=0$, which means that edges of the dual string are fixed. This is by definition the Dirichlet boundary conditions for a T-dual string.

2) Similarly, if the end points of an initial string satisfy Dirichlet boundary conditions (it means that the edges of the string are fixed) then $\dot{x}^i/_{\partial\Sigma}=0$ where $i=p+1,\cdots,D-1$. Together with Eq. (27) it produces ${}^\star\gamma^i_{(0)}(y)/_{\partial\Sigma}=0$, which according to Eq. (25) means that variations of the corresponding dual string end points $\delta y_i/_{\partial\Sigma}$ are arbitrary. This is by definition the Neumann boundary conditions for a T-dual string.

3.2 Neumann and Dirichlet vector background fields

The action of closed string theory (1) is invariant under local gauge transformations,

$$\delta_\Lambda G_{\mu\nu}=0,\qquad \delta_\Lambda B_{\mu\nu}=\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu. \tag{29}$$

The open string theory is not invariant under these transformations. In Ref. [14] has been shown that for open strings,

$$\delta_\Lambda S[x]=2\kappa\int \mathrm{d}\tau(\Lambda_\mu\dot{x}^\mu/_{\sigma=\pi}-\Lambda_\mu\dot{x}^\mu/_{\sigma=0}). \tag{30}$$

We already denoted the coordinates with Neumann boundary conditions with x^a and those with Dirichlet boundary conditions with x^i. This means that $\delta x^a/_{\sigma=\pi}$ and $\delta x^a/_{\sigma=0}$ are arbitrary, which produces $\gamma^{(0)}_a(x)/_{\sigma=\pi}=\gamma^{(0)}_a(x)/_{\sigma=0}=0$. On the other hand, $\delta x^i/_{\sigma=\pi}=0$ and $\delta x^i/_{\sigma=0}=0$, so that $\gamma^{(0)}_i(x)/_{\sigma=\pi}$ and $\gamma^{(0)}_i(x)/_{\sigma=0}$ are arbitrary. So, because both string end points for x^i coordinates satisfy Dirichlet boundary conditions, we have $\dot{x}^i/_{\sigma=0}=\dot{x}^i/_{\sigma=\pi}=0$ and consequently,

$$\delta_\Lambda S[x]=2\kappa\int \mathrm{d}\tau(\Lambda_a\dot{x}^a/_{\sigma=\pi}-\Lambda_a\dot{x}^a/_{\sigma=0}). \tag{31}$$

To obtain gauge invariant action we should add the term

$$S_{A_\Lambda}[x]=2\kappa\int \mathrm{d}\tau(A_a\dot{x}^a/_{\sigma=\pi}-A_a\dot{x}^a/_{\sigma=0}), \tag{32}$$

where the newly introduced vector field A_a transforms with the same gauge parameter Λ_a,

$$\delta_\Lambda A_a=-\Lambda_a. \tag{33}$$

Therefore, adding the term $S_{A_\Lambda}[x]$, we obtain the open string action invariant under local gauge transformations with parameter Λ_a.

It is natural to ask:

1) whether T-dual transformations of local gauge transformations exist (29);

2) whether we can add some term $S_{A_\xi}[x]$ in order to obtain open string action invariant under such T-dual transformation;

3) whether the terms $S_{A_\Lambda}[x]$ and $S_{A_\xi}[x]$ are connected by T-duality transformations as well as their origins.

In this article we will show that the answers to all these questions are affirmative. We can expect such a conclusion, because if T-duality is valid in the case of an open string then any step in the original theory should have a partner in the T-dual version.

We expect that for every characteristic of the initial theory we can find the corresponding one in T-dual theory. In fact, we have the following table of related terms in initial and T-dual theory:

$G_{\mu\nu}$	$B_{\mu\nu}$	$-\kappa\dot{x}^a$	$\gamma^{(0)}_i(x)$	LGT	$A^N_a(x)$	$A^D_i(x)$
${}^\star G^{\mu\nu}$	${}^\star B^{\mu\nu}$	${}^\star\gamma^a_{(0)}(y)$	$-\kappa\dot{y}_i$	?	${}^\star A^a_D(V)$	${}^\star A^i_N(V)$

where LGT is the abbreviation for local gauge transformations and the question mark is for an unknown symmetry which we expect to be "transformation T-dual to the local gauge transformation". It will allow us to introduce the Dirichlet vector fields A^D_i in analogy with the same procedure in which Neumann vector fields A^N_a were introduced in Ref.[14]. An interesting result has been obtained, that Dirichlet and Neumann vector fields, A^D_i and A^N_a, are coupled to the T-dual expressions $\gamma^{(0)}_i(x)$ and $-\kappa\dot{x}^a$ respectively. Later we will find that Dirichlet and Neumann vector fields are also T-dual to each other, which will complete the table above.

In Refs. [18–20] it has been shown that the generator of the local gauge transformations is σ-derivative of the coordinates x'^μ. Briefly, if the variation of the energy-momentum tensor $\delta T_\pm$ can be written as the Poisson bracket of some generator Γ with energy-momentum tensor $T_\pm$, namely if the relation

$$\delta T_\pm=\{\Gamma,T_\pm\}, \tag{34}$$

is satisfied, then the corresponding transformation of background fields is the target-space symmetry of the theory. For $\Gamma\to\Gamma_\Lambda=2\kappa\int\mathrm{d}\sigma\Lambda_\mu x'^\mu$, we just obtain transformations (29).

According to Eq. (23) the corresponding T-dual generator is $\Gamma_\xi=2\int\mathrm{d}\sigma\xi^\mu\pi_\mu$, with the following transformations of background fields:

$$\begin{aligned}\delta_\xi G_{\mu\nu} &= -2(D_\mu\xi_\nu+D_\nu\xi_\mu),\\ \delta_\xi B_{\mu\nu} &= -2\xi^\rho B_{\rho\mu\nu}+2\partial_\mu(B_{\nu\rho}\xi^\rho)-2\partial_\nu(B_{\mu\rho}\xi^\rho).\end{aligned} \tag{35}$$

These transformations have exactly the form of general coordinate transformations for background fields and they are symmetry transformations of the space-time action.

Are they symmetries of the σ-model action? The difference is transformation of the coordinates, which does not appear explicitly in target-space but is present in σ-model action. In fact, the energy-momentum tensor does not depend explicitly on the coordinates, and the above consideration does not give us information about transformations of the coordinates. In order to distinguish from standard general coordinate transformations (which includes transformation of x^μ), those without transformation of x^μ we will call transformations generated by π_μ.

To better understand what one we have to choose as a T-dual to local gauge transformations in the σ-model action, it is useful to make transformations (35) of the background fields (metric tensor $G_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$) with parameter ξ_μ and the transformations of the string coordinates x^μ with a different parameter $\delta x^\mu=\bar{\xi}^\mu$. Then, using the equation of motion (4) we obtain

$$\delta_\xi S[x] = -2\int \mathrm{d}\tau\Big[(\xi_\mu-\bar{\xi}_\mu)G^{-1\mu\nu}\gamma^{(0)}_\nu(x)/_{\sigma=\pi} -(\xi_\mu-\bar{\xi}_\mu)G^{-1\mu\nu}\gamma^{(0)}_\nu(x)/_{\sigma=0}\Big], \tag{36}$$

where σ-momentum $\gamma^{(0)}_\mu(x)$, defined in Eq. (6), is an expression which appears in the boundary conditions of the original theory.

First, we can conclude that at interior points of the string, on the equations of motions, the action is invariant even under separate transformations of background fields and of the string coordinates. For general coordinate transformations we have $\xi_\mu=\bar{\xi}_\mu$, and the whole action is invariant. This is true even without using equations of motion. So, in the case of σ-model action for an open string, we cannot accept reparametrization as a T-dual of local gauge transformations. Such a choice does not allow us to add the corresponding vector fields, which should be T-dual to the fields A_a, introduced above.

Therefore, as a T-dual to local gauge transformations we will try to impose the transformations (35), the part of general coordinate transformations, which include the transformations of background fields but do not include the transformations of the string coordinates x^μ. Then we have $\bar{\xi}_\mu/_{\sigma=\pi}=\bar{\xi}_\mu/_{\sigma=0}=0$ and

$$\delta_\xi S[x] = -2\int \mathrm{d}\tau\Big(\xi_\mu G^{-1\mu\nu}\gamma^{(0)}_\nu(x)/_{\sigma=\pi} -\xi_\mu G^{-1\mu\nu}\gamma^{(0)}_\nu(x)/_{\sigma=0}\Big). \tag{37}$$

Note that this relation is a strong indication that we are on the right track, because according to Eq. (27) $\dot{x}^\mu$ and $\gamma^{(0)}_\mu(x)$ are expressions T-dual to each other. So, we obtained non-trivial transformations as we need and the transformations (29) and (35) are connected by T-duality. From now on, for transformation (35) we will use expression: T-dual to local gauge transformations.

Let us for simplicity assume that both metric tensor and Kalb-Ramond fields have a form

$$G_{\mu\nu}=\begin{pmatrix} G_{ab} & 0 \\ 0 & G_{ij}\end{pmatrix}, \qquad B_{\mu\nu}=\begin{pmatrix} B_{ab} & 0 \\ 0 & B_{ij}\end{pmatrix}. \tag{38}$$

Then for Neumann boundary conditions we have $\gamma^{(0)}_a(x)/_{\sigma=\pi}=\gamma^{(0)}_a(x)/_{\sigma=0}=0$, and thus on the equation of motion (4) we are left only with coordinates that satisfy Dirichlet boundary conditions,

$$\delta_\xi S[x] = -2\int \mathrm{d}\tau\Big(\xi_i G^{-1ij}\gamma^{(0)}_j(x)/_{\sigma=\pi} -\xi_i G^{-1ij}\gamma^{(0)}_j(x)/_{\sigma=0}\Big). \tag{39}$$

Consequently, on the equations of motion the closed string is invariant under transformations (35), while it is violated on the open string endpoints with Dirichlet boundary conditions. It remains to discuss the relation of transformations (35) as a symmetry T-dual to the local gauge transformations, with well known general coordinate transformations. The similarity is obvious, because the first transformations are part of the second ones. There are two differences. One is a lack of transformation of the coordinates and the other is that transformations (35) are symmetric only on the equations of motion. From the point of view of T-duality the second one is not a big surprise because, as it is well known, the equations of motion and Bianchi identity are T-dual to each other. So, local gauge transformations are symmetries of the action without equations of motion, while its T-dual residual general coordinate transformations are symmetries of the action on the equations of motion.

The next steps are similar to those in the case of local gauge transformations. To obtain action invariant under residual general coordinate transformations we should add the term

$$S_{A_\xi}[x] = -2\int \mathrm{d}\tau\Big(A_i G^{-1ij}\gamma^{(0)}_j(x)/_{\sigma=\pi} -A_i G^{-1ij}\gamma^{(0)}_j(x)/_{\sigma=0}\Big), \tag{40}$$

where the vector field A_i transforms with the gauge parameter of the residual general coordinate transformations ξ_i

$$\delta_\xi A_i=-\xi_i. \tag{41}$$

Note that variation of $S_{A_\xi}[x]$ does not include variation of metric G^{-1ij} and σ-momentum $\gamma^{(0)}_j(x)$. In fact, A_i is infinitesimal and variation of G^{-1ij} or $\gamma^{(0)}_j(x)$ will produce infinitesimals of the second order, which we will neglect.

So, the full gauge invariant action for an open string is

$$S_{\rm open}[x] = S[x]+S_{A_A}[x]+S_{A_\xi}[x]=\kappa\int_\Sigma {\rm d}^2\xi\sqrt{-g}\Big[\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}[x]+\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}[x]\Big]\partial_\alpha x^\mu\partial_\beta x^\nu$$
$$+2\kappa\int{\rm d}\tau\Big[\Big(A_a^N[x]\dot{x}^a-\frac{1}{\kappa}A_i^D[x]G^{-1ij}\gamma_j^{(0)}(x)\Big)/_{\sigma=\pi}-\Big(A_a^N[x]\dot{x}^a-\frac{1}{\kappa}A_i^D[x]G^{-1ij}\gamma_j^{(0)}(x)\Big)/_{\sigma=0}\Big]. \quad (42)$$

Consequently, the nontrivial background fields are $A_a\to A_a^N$ and $A_i\to A_i^D$, where we introduced the indices N and D for vector fields corresponding to Neumann and Dirichlet boundary conditions.

Note that the variables

$$\begin{aligned}\mathcal{B}_{ab} &= B_{ab}+\partial_a A_b^N-\partial_b A_a^N, \qquad \mathcal{G}_{ab}=G_{ab},\\ \mathcal{B}_{ij} &= B_{ij}-2A_D^k B_{kij}+2\partial_i(B_{jk}G^{-1kq}A_q^D)\\ &\quad -2\partial_j(B_{ik}G^{-1kq}A_q^D),\\ \mathcal{G}_{ij} &= G_{ij}-2(\partial_i A_j^D+\partial_j A_i^D),\end{aligned} \quad (43)$$

are gauge invariant under Eq. (35), Eq. (29) and transformations of the vector fields $\delta A_a^N=-\Lambda_a$ and $\delta A_i^D=-\xi_i$, and consequently they are physical. For further benefit let us introduce notations

$$F_{ab}^{(a)}=\partial_a A_b^N-\partial_b A_a^N, \quad F_{ij}^{(s)}=-2(\partial_i A_j^D+\partial_j A_i^D). \quad (44)$$

We are going to use the conformal gauge and the light-cone coordinates, so that the first term in S_{open} obtains the form of the action (2). For constant metric and Kalb-Ramond fields we have

$$S_{\rm open}[x]=\kappa\int_\Sigma {\rm d}^2\xi\partial_+ x^\mu\Pi_{+\mu\nu}\partial_- x^\nu$$
$$+2\kappa\int{\rm d}\tau\Big[\Big(A_a^N[x]\dot{x}^a-\frac{1}{\kappa}A_i^D[x]G^{-1ij}\gamma_j^{(0)}(x)\Big)/_{\sigma=\pi}$$
$$-\Big(A_a^N[x]\dot{x}^a-\frac{1}{\kappa}A_i^D[x]G^{-1ij}\gamma_j^{(0)}(x)\Big)/_{\sigma=0}\Big]. \quad (45)$$

In the literature $A_a^N[x]$ is known as a massless vector field on the Dp-brane while $A_i^D[x]$ is known as massless scalar oscillations orthogonal to the Dp-brane. These are terms in relation to the Lorentz transformations that preserve the Dp-brane.

Note that inclusion of vector background fields changes the σ-momentum $\gamma_i^{(0)}$, defined in Eq. (6). In fact, we will get an additional infinitesimal term linear in the vector background fields. It is multiplied by another infinitesimal, A_i^D, and consequently we will neglect it.

It is common to take both the vector and the massless scalar fields to be constant, when the Buscher procedure can be applied. The constant massless scalar field performs uniform translation of the Dp-brane [16]. We are going to use the generalized procedure [5, 6], so we are able to consider vector and massless scalar fields linear in coordinates with infinitesimal coefficients. As explained in Ref. [16], coordinate-dependent massless scalar fields produce coordinate-dependent translations, which curve the Dp-brane. Consequently, our approach is able to describe an infinitesimally curved Dp-brane. We are not going to do this in the present article, because for simplicity we will assume later in Eq. (46) that $A_i^D(x)$ depends only on x^i coordinates and not on x^a.

3.3 T-dual background fields of open string

Let us perform the T-dualization procedure on the theory described by the action (45). The first term contains constant background fields and so we can apply the standard Buscher's procedure of Section 2.2. The remaining two terms are nontrivial because the background fields A_a^N and A_i^D are coordinate-dependent. To simplify the situation we will assume that the vector fields are linear in coordinates,

$$A_a^N(x)=A_a^0-\frac{1}{2}F_{ab}^{(a)}x^b, \quad A_i^D(x)=A_i^0-\frac{1}{4}F_{ij}^{(s)}x^j, \quad (46)$$

so that the corresponding field strengths are constant. The coefficients $F_{ab}^{(a)}$ and $F_{ij}^{(s)}$ are defined in Eq. (44). The first coefficient is antisymmetric under a,b indices while the second is symmetric under i,j indices.

These forms of background fields satisfy the additional space-time equations of motion for open strings [15]. In our notation they take the form

$$\begin{aligned}\beta_a &= -\frac{1}{2}\mathcal{B}_a{}^b\partial_b\Phi+\mathcal{G}_E^{-1bc}\partial_c\mathcal{B}_{ba}\\ &\quad +\mathcal{G}_E^{-1bc}\Big(\frac{1}{2}\mathcal{B}_a{}^d B_{dbe}\mathcal{B}^e{}_c+K_{ac}^\mu B_{\mu\nu}\partial_b f^\nu\Big),\\ [1mm]\beta_\mu &= \frac{1}{2}\partial_\mu\Phi+\mathcal{G}_E^{-1ab}(\frac{1}{2}\mathcal{B}_b{}^c B_{\mu ac}-K_{\mu ab}),\end{aligned} \quad (47)$$

where

$$\mathcal{B}_{ab}=B_{ab}+F_{ab}^{(a)}, \quad \mathcal{G}_{ab}^E=G_{ab}-4\mathcal{B}_{ac}G^{-1cd}\mathcal{B}_{db}, \quad (48)$$

$B_{\mu\nu\rho}$ is the field strength of the Kalb-Ramond field $B_{\mu\nu}$ defined in Section 2.1 and K_{ab}^μ is the extrinsic curvature. According to our assumptions Φ=const and $\mathcal{B}_{ab}$=const. So, $\mathcal{G}_{ab}^E$=const and B_{abc}=0. Since we are working with a plane Dp-brane the extrinsic curvature is zero and both β-functions vanish.

Note that the part with Dirichlet vector field

$$\begin{aligned} S^D_A[x] &= -2\int \mathrm{d}\tau\Big[\Big(A^D_i[x]G^{-1ij}\gamma^{(0)}_j(x)\Big)/_{\sigma=\pi} \\ &\quad -\Big(A^D_i[x]G^{-1ij}\gamma^{(0)}_j(x)\Big)/_{\sigma=0}\Big] \\ &= 2\kappa\int \mathrm{d}\tau\Big[\Big(2\dot{x}^i(BG^{-1})_i{}^jA^D_j[x]+x'^iA^D_i[x]\Big)/_{\sigma=\pi} \\ &\quad -\Big(2\dot{x}^i(BG^{-1})_i{}^jA^D_j[x]+x'^iA^D_i[x]\Big)/_{\sigma=0}\Big], \end{aligned} \quad (49)$$

using the form of the vector field (46) after partial integration over τ, can be rewritten as

$$\begin{aligned} S^D_A[x] &= 2\kappa\int \mathrm{d}\tau\Big[\Big(2\dot{x}^iA^D_i[G^{-1}Bx]+x'^iA^D_i[x]\Big)/_{\sigma=\pi} \\ &\quad -\Big(2\dot{x}^iA^D_i[G^{-1}Bx]+x'^iA^D_i[x]\Big)/_{\sigma=0}\Big]. \end{aligned} \quad (50)$$

So, we can conclude that following forms of the Dirichlet vector field are equivalent,

$$(BG^{-1})_i{}^jA^D_j[x]\cong A^D_i[G^{-1}Bx]. \quad (51)$$

3.3.1 Auxiliary action

Because parts with vector fields depend on the coordinate x^μ itself and not on its derivatives with respect to τ and σ, it is not possible to apply the standard Buscher's procedure. So, we will need generalized T-duality, developed in Ref. [5]. Even more, the part with $A^D_i(x)$ does not have the global shift symmetry, because the expression $\gamma^{(0)}_i$ contains the part $G_{ij}x'^j$ which is not the total derivative with respect to integration variable τ. So, we should apply the T-dualization procedure of Ref. [6], which works in the absence of global symmetry.

Following Ref. [6], let us introduce the auxiliary action,

$$\begin{aligned} S_{\rm aux}[v_\pm,y] &= \kappa\int_\Sigma \mathrm{d}^2\xi\Big[v^\mu_+\Pi_{+\mu\nu}v^\nu_-+\frac{1}{2}(v^\mu_+\partial_-y_\mu-\partial_+y_\mu v^\mu_-)\Big]+2\kappa\int\mathrm{d}\tau\Big\{\Big[A^N_a(\Delta V)v^a_0-\frac{1}{\kappa}A^D_i(\Delta V)G^{-1ij}\gamma^{(0)}_j(V)\Big]/_{\sigma=\pi} \\ &\quad -\Big[A^N_a(\Delta V)v^a_0-\frac{1}{\kappa}A^D_i(\Delta V)G^{-1ij}\gamma^{(0)}_j(V)\Big]/_{\sigma=0}\Big\}, \end{aligned} \quad (52)$$

where $\gamma^{(0)}_i(V)\equiv\kappa(2B_{ij}\dot{V}^j-G_{ij}V'^j)=\kappa(2B_{ij}v^j_0-G_{ij}v^j_1)$ have been defined in accordance with Eq. (6). It can be obtained from the action (45), by making substitutions,

$$\partial_\pm x^\mu\to v^\mu_\pm,\quad \dot{x}^\mu\to v^\mu_0,\quad x'^\mu\to v^\mu_1,\quad x^\mu\to\Delta V^\mu, \quad (53)$$

and adding the Lagrange multiplier term with Lagrange multiplier y_μ. This action is constructed in the form of the gauge fixed action. Here $v^\mu_\pm$ are some auxiliary fields, which take over the role of the gauge fields. Similarly to Refs. [5–7], the argument of the background fields is the line integral of the auxiliary fields taken along a path P (from ξ_0 to ξ),

$$\Delta V^\mu[v_+,v_-]\equiv\int_P \mathrm{d}\xi^\alpha v^\mu_\alpha=\int_P(\mathrm{d}\xi^+v^\mu_++\mathrm{d}\xi^-v^\mu_-). \quad (54)$$

It is easy to show that the auxiliary action $S_{\rm aux}[x]$ (52) turns to the initial action $S_{\rm open}[x]$ (45). Note that, also in Refs. [5, 6], the equation of motion with respect to y_μ forces the "field strength" to vanish,

$$\partial_+v^\mu_--\partial_-v^\mu_+=0, \quad (55)$$

which is just the condition for the path independence of ΔV^μ. Using the solution of Eq. (55),

$$v^\mu_\pm=\partial_\pm x^\mu, \quad (56)$$

one obtains $\Delta V^\mu(\xi)=x^\mu(\xi)-x^\mu(\xi_0)$, and taking $x^\mu(\xi_0)=0$ the auxiliary action reduces to the initial one (45).

3.3.2 T-dual action

The next step is to find the equations of motion with respect to the auxiliary fields $v^\mu_\pm$. To prepare this, let us first rewrite the part of the action (52) with integration over $\mathrm{d}\tau$ to the integration over $\mathrm{d}^2\xi=\mathrm{d}\tau\mathrm{d}\sigma$. We obtain

$$\begin{aligned} S_{\rm aux}[v_\pm,y] &= \kappa\int_\Sigma \mathrm{d}^2\xi\Big[v^\mu_+\Pi_{+\mu\nu}v^\nu_-+\frac{1}{2}(v^\mu_+\partial_-y_\mu-\partial_+y_\mu v^\mu_-)\Big] \\ &\quad +\kappa\int_\Sigma\mathrm{d}^2\xi\Big\{\Big[A^N_a(V)(v^a_++v^a_-) \\ &\quad -2A^D_i(V)G^{-1ij}\Big(\Pi_{-jk}v^k_++\Pi_{+jk}v^k_-\Big)\Big]\Delta(\sigma)\Big\}, \end{aligned} \quad (57)$$

where $\Delta(\sigma)\equiv\delta(\sigma-\pi)-\delta(\sigma)$ and we used the relations

$$v^\mu_\pm=v^\mu_0\pm v^\mu_1, \quad (58)$$

and Eq. (6) for $\gamma^{(0)}_i(x)$.

Let us first calculate the variation over the arguments V^μ of the vector background fields. Using the form (46) of these fields and zero order equation of motion we can reexpress the term with vector fields from Eq. (57) in the form

$$\begin{aligned} S_A[v_\pm,y] &= \kappa\int_\Sigma\mathrm{d}^2\xi\Big\{\Big[A^N_a(V)(v^a_++v^a_-)+2A^D_i(G^{-1}\Pi_+V)v^i_+ \\ &\quad +2A^D_i(G^{-1}\Pi_-V)v^i_-\Big]\Delta(\sigma)\Big\}. \end{aligned} \quad (59)$$

It helps us to find the variation with respect to arguments V^μ of the background fields,

$$\begin{aligned}\delta_V S_{\rm aux}[v_\pm,y] &= \kappa\int_\Sigma {\rm d}^2\xi\big\{(\delta v_+^a+\delta v_-^a)A_a^N(V)\\&\quad+2\delta v_+^i A_i^D(G^{-1}\Pi_+V)\\&\quad+2\delta v_-^i A_i^D(G^{-1}\Pi_-V)\big\}\Delta(\sigma).\end{aligned}\tag{60}$$

Now, the equations of motion after variation with respect to the auxiliary fields $v_\mp^\mu$ are:

$$\begin{aligned}&\Pi_{\mp\mu\nu}v_\pm^\nu+\frac{1}{2}\partial_\pm y_\mu+\Big[\mp2A_a^N(V)\mp2\Pi_{\mp ij}G^{-1jk}A_k^D(V)\\&\mp2A_i^D(G^{-1}\Pi_\mp V)\Big]\Delta(\sigma)=0.\end{aligned}\tag{61}$$

Introducing new variables $\mathcal{A}_{\pm\mu}(V)=\{\mathcal{A}_{\pm a}(V),\mathcal{A}_{\pm i}(V)\}$,

$$\begin{aligned}\mathcal{A}_{\pm a}(V) &\equiv A_a^N(V),\\\mathcal{A}_{\pm i}(V) &\equiv \Pi_{\mp ij}G^{-1jk}A_k^D(V)+A_i^D(G^{-1}\Pi_\mp V)\\&= -\frac{1}{4}\big(BG^{-1}F^{(s)}+F^{(s)}G^{-1}B\mp F^{(s)}\big)_{ij}V^j,\end{aligned}\tag{62}$$

or in components,

$$\begin{aligned}\mathcal{A}_{0a}(V) &= A_a^N(V),\quad \mathcal{A}_{1a}(V)=0,\\\mathcal{A}_{0i}(V) &= (BG^{-1})_i{}^jA_j^D(V)+A_i^D(G^{-1}BV)\\&= \mathcal{A}_{0i}^{(0)}-\frac{1}{4}(BG^{-1}F^{(s)}+F^{(s)}G^{-1}B)_{ij}V^j,\\\mathcal{A}_{1i}(V) &= -A_i^{(0)D}+\frac{1}{4}F_{ij}^{(s)}V^j=-A_i^D(V),\end{aligned}\tag{63}$$

we can rewrite the above equation as

$$\Pi_{\mp\mu\nu}v_\pm^\nu+\frac{1}{2}\partial_\pm y_\mu\mp2\mathcal{A}_{\pm\mu}(V)\Delta(\sigma)=0.\tag{64}$$

We introduced a pair of effective vector fields $\mathcal{A}_{\alpha\mu}=\{\mathcal{A}_{0\mu},\mathcal{A}_{1\mu}\}$ instead of the initial one $A_\mu=\{A_a^N,A_i^D\}$. So, we doubled the number of vector fields, but there are two constraints on the effective vector fields,

$$\mathcal{A}_{1a}(V)=0,\quad \mathcal{A}_{0i}(V)=-(BG^{-1})_i{}^j\mathcal{A}_{1j}(V)-\mathcal{A}_{1i}(G^{-1}BV).\tag{65}$$

The second constraint we can also rewrite in the forms

$$\begin{aligned}&(\Pi_+G^{-1})_i{}^j\mathcal{A}_{+j}(V)+\mathcal{A}_{+i}(G^{-1}\Pi_+V)\\&=(\Pi_-G^{-1})_i{}^j\mathcal{A}_{-j}(V)+\mathcal{A}_{-i}(G^{-1}\Pi_-V).\end{aligned}\tag{66}$$

Using Eq. (51), from now on the i-components of Eqs. (62) and (63) we will express as

$$\mathcal{A}_{\pm i}(V)=2\Pi_{\mp ij}G^{-1jk}A_k^D(V),\ \mathcal{A}_{0i}(V)=2(BG^{-1})_i{}^jA_j^D(V).\tag{67}$$

Multiplying Eq. (64) from the left with $2\kappa\theta_\pm$, we can solve it in terms of $v_\pm^\mu$,

$$\begin{aligned}v_\pm^\mu &= -\kappa\theta_\pm^{\mu\nu}\partial_\pm y_\nu\pm4\kappa\theta_\pm^{\mu\nu}\mathcal{A}_{\pm\nu}(V)\Delta(\sigma)\\&= -\kappa\theta_\pm^{\mu\nu}\Big(\partial_\pm y_\nu\mp4\mathcal{A}_{\pm\nu}(V)\Delta(\sigma)\Big),\end{aligned}\tag{68}$$

or in components,

$$\begin{aligned}v_0^\mu &= -\kappa\theta^{\mu\nu}\Big[\dot y_\nu-4\mathcal{A}_{1\nu}\Delta(\sigma)\Big]+G_E^{-1\,\mu\nu}\Big[y'_\nu-4\mathcal{A}_{0\nu}\Delta(\sigma)\Big],\\v_1^\mu &= -\kappa\theta^{\mu\nu}\Big[y'_\nu-4\mathcal{A}_{0\nu}\Delta(\sigma)\Big]+G_E^{-1\,\mu\nu}\Big[\dot y_\nu-4\mathcal{A}_{1\nu}\Delta(\sigma)\Big].\end{aligned}\tag{69}$$

Substituting Eq. (68) in Eq. (54) we obtain

$$V^\mu=-\kappa\theta^{\mu\nu}(y_\nu-4\tilde{\mathcal{A}}_\nu)+G_E^{-1\,\mu\nu}(\tilde y_\nu-4\mathcal{A}_\nu)=V_0^\mu+V_1^\mu,\tag{70}$$

where

$$\begin{aligned}\tilde y_\mu &\equiv -\varepsilon_\alpha{}^\beta\int{\rm d}\xi^\alpha\partial_\beta y_\mu=\int({\rm d}\tau y'_\mu+{\rm d}\sigma\dot y_\mu),\\\mathcal{A}_\mu &\equiv \int{\rm d}\xi^\alpha\mathcal{A}_{\alpha\mu}\Delta(\sigma)=\int({\rm d}\tau\mathcal{A}_{0\mu}+{\rm d}\sigma\mathcal{A}_{1\mu})\Delta(\sigma),\\\tilde{\mathcal{A}}_\mu &\equiv -\varepsilon_\alpha{}^\beta\int{\rm d}\xi^\alpha\mathcal{A}_{\beta\mu}\Delta(\sigma)=\int({\rm d}\tau\mathcal{A}_{1\mu}+{\rm d}\sigma\mathcal{A}_{0\mu})\Delta(\sigma).\end{aligned}\tag{71}$$

The finite part V_0^μ and the infinitesimal one V_1^μ take a form

$$V_0^\mu=-\kappa\theta^{\mu\nu}y_\nu+G_E^{-1\,\mu\nu}\tilde y_\nu,\quad V_1^\mu=4\kappa\theta^{\mu\nu}\tilde{\mathcal{A}}_\nu-4G_E^{-1\,\mu\nu}\mathcal{A}_\nu.\tag{72}$$

We are going to substitute the solution (68) back into the action (57). First we calculate

$$\begin{aligned}v_+^\mu\Pi_{+\mu\nu}v_-^\nu &= -\frac{\kappa}{2}\partial_+y_\mu\theta_-^{\mu\nu}\partial_-y_\nu+2\kappa\mathcal{A}_{+\mu}(V)\theta_-^{\mu\nu}\partial_-y_\nu\Delta(\sigma)\\&\quad-2\kappa\partial_+y_\mu\theta_-^{\mu\nu}\mathcal{A}_{-\nu}(V)\Delta(\sigma)\\&\quad+8\kappa\mathcal{A}_{+\mu}(V)\theta_-^{\mu\nu}\mathcal{A}_{-\nu}(V)\Delta^2(\sigma),\end{aligned}\tag{73}$$

and

$$\begin{aligned}&\frac{1}{2}(v_+^\mu\partial_-y_\mu-\partial_+y_\mu v_-^\mu)\\&=\kappa\partial_+y_\mu\theta_-^{\mu\nu}\partial_-y_\nu-2\kappa\mathcal{A}_{+\mu}(V)\theta_-^{\mu\nu}\partial_-y_\nu\Delta(\sigma)\\&\quad+2\kappa\partial_+y_\mu\theta_-^{\mu\nu}\mathcal{A}_{-\nu}(V)\Delta(\sigma).\end{aligned}\tag{74}$$

Consequently, the first part of the T-dual action is

$$\begin{aligned}&v_+^\mu\Pi_{+\mu\nu}v_-^\nu+\frac{1}{2}(v_+^\mu\partial_-y_\mu-\partial_+y_\mu v_-^\mu)\\&=\frac{\kappa}{2}\partial_+y_\mu\theta_-^{\mu\nu}\partial_-y_\nu+8\kappa\mathcal{A}_{+\mu}(V)\theta_-^{\mu\nu}\mathcal{A}_{-\nu}(V)\Delta^2(\sigma).\end{aligned}\tag{75}$$

Substituting the solution (68) into the part of the action (57) with vector background fields, we have

$$\begin{aligned}&\big[A_a^N(V)(v_+^a+v_-^a)-2A_i^D(V)G^{-1ij}B_{jk}(v_+^k+v_-^k)+A_i^D(V)(v_+^i-v_-^i)\big]\Delta(\sigma)\\&=2\Big[A_i^D(V)G^{-1ij}\dot y_j-\frac{1}{\kappa}A_a^N(V)\,{}^\star\gamma_{(0)}^a(y)\Big]\Delta(\sigma)+4\kappa A_a^N(\theta_+^{ab}\mathcal{A}_{+b}-\theta_-^{ab}\mathcal{A}_{-b})\Delta^2(\sigma)-4A_i^DG^{-1ij}(\mathcal{A}_{+j}-\mathcal{A}_{-j})\Delta^2(\sigma),\end{aligned}\tag{76}$$

where ${}^\star\gamma^a_{(0)}(y)$ has been defined in Eq. (26). Similarly to the case of the initial theory, open string T-dual σ-momentum has an additional infinitesimal term proportional to the T-dual vector fields. We will neglect it because it is multiplied by another infinitesimal $A^N_a(V)$.

Since the vector fields are infinitesimal, we can neglect all terms bilinear in the vector fields which helps us to avoid trouble with $\Delta^2(\sigma)$. Consequently, the T-dual action takes the form

$$\begin{aligned}{}^\star S[y] \,=\, & \frac{\kappa^2}{2}\int \mathrm{d}^2\xi\ \partial_+ y_\mu \theta^{\mu\nu}_- \partial_- y_\nu + 2\kappa \int \mathrm{d}\tau \Big[\Big(A^D_i(V) G^{-1ij}\dot{y}_j \\ & -\frac{1}{\kappa} A^N_a(V)\,{}^\star\gamma^a_{(0)}(V)\Big)\Big/_{\sigma=\pi} \\ & -\Big(A^D_i(V) G^{-1ij}\dot{y}_j - \frac{1}{\kappa} A^N_a(V)\,{}^\star\gamma^a_{(0)}\Big)\Big/_{\sigma=0}(V)\Big]. \end{aligned} \tag{77}$$

3.3.3 T-dual background fields

Because T-dual action should have the same form as the initial one (45) but in terms of T-dual fields,

$$\begin{aligned}{}^\star S[y] \,=\, & \kappa \int \mathrm{d}^2\xi \partial_+ y_\mu\, {}^\star\Pi^{\mu\nu}_+ \partial_- y_\nu + 2\kappa \int \mathrm{d}\tau \Big[\Big({}^\star A^i_N(V)\dot{y}_i \\ & -\frac{1}{\kappa}\,{}^\star A^a_D(V)\,{}^\star G^{-1}_{ab}\,{}^\star\gamma^b_{(0)}(V)\Big)\Big/_{\sigma=\pi} \\ & -\Big({}^\star A^i_N(V)\dot{y}_i - \frac{1}{\kappa}\,{}^\star A^a_D(V)\,{}^\star G^{-1}_{ab}\,{}^\star\gamma^b_{(0)}(V)\Big)\Big/_{\sigma=0}\Big], \end{aligned} \tag{78}$$

we can express T-dual background fields in terms of the initial ones,

$$\begin{aligned} & {}^\star\Pi^{\mu\nu}_+ = \frac{\kappa}{2}\theta^{\mu\nu}_-, \qquad {}^\star A^a_D(V) = G^{-1ab}_E A^N_b(V), \\ & {}^\star A^i_N(V) = G^{-1ij} A^D_j(V). \end{aligned} \tag{79}$$

As one might expect, the T-dual metric and T-dual Kalb-Ramond fields remain the same as in the closed string case, Eq. (9).

With the help of last two relations we can find effective T-dual vector fields in analogy with first relation in Eq. (67) and first relation in Eq. (62),

$$\begin{aligned} {}^\star\mathcal{A}^a_\pm(V) = 2\,{}^\star\Pi^{ab}_\mp\,{}^\star G^{-1}_{bc}\,{}^\star A^c_D(V) = \kappa\theta^{ab}_\pm A^N_b(V), \\ {}^\star\mathcal{A}^i_\pm(V) = {}^\star A^i_N(V) = G^{-1ij} A^D_j(V). \end{aligned} \tag{80}$$

In analogy with second relation in Eq. (67), or from the previous relations, we have,

$$\begin{aligned} {}^\star\mathcal{A}^a_0(V) \,&=\, 2({}^\star B\,{}^\star G^{-1})^a{}_b\,{}^\star A^b_D(V) = \kappa\theta^{ab} A^N_b(V), \\ {}^\star\mathcal{A}^a_1(V) \,&=\, -{}^\star A^a_D(V) = -G^{-1ab}_E A^N_b(V), \\ {}^\star\mathcal{A}^i_0(V) \,&=\, {}^\star A^i_N(V) = G^{-1ij} A^D_j(V), \quad {}^\star\mathcal{A}^i_1(V) = 0. \end{aligned} \tag{81}$$

We introduced two effective T-dual vector fields ${}^\star\mathcal{A}^\mu_\alpha = \{{}^\star\mathcal{A}^\mu_0, {}^\star\mathcal{A}^\mu_1\}$ instead of the initial one ${}^\star A^\mu = \{{}^\star A^a_D, {}^\star A^i_N\}$, but we have two constraints,

$$\begin{aligned} {}^\star\mathcal{A}^a_0(V) = -2({}^\star B\,{}^\star G^{-1})^a{}_b\,{}^\star\mathcal{A}^b_1(V) = 2(G^{-1}B)^a{}_b\,{}^\star\mathcal{A}^b_1(V), \\ {}^\star\mathcal{A}^i_1(V) = 0. \end{aligned} \tag{82}$$

The first relation we can rewrite in the forms

$$\begin{aligned} {}^\star\Pi^{ab}_+\,{}^\star G^{-1}_{bc}\,{}^\star\mathcal{A}^c_+(V) \,&=\, {}^\star\Pi^{ab}_-\,{}^\star G^{-1}_{bc}\,{}^\star\mathcal{A}^c_-(V), \\ \Pi_{-ab}\,{}^\star\mathcal{A}^b_+(V) \,&=\, \Pi_{+ab}\,{}^\star\mathcal{A}^b_-(V). \end{aligned} \tag{83}$$

Let us make two observations. First, vector fields corresponding to Neumann (Dirichlet) boundary conditions of the initial theory $A^N_a(x)$ in front of $\dot{x}^a$ in Eq. (45) ($A^D_i(x)$ in front of $\gamma^{(0)}_i$ in Eq. (45)) after T-dualization turn to the fields corresponding to Dirichlet (Neumann) boundary conditions of the T-dual theory ${}^\star A^a_D(V)$ in front of ${}^\star\gamma^a_{(0)}$ in Eq. (78) (${}^\star A^i_N(V)$ in front of $\dot{y}_i$ in Eq. (78)). Therefore, T-duality interchanges Neumann with Dirichlet gauge fields. Second, the T-dual vector background fields depend not on y_μ but on the finite part of Eq. (72),

$$V^\mu \to V^\mu_0 = -\kappa\theta^{\mu\nu} y_\nu + G^{-1\mu\nu}_E \tilde{y}_\nu. \tag{84}$$

We can neglect the infinitesimal part V^μ_1 because it always appears in the argument of the vector background fields, with an infinitesimal coefficient. So it will produce the square of the vector fields, which we will neglect.

The variable $\tilde{y}_\mu$ naturally appears in Buscher's approach, as a part of variable V^μ when we perform T-dualization along coordinates on which background fields depend. Then we must introduce gauge invariant coordinates which are line integrals of the covariant derivatives. The corresponding argument of T-dual background fields V^μ is a solution of T-dual transformation laws and depends not only on y_μ, but is a linear combination of both y_μ and $\tilde{y}_\mu$. Therefore, the variable $V^\mu(y_\mu, \tilde{y})$, and not variable y_μ, is T-dual to x^μ.

The variable $\tilde{y}_\mu$ is defined in terms of y_μ, see Eq. (3.45), as a line integral of σ and τ derivatives of y_μ. It produces non-locality of the arguments of background fields, which in our formulation is the source of non-geometry. On the finite part of the equation of motion (a case that always happens) it does not depend on the integration path. Because it always appears as a part of variable V^μ, we can take for it the same boundary conditions as for variable y_μ. Then the variable V^μ has definite boundary conditions.

The variable $\tilde{y}_\mu$, as a part of V^μ, is significant because it distinguishes non-geometric from geometric theories. In the literature, these kinds of theories are recognized as theories with R-flux. Some authors refer to them as exotic configurations. I expect that just background field dependence on $V^\mu(y_\mu, \tilde{y}_\mu)$ is the source of these exotic non-geometric behavior. In fact, as was shown in Ref. [5], the presence of $\tilde{y}_\mu$ produces non-commutativity of the closed string variables and non-associativity.

Later, in Sections 5 and 6, we will see that $\tilde{y}_\mu$ has a central role in the definition of field strength for non-geometric theories, and is a basic variable for truly non-geometric theories.

Note that the T-dual of the T-dual produces the initial background fields. For example,

$$^{\star\star}A_a^N(x)=^{\star}G_{ab}^{-1\star}A_D^b(V)=G_{ab}^E G_E^{-1bc}A_c^N(x)=A_a^N(x),$$
$$^{\star\star}A_i^D(x)=^{\star}G_{Eij}^{-1\star}A_N^j(V)=G_{ij}G^{-1jk}A_k^D(x)=A_i^D(x). \quad (85)$$

From Eqs. (56) and (68) we can find the T-dual transformation law,

$$\partial_\pm x^\mu \cong -\kappa\theta_\pm^{\mu\nu}\partial_\pm y_\nu \pm 4\kappa\theta_\pm^{\mu\nu}\mathcal{A}_{\pm\nu}(V)\Delta(\sigma), \quad (86)$$

while its inverse is

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu \pm 4\mathcal{A}_{\pm\mu}(x)\Delta(\sigma). \quad (87)$$

In fact the last transformation can be obtained after T-dualization of the T-dual action (77). Both transformations differ from the closed string ones by the infinitesimal term which contains vector background fields $\mathcal{A}_{\pm\mu}$.

3.4 Relation with standard approach

There are significant differences between present and standard T-duality transformations of the open string. First, we are working with constant field strength (gauge field linear in compactified coordinates) while in the standard approach the field strength is zero (gauge field is independent of compactified coordinates). Second, and most important, in the present article both Neumann and Dirichlet gauge fields are introduced through the boundary coupling in the action: the Neumann one through coupling with $\dot{x}^a$ and the Dirichlet one through coupling with $\gamma_i^{(0)}$. As a difference of the standard approach they are treated in the same way. The Lagrangian treatment of Dirichlet gauge fields through the term $A_i^D G^{-1ij}\gamma_i^{(0)}$ has not previously been presented in the literature. We will see that the problem with the standard approach is that it misses such a Dirichlet part in the action. Third, which is a more technical difference, in the standard approach T-duality has been performed along one direction while in our approach it is performed along an arbitrary set of directions.

Let us start with the choice of auxiliary action. For discussion of this subsection it is useful to introduce T-dual coordinates y_a through the part of the action

$$\Delta S_{\rm aux}=\frac{\kappa}{2}\int_\Sigma \mathrm{d}^2\xi y_a F_{+-}^a=\frac{\kappa}{2}\int_\Sigma \mathrm{d}^2\xi y_a(\partial_+ v_-^a-\partial_- v_+^a), \quad (88)$$

which can be reexpressed as

$$\Delta S_{\rm aux}=\frac{\kappa}{2}\int_\Sigma \mathrm{d}^2\xi(v_+^a\partial_- y_a-\partial_+ y_a v_-^a)+\kappa\int_{\partial\Sigma}\mathrm{d}\tau y_a v_0^a. \quad (89)$$

The last term changes boundary part (59) in such a way that

$$A_a^N\to A_a^N+\frac{1}{2}y_a. \quad (90)$$

Note that according to the relation $\delta A_a^N=-\Lambda_a$, this additional term is just a gauge transformation with $\Lambda_a=-\frac{1}{2}y_a$. So, up to gauge transformation this choice of auxiliary action is equivalent to the previous one (52).

Let us for the sake of discussion preserve this boundary term (not gauge it away). To understand the relationship between ours and previous approaches [25, 26], let us reduce our case to the standard one. We will suppose that the field strengths for both Neumann and Dirichlet gauge fields are zero, which means that these fields are constant. Finally, because this is the first time Lagrangian treatment of Dirichlet gauge fields has been performed, we should put the Dirichlet field of the initial action equal to zero, $A_i^D=0$.

The T-dual action must have the same form as the initial one but in terms of T-dual fields. According to Eq. (79) and taking into account the new term (90), we obtain an expression for T-dual vector fields,

$$^{\star}A_D^a=G_E^{-1ab}\left(A_b^N+\frac{1}{2}y_b\right), \quad ^{\star}A_N^i=G^{-1ij}A_j^D. \quad (91)$$

Because the standard approach started with $A_a^N=$const and $A_i^D=0$, it produces

$$^{\star}A_D^a=G_E^{-1ab}(A_b^N+\frac{1}{2}y_b), \quad ^{\star}A_N^i=0. \quad (92)$$

Let us stress that in the standard approach, T-dual action does not contain either Neumann or Dirichlet vector fields. Generally, a term with T-dual Neumann fields $^{\star}A_N^i$ could be recognized as a coefficient in front of $\dot{y}_i$, but according to the last relation it is zero. This is a consequence of the fact that one did not know how to include initial Dirichlet fields and had to put it to zero. It follows that the T-dual Neumann field is also zero.

In the standard approach one cannot recognize the T-dual Dirichlet vector field $^{\star}A_D^a$, which according to the present paper should be in front of $^{\star}\gamma_{(0)}^\mu(y)$. So, for consistency of the standard approach, one should require that it vanishes, $^{\star}A_D^a=0$. According to Eq. (92), this is in fact the Dirichlet boundary condition of the standard approach, $y_a=-2A_a^N$.

Consequently, the problem of the standard approach, which has been solved in the present article, is ignorance in introducing the Dirichlet background field in both initial and T-dual Lagrangians. It means that the consistency of the standard approach $^{\star}A_D^a=0$ produces $y_a=-2A_a^N$. This has the interpretation of Dirichlet boundary conditions for T-dual theory. This is an external condition which has not been obtained from the Lagrangian. If we perform T-duality along one direction (let us say $a=1$), than we obtain the well known result of the standard approach, $y=-2A^N$.

4 T-duality in terms of field strengths

In the previous section we investigated the T-duality of the vector fields. In the initial (geometric) the-

ory we considered gauge fields linear in coordinates x^μ. We obtained that the gauge fields of the T-dual (non-geometric) theory are linear in the new variable V^μ, which is a function of the T-dual coordinate y_μ and its double $\tilde{y}_\mu$.

Generally, it is not clear how to define the field strength for non-geometric theories. So we will go a roundabout way. It is known that in geometric theories, if both ends of the open string are attached to the same Dp-brane, the term in the action which contains the vector background field with integration over τ can be transformed to the term in the action which contains corresponding field strength with integration over $d^2\xi=d\tau d\sigma$. We are going to generalize such a relation to non-geometric theories.

4.1 Field strengths of initial theory

After some direct calculation for Neumann vector fields we obtain

$$\begin{aligned} S^N_A[x] &= 2\kappa\int \mathrm{d}\tau\Big[\Big(A^N_a[x]\dot{x}^a\Big)/_{\sigma=\pi}-\Big(A^N_a[x]\dot{x}^a\Big)/_{\sigma=0}\Big], \\ &= -2\kappa\int \mathrm{d}^2\xi\dot{x}^a\mathcal{F}_{ab}x'^b=\kappa\int \mathrm{d}^2\xi\partial_+x^a\mathcal{F}_{ab}\partial_-x^b, \end{aligned} \tag{93}$$

where only the antisymmetric part contributes,

$$\begin{aligned} \mathcal{F}_{ab} &= \mathcal{F}^{(a)}_{ab}=F^{(a)}_{ab}=\partial_aA^N_b(x)-\partial_bA^N_a(x) \\ &= \partial_a\mathcal{A}_{0b}(x)-\partial_b\mathcal{A}_{0a}(x), \qquad \mathcal{F}^{(s)}_{ab}=0. \end{aligned} \tag{94}$$

We can trivially reexpress Eq. (93) in a form where the effective background vector field $\mathcal{A}_{0a}$ is multiplied with $\dot{x}^a$,

$$S^N_A[x]=2\kappa\int \mathrm{d}\tau\Big(\mathcal{A}_{0a}[x]\dot{x}^a/_{\sigma=\pi}-\mathcal{A}_{0a}[x]\dot{x}^a/_{\sigma=0}\Big). \tag{95}$$

We introduce the names *effective background vector field* $\mathcal{A}_a$ and corresponding *effective field strength* $\mathcal{F}_{ab}$ for variables obtained in this way. In this simplest case we have a standard picture: one effective vector field $\mathcal{A}_{0a}=A^N_a$ and corresponding antisymmetric effective field strength $\mathcal{F}_{ab}=\mathcal{F}^{(a)}_{ab}$. In the next cases the situation will be more complicated.

Instead of Eq. (94) we can also accept the relation (93) as definitions of the field strength for geometric theories. Let us extend this definition to non-standard theories. Unlike Neumann vector fields, which are coupled with $\dot{x}^a$, Dirichlet vector fields are coupled with σ-momentum $\gamma^{(0)}_j(x)$, which will produce additional problems. Using the finite part of the equation of motion $\ddot{x}^i=x''^i$, we have

$$\begin{aligned} S^D_A[x] &= 2\kappa\int \mathrm{d}\tau\Big[\Big(-\frac{1}{\kappa}A^D_i[x]G^{-1ij}\gamma^{(0)}_j(x)\Big)/_{\sigma=\pi} \\ &\quad -\Big(-\frac{1}{\kappa}A^D_i[x]G^{-1ij}\gamma^{(0)}_j(x)\Big)/_{\sigma=0}\Big], \\ &= \kappa\int \mathrm{d}^2\xi\partial_+x^i\mathcal{F}_{ij}\partial_-x^j. \end{aligned} \tag{96}$$

Now, both symmetric and antisymmetric parts contribute

$$\mathcal{F}_{ij}=\mathcal{F}^{(a)}_{ij}+\frac{1}{2}\mathcal{F}^{(s)}_{ij}, \tag{97}$$

where

$$\begin{aligned} \mathcal{F}^{(a)}_{ij} &= \Big[\partial_i\Big(2B_{jk}G^{-1kq}A^D_q\Big)-\partial_j\Big(2B_{ik}G^{-1kq}A^D_q\Big)\Big] \\ &= \frac{1}{2}\Big(B_{ik}G^{-1kq}F^{(s)}_{qj}+F^{(s)}_{ik}G^{-1kq}B_{qj}\Big) \\ &= F^{(a)}_{ij}=\partial_i\mathcal{A}_{0j}(x)-\partial_j\mathcal{A}_{0i}(x), \end{aligned} \tag{98}$$

and

$$\mathcal{F}^{(s)}_{ij}=-2(\partial_iA^D_j+\partial_jA^D_i)=F^{(s)}_{ij}=2\Big(\partial_i\mathcal{A}_{1j}(x)+\partial_j\mathcal{A}_{1i}(x)\Big). \tag{99}$$

For the Dirichlet sector, an analogy with the standard approach does not exist. In that case both components of the effective background vector field, $\mathcal{A}_{0i}$ and $\mathcal{A}_{1i}$, as well as both $\dot{x}^i$ and x'^i, contribute. So, we can reexpress Eq. (96) as

$$\begin{aligned} S^D_A[x] &= 2\kappa\int \mathrm{d}\tau\Big[\Big(\mathcal{A}_{0i}[x]\dot{x}^i-\mathcal{A}_{1i}[x]x'^i\Big)/_{\sigma=\pi} \\ &\quad -\Big(\mathcal{A}_{0i}[x]\dot{x}^i-\mathcal{A}_{1i}[x]x'^i\Big)/_{\sigma=0}\Big], \\ &= 2\kappa\eta^{\alpha\beta}\int \mathrm{d}\tau\Big(\mathcal{A}_{\alpha i}[x]\partial_\beta x^i/_{\sigma=\pi}-\mathcal{A}_{\alpha i}[x]\partial_\beta x^i/_{\sigma=0}\Big) \\ &= 2\kappa\int \mathrm{d}\tau\Big(\mathcal{A}_{\alpha i}[x]\dot{x}^{\alpha i}/_{\sigma=\pi}-\mathcal{A}_{\alpha i}[x]\dot{x}^{\alpha i}/_{\sigma=0}\Big), \end{aligned} \tag{100}$$

where $\mathcal{A}_{0i}[x]$ has been defined in Eq. (67), $\mathcal{A}_{1i}[x]$ in Eq. (63), and we introduced the notation $\dot{x}^{\alpha i}=\{\dot{x}^i,-x'^i\}=\eta^{\alpha\beta}\partial_\beta x^i$.

Note that, although we work with an initial theory, this action does depend on $\mathcal{A}_{1i}[x]$ and the vector field couples not only with $\dot{x}^i$ but also with x'^i. This is consequence of the fact that the original vector field $A^D_i(x)$ is not multiplied by $\dot{x}^i$ but by σ-momentum $G^{-1ij}\gamma^{(0)}_j(x)$.

4.2 Field strengths of T-dual theory

The case with T-dual theory is more complicated because the vector fields depend on V^μ, which is a function of two variables, y_μ and $\tilde{y}_\mu$.

4.2.1 The case of Dirichlet vector fields

For Dirichlet vector fields, with the help of the finite part of the equation of motion $\ddot{y}_a=y''_a$, we find

$$\begin{aligned}{}^\star S^D_A[y] &= 2\kappa\int \mathrm{d}\tau\Big[\Big(-\frac{1}{\kappa}{}^\star A^a_D(V){}^\star G^{-1}_{ab}{}^\star\gamma^b_{(0)}(y)\Big)/_{\sigma=\pi}\\ &\quad -\Big(-\frac{1}{\kappa}{}^\star A^a_D(V){}^\star G^{-1}_{ab}{}^\star\gamma^b_{(0)}(y)\Big)/_{\sigma=0}\Big],\\ &= \kappa\int \mathrm{d}^2\xi\partial_+ y_a{}^\star\mathcal{F}^{ab}\partial_- y_b. \end{aligned}\tag{101}$$

Here we have

$${}^\star\mathcal{F}^{ab}={}^\star\mathcal{F}^{ab}_{(a)}+\frac{1}{2}{}^\star\mathcal{F}^{ab}_{(s)},\tag{102}$$

where the antisymmetric part,

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(a)} &= \kappa\big({}^\star G_E{}^\star\theta{}^\star F{}^\star B-{}^\star B{}^\star F^T{}^\star\theta{}^\star G_E\big)^{ab}\\ &\quad -\frac{1}{2}({}^\star F^a{}_c{}^\star G^{cb}-{}^\star G^{ac}{}^\star F^T{}_c{}^b)\\ &= 2{}^\star B^{ac}({}^\star F^T{}^\star G^{-1}-{}^\star G^{-1}{}^\star F)_{cd}{}^\star B^{db}\\ &\quad -\frac{1}{2}({}^\star F^a{}_c{}^\star G^{cb}-{}^\star G^{ac}{}^\star F^T{}_c{}^b), \end{aligned}\tag{103}$$

and symmetric part,

$${}^\star\mathcal{F}^{ab}_{(s)}=-4({}^\star F{}^\star B+{}^\star B{}^\star G^{-1}{}^\star F{}^\star G)^{ab},\tag{104}$$

are expressed in terms of coefficient ${}^\star F^a{}_b$, defined with the relation

$${}^\star A^a_D(V)={}^\star A^a_0-\frac{1}{2}{}^\star F^a{}_b V^b.\tag{105}$$

Taking into account that with the help of Eq. (79), from the first relation in Eq. (46) and Eq. (105) we have ${}^\star F^a{}_b=G^{-1ac}_E F_{cb}$, and it follows that:

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(a)} &= -\kappa^2\theta^{ac}F^{(a)}_{cd}\theta^{db}-G^{-1ac}_E F^{(a)}_{cd}G^{-1db}_E\\ &= -\frac{\kappa^2}{2}\Big[\theta^{ac}_+F^{(a)}_{cd}\theta^{db}_++\theta^{ac}_-F^{(a)}_{cd}\theta^{db}_-\Big], \end{aligned}\tag{106}$$

and

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(s)} &= -2\kappa\Big[G^{-1ac}_E F^{(a)}_{cd}\theta^{db}+\theta^{ac}F^{(a)}_{cd}G^{-1db}_E\Big]\\ &= \kappa^2\Big[\theta^{ac}_+F^{(a)}_{cd}\theta^{db}_+-\theta^{ac}_-F^{(a)}_{cd}\theta^{db}_-\Big]. \end{aligned}\tag{107}$$

Note that neither of these depend on the symmetric part $F^{(s)}_{ab}$. Because, according to Eq. (94) $\mathcal{F}^{(a)}_{ab}=F^{(a)}_{ab}$, we can rewrite the above equations in terms of effective field strength $\mathcal{F}^{(a)}_{ab}$,

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(a)} &= -\kappa^2\theta^{ac}\mathcal{F}^{(a)}_{cd}\theta^{db}-G^{-1ac}_E\mathcal{F}^{(a)}_{cd}G^{-1db}_E\\ &= -\frac{\kappa^2}{2}\Big[\theta^{ac}_+\mathcal{F}^{(a)}_{cd}\theta^{db}_++\theta^{ac}_-\mathcal{F}^{(a)}_{cd}\theta^{db}_-\Big], \end{aligned}\tag{108}$$

and

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(s)} &= -2\kappa\Big[G^{-1ac}_E\mathcal{F}^{(a)}_{cd}\theta^{db}+\theta^{ac}\mathcal{F}^{(a)}_{cd}G^{-1db}_E\Big]\\ &= \kappa^2\Big[\theta^{ac}_+\mathcal{F}^{(a)}_{cd}\theta^{db}_+-\theta^{ac}_-\mathcal{F}^{(a)}_{cd}\theta^{db}_-\Big]. \end{aligned}\tag{109}$$

In the Dirichlet sector of T-dual theory we can re-express the term ${}^\star S^D_A[y]$ in the "standard" form where the effective vector fields ${}^\star\mathcal{A}^a_\alpha$ are multiplied by the $\dot{y}^\alpha_a=\{\dot{y}_a,-y'_a\}=\eta^{\alpha\beta}\partial_\beta y_a$, so that the term with vector background fields takes the form

$$\begin{aligned}{}^\star S^D_A[y] &= 2\kappa\eta^{\alpha\beta}\int \mathrm{d}\tau\Big({}^\star\mathcal{A}^a_\alpha[V]\partial_\beta y_a/_{\sigma=\pi}-{}^\star\mathcal{A}^a_\alpha[V]\partial_\beta y_a/_{\sigma=0}\Big)\\ &= 2\kappa\int \mathrm{d}\tau\Big({}^\star\mathcal{A}^a_\alpha[V]\dot{y}^\alpha_a/_{\sigma=\pi}-{}^\star\mathcal{A}^a_\alpha[V]\dot{y}^\alpha_a/_{\sigma=0}\Big), \end{aligned}\tag{110}$$

where ${}^\star\mathcal{A}^a_0[V]$ and ${}^\star\mathcal{A}^a_1[V]$ have been defined in Eq. (81).

4.2.2 The case of Neumann vector fields

For the T-dual case, corresponding to the Neumann vector field $A^i_N(V)$, we define the field strength

$${}^\star\mathcal{F}^{ij}={}^\star\mathcal{F}^{ij}_{(a)}+\frac{1}{2}{}^\star\mathcal{F}^{ij}_{(s)},\tag{111}$$

with the relation

$$\begin{aligned}{}^\star S^N_A[y] &= 2\kappa\int \mathrm{d}\tau\Big[\Big({}^\star A^i_N(V)\dot{y}_i\Big)/_{\sigma=\pi}-\Big({}^\star A^i_N(V)\dot{y}_i\Big)/_{\sigma=0}\Big]\\ &= \kappa\int \mathrm{d}^2\xi\partial_+ y_i{}^\star\mathcal{F}^{ij}\partial_- y_j. \end{aligned}\tag{112}$$

After some calculations we obtain

$${}^\star\mathcal{F}^{ij}_{(a)} = -{}^\star B^{ik}({}^\star F^T)_k{}^j-{}^\star F^i{}_k{}^\star B^{kj},\tag{113}$$

and

$${}^\star\mathcal{F}^{ij}_{(s)} = -{}^\star F^i{}_k{}^\star G^{kj}-{}^\star G^{ik}({}^\star F^T)_k{}^j,\tag{114}$$

where the coefficient ${}^\star F^i{}_j$ is defined as

$${}^\star A^i_N(V)={}^\star A^i_0-\frac{1}{2}{}^\star F^i{}_j V^j.\tag{115}$$

Using second equation in Eq. (46), Eq. (79) and Eq. (115), we obtain ${}^\star F^i{}_j=G^{-1ik}F^{(s)}_{kj}$ and consequently,

$${}^\star\mathcal{F}^{ij}_{(a)}=-\frac{\kappa}{4}\Big(\theta^{ik}F^{(s)}_{kq}G^{-1qj}+G^{-1ik}F^{(s)}_{kq}\theta^{qj}\Big),\tag{116}$$

and

$${}^\star\mathcal{F}^{ij}_{(s)}=-\frac{1}{2}\Big(G^{-1ik}_E F^{(s)}_{kq}G^{-1qj}+G^{-1ik}F^{(s)}_{kq}G^{-1qj}_E\Big).\tag{117}$$

We can eliminate $F^{(a)}_{ij}$ and $F^{(s)}_{ij}$ from Eqs. (98), (99) and (116) and express ${}^\star\mathcal{F}^{ij}_{(a)}$ in terms of $\mathcal{F}^{(a)}_{ij}$ and $\mathcal{F}^{(s)}_{ij}$. This is not a direct calculation, but we can check that expression

$$\begin{aligned}{}^\star\mathcal{F}^{ij}_{(a)} &= -\kappa^2\theta^{ik}\mathcal{F}^{(a)}_{kq}\theta^{qj}-G^{-1ik}_E\mathcal{F}^{(a)}_{kq}G^{-1qj}_E\\ &\quad -\frac{\kappa}{2}\Big(G^{-1ik}_E\mathcal{F}^{(s)}_{kq}\theta^{qj}+\theta^{ik}\mathcal{F}^{(s)}_{kq}G^{-1qj}_E\Big), \end{aligned}\tag{118}$$

is a proper solution. Similarly, we can eliminate the same variables $F^{(a)}_{ij}$ and $F^{(s)}_{ij}$ from Eqs. (98), (99) and (117) and express ${}^\star\mathcal{F}^{ij}_{(s)}$ in terms of $\mathcal{F}^{(a)}_{ij}$ and $\mathcal{F}^{(s)}_{ij}$:

$$\begin{aligned}{}^\star\mathcal{F}^{ij}_{(s)} &= -\kappa^2\theta^{ik}\mathcal{F}^{(s)}_{kq}\theta^{qj}-G^{-1ik}_E\mathcal{F}^{(s)}_{kq}G^{-1qj}_E\\ &\quad -2\kappa\Big(G^{-1ik}_E\mathcal{F}^{(a)}_{kq}\theta^{qj}+\theta^{ik}\mathcal{F}^{(a)}_{kq}G^{-1qj}_E\Big). \quad (119)\end{aligned}$$

Similarly, in the Neumann sector of T-dual theory we can reexpress the term ${}^\star S^N_A[y]$ in the form

$$\begin{aligned}{}^\star S^N_A[y] &= 2\kappa\eta^{\alpha\beta}\int\mathrm{d}\tau\Big({}^\star\mathcal{A}^i_\alpha[V]\partial_\beta y_i/_{\sigma=\pi}-{}^\star\mathcal{A}^i_\alpha[V]\partial_\beta y_i/_{\sigma=0}\Big)\\ &= 2\kappa\int\mathrm{d}\tau\Big({}^\star\mathcal{A}^i_\alpha[V]\dot y^\alpha_i/_{\sigma=\pi}-{}^\star\mathcal{A}^i_\alpha[V]\dot y^\alpha_i/_{\sigma=0}\Big), \quad (120)\end{aligned}$$

where the effective vector fields ${}^\star\mathcal{A}^i_\alpha[V] = \{{}^\star\mathcal{A}^i_0[V],{}^\star\mathcal{A}^i_1[V]\}$ introduced in Eq. (81) are multiplied by $\dot y^\alpha_i=\{\dot y_i,-y'_i\}=\eta^{\alpha\beta}\partial_\beta y_i$. We have put it in the suggestive form of Eq. (110) although it is much simpler. Because, according to Eq. (81), ${}^\star\mathcal{A}^i_1[V]=0$, it has a form

$${}^\star S^N_A[y]=2\kappa\int\mathrm{d}\tau\Big[\Big({}^\star\mathcal{A}^i_0(V)\dot y_i\Big)/_{\sigma=\pi}-\Big({}^\star\mathcal{A}^i_0(V)\dot y_i\Big)/_{\sigma=0}\Big]. \quad (121)$$

4.3 T-dual field strength in terms of initial one

Let us introduce the complete field strengths

$$\mathcal{F}_{\mu\nu}=\mathcal{F}^{(a)}_{\mu\nu}+\frac{1}{2}\mathcal{F}^{(s)}_{\mu\nu},\ \mathcal{F}^{(a)}_{\mu\nu}=\begin{pmatrix}\mathcal{F}^{(a)}_{ab} & 0\\ 0 & \mathcal{F}^{(a)}_{ij}\end{pmatrix},\ \mathcal{F}^{(s)}_{\mu\nu}=\begin{pmatrix}0 & 0\\ 0 & \mathcal{F}^{(s)}_{ij}\end{pmatrix}, \quad (122)$$

which contain the Neumann parts $\mathcal{F}^{(a)}_{ab}$ defined in Eq. (94), as well as the Dirichlet ones $\mathcal{F}^{(a)}_{ij}$ and $\mathcal{F}^{(s)}_{ij}$ defined in Eqs. (98) and (99). Then, taking into account that according to Eq. (65) $\mathcal{A}_{1a}(x)=0$, we can rewrite the action with vector background fields as:

$$\begin{aligned}S_A(x) &= S^N_A(x)+S^D_A(x)\\ &= 2\kappa\eta^{\alpha\beta}\int\mathrm{d}\tau\Big(\mathcal{A}_{\alpha\mu}[x]\partial_\beta x^\mu/_{\sigma=\pi}-\mathcal{A}_{\alpha\mu}[x]\partial_\beta x^\mu/_{\sigma=0}\Big)\\ &= \kappa\int\mathrm{d}^2\xi\partial_+x^\mu\mathcal{F}_{\mu\nu}\partial_-x^\nu, \quad (123)\end{aligned}$$

where the expressions for the terms $S^N_A(x)$ and $S^D_A(x)$ have been defined in Eqs. (95) and (100). Note that $S_A(x)$ has the same form as the initial action and according to Eq. (122) contains both symmetric and antisymmetric parts. So, the other way to introduce vector background fields is to substitute the Kalb-Ramond field $B_{\mu\nu}$ and metric $G_{\mu\nu}$ with

$$B_{\mu\nu}\to\mathcal{B}_{\mu\nu}=B_{\mu\nu}+\mathcal{F}^{(a)}_{\mu\nu},\quad G_{\mu\nu}\to\mathcal{G}_{\mu\nu}=G_{\mu\nu}+\mathcal{F}^{(s)}_{\mu\nu}. \quad (124)$$

Note that according to Eq. (43) the new variables are just gauge invariant ones. Then the open string action takes the form

$$S_{\rm open}=\kappa\int\mathrm{d}^2\xi\partial_+x^\mu\Big(\mathcal{B}_{\mu\nu}+\frac{1}{2}\mathcal{G}_{\mu\nu}\Big)\partial_-x^\nu. \quad (125)$$

Because all background fields in this action are constant, we already know the form of T-dual fields for such an action. In analogy with Eq. (12) we have

$${}^\star\mathcal{G}^{\mu\nu}=(\mathcal{G}^{-1}_E)^{\mu\nu},\quad {}^\star\mathcal{B}^{\mu\nu}=\frac{\kappa}{2}\Theta^{\mu\nu}, \quad (126)$$

where according to Eq. (162),

$$\mathcal{G}^E_{\mu\nu}\equiv\mathcal{G}_{\mu\nu}-4(\mathcal{B}\mathcal{G}^{-1}\mathcal{B})_{\mu\nu},\quad \Theta^{\mu\nu}\equiv-\frac{2}{\kappa}(\mathcal{G}^{-1}_E\mathcal{B}\mathcal{G}^{-1})^{\mu\nu}. \quad (127)$$

Taking into account that the vector background fields and consequently their field strengths are infinitesimal, we can separate the infinitesimal part of ${}^\star\mathcal{B}^{\mu\nu}$,

$$\begin{aligned}{}^\star\mathcal{F}^{\mu\nu}_{(a)} &= -G^{-1\mu\rho}_E\mathcal{F}^{(a)}_{\rho\sigma}G^{-1\sigma\nu}_E-\kappa^2\theta^{\mu\rho}\mathcal{F}^{(a)}_{\rho\sigma}\theta^{\sigma\nu}\\ &\quad -\frac{\kappa}{2}\big(G^{-1\mu\rho}_E\mathcal{F}^{(s)}_{\rho\sigma}\theta^{\sigma\nu}+\theta^{\mu\rho}\mathcal{F}^{(s)}_{\rho\sigma}G^{-1\sigma\nu}_E\big)\\ &= -\frac{\kappa^2}{2}\big(\theta^{\mu\rho}_+\mathcal{F}^{(a)}_{\rho\sigma}\theta^{\sigma\nu}_++\theta^{\mu\rho}_-\mathcal{F}^{(a)}_{\rho\sigma}\theta^{\sigma\nu}_-\big)\\ &\quad +\frac{\kappa^2}{4}\big(\theta^{\mu\rho}_+\mathcal{F}^{(s)}_{\rho\sigma}\theta^{\sigma\nu}_+-\theta^{\mu\rho}_-\mathcal{F}^{(s)}_{\rho\sigma}\theta^{\sigma\nu}_-\big). \quad (128)\end{aligned}$$

and the infinitesimal part of ${}^\star\mathcal{G}^{\mu\nu}$,

$$\begin{aligned}{}^\star\mathcal{F}^{\mu\nu}_{(s)} &= -G^{-1\mu\rho}_E\mathcal{F}^{(s)}_{\rho\sigma}G^{-1\sigma\nu}_E-\kappa^2\theta^{\mu\rho}\mathcal{F}^{(s)}_{\rho\sigma}\theta^{\sigma\nu}\\ &\quad -2\kappa\big(G^{-1\mu\rho}_E\mathcal{F}^{(a)}_{\rho\sigma}\theta^{\sigma\nu}+\theta^{\mu\rho}\mathcal{F}^{(a)}_{\rho\sigma}G^{-1\sigma\nu}_E\big)\\ &= -\frac{\kappa^2}{2}\big(\theta^{\mu\rho}_+\mathcal{F}^{(s)}_{\rho\sigma}\theta^{\sigma\nu}_++\theta^{\mu\rho}_-\mathcal{F}^{(s)}_{\rho\sigma}\theta^{\sigma\nu}_-\big)\\ &\quad +\kappa^2\big(\theta^{\mu\rho}_+\mathcal{F}^{(a)}_{\rho\sigma}\theta^{\sigma\nu}_+-\theta^{\mu\rho}_-\mathcal{F}^{(a)}_{\rho\sigma}\theta^{\sigma\nu}_-\big). \quad (129)\end{aligned}$$

Here $\theta^{\mu\nu}_\pm$ has been defined in Eq. (163) and the complete T-dual field strengths

$$\begin{aligned}{}^\star\mathcal{F}^{\mu\nu} &= {}^\star\mathcal{F}^{\mu\nu}_{(a)}+\frac{1}{2}{}^\star\mathcal{F}^{\mu\nu}_{(s)},\\ {}^\star\mathcal{F}^{\mu\nu}_{(a)} &= \begin{pmatrix}{}^\star\mathcal{F}^{ab}_{(a)} & 0\\ 0 & {}^\star\mathcal{F}^{ij}_{(a)}\end{pmatrix}, \quad (130)\\ {}^\star\mathcal{F}^{\mu\nu}_{(s)} &= \begin{pmatrix}{}^\star\mathcal{F}^{ab}_{(s)} & 0\\ 0 & {}^\star\mathcal{F}^{ij}_{(s)}\end{pmatrix},\end{aligned}$$

contain Neumann parts ${}^\star\mathcal{F}^{ab}_{(a)}$ and ${}^\star\mathcal{F}^{ab}_{(s)}$ as well as Dirichlet ones ${}^\star\mathcal{F}^{ij}_{(a)}$ and ${}^\star\mathcal{F}^{ij}_{(s)}$. Therefore, taking into account that $\mathcal{F}^{(s)}_{ab}=0$, Eq. (128) is in complete agreement with Eq. (108) and Eq. (118), as well as Eq.(129) being in complete agreement with Eqs. (109) and (119).

5 Field strength for non-geometric theories

In our approach, a characteristic feature of non-geometric theories are background dependence on vari-

able V^μ, which includes dependence on both T-dual coordinate y_μ and its double $\tilde{y}_\mu$. In Refs. [7] and [8] it was shown that V^μ-dependence produces non-commutativity and non-associativity of the closed string coordinates. It is also the origin of difficulties in defining field strength for non-geometric theories. In this section we offer a solution to this problem and define the field strength of non-geometric theories which include derivation with respect to both y_μ and $\tilde{y}_\mu$.

In geometric theories the field strength for an Abelian vector field $\mathcal{A}_\mu(x)$ is defined simply as $\mathcal{F}_{\mu\nu}=\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu$, where with ∂_μ we denote derivation with respect to the variable x^μ. In non-geometric theories the vector field $\mathcal{A}_\mu(V)$ depends on $V^\mu=-\kappa\theta^{\mu\nu}y_\nu+G_E^{-1\mu\nu}\tilde{y}_\nu$, which means that it depends on two variables y_μ and $\tilde{y}_\mu$. Generally speaking, in order to obtain the field strength for non-geometric theories, we should have the derivative with respect to both variables y_μ and $\tilde{y}_\mu$.

5.1 Non-geometric field strengths in terms of effective gauge fields

For the initial theory, according to Eqs. (94), (98) and (99) and taking into account that $\mathcal{A}_{1a}=0$, we have

$$\begin{aligned}\mathcal{F}^{(a)}_{\mu\nu} &= \partial_\mu\mathcal{A}_{0\nu}(x)-\partial_\nu\mathcal{A}_{0\mu}(x),\\ \mathcal{F}^{(s)}_{\mu\nu} &= 2\Big[\partial_\mu\mathcal{A}_{1\nu}(x)+\partial_\nu\mathcal{A}_{1\mu}(x)\Big].\end{aligned}\tag{131}$$

The antisymmetric part has a standard form, but we also obtain a non-trivial symmetric part.

Let us consider the field strengths of T-dual non-geometric theories. Until now we obtained the complete expressions for T-dual field strengths ${}^\star\mathcal{F}^{\mu\nu}$ for non-geometric theories. The next step is to write out these expressions in terms of derivatives of T-dual gauge fields ${}^\star\mathcal{A}^a_0(V)$ and ${}^\star\mathcal{A}^a_1(V)$ with respect to variables y_μ and $\tilde{y}_\mu$, and find local gauge symmetries in such cases.

With the help of the first line in Eq. (81) and Eq. (84) we find the antisymmetric parts,

$$\begin{aligned}{}^\star_y\mathcal{F}^{ab}_{0(a)} &\equiv \partial^a_y\,{}^\star\mathcal{A}^b_0(V)-\partial^b_y\,{}^\star\mathcal{A}^a_0(V)=-\kappa^2(\theta F_{(a)}\theta)^{ab},\\ [1mm]^\star_{\tilde{y}}\mathcal{F}^{ab}_{1(a)} &\equiv \partial^a_{\tilde{y}}\,{}^\star\mathcal{A}^b_1(V)-\partial^b_{\tilde{y}}\,{}^\star\mathcal{A}^a_1(V)=-(G_E^{-1}F_{(a)}G_E^{-1})^{ab},\end{aligned}\tag{132}$$

and symmetric parts,

$$\begin{aligned}{}^\star_y\mathcal{F}^{ab}_{1(s)} &\equiv \partial^a_y\,{}^\star\mathcal{A}^b_1(V)+\partial^b_y\,{}^\star\mathcal{A}^a_1(V)\\ [1mm] &= -\frac{\kappa}{2}(G_E^{-1}F_{(a)}\theta+\theta F_{(a)}G_E^{-1})^{ab},\\ [1mm]^\star_{\tilde{y}}\mathcal{F}^{ab}_{0(s)} &\equiv \partial^a_{\tilde{y}}\,{}^\star\mathcal{A}^b_0(V)+\partial^b_{\tilde{y}}\,{}^\star\mathcal{A}^a_0(V)\\ [1mm] &= -\frac{\kappa}{2}(G_E^{-1}F_{(a)}\theta+\theta F_{(a)}G_E^{-1})^{ab},\end{aligned}\tag{133}$$

where with ∂^a_y and $\partial^a_{\tilde{y}}$ we denote partial derivations with respect to y_a and $\tilde{y}_a$.

With the help of the last line in Eq. (81) and Eq. (84), for the ij sector, we have

$$\begin{aligned}{}^\star_y\mathcal{F}^{ij}_{0(a)} &\equiv \partial^i_y\,{}^\star\mathcal{A}^j_0(V)-\partial^j_y\,{}^\star\mathcal{A}^i_0(V)\\ [1mm] &= -\frac{\kappa}{4}(G^{-1}F_{(s)}\theta+\theta F_{(s)}G^{-1})^{ij},\\ [1mm]^\star_{\tilde{y}}\mathcal{F}^{ij}_{0(s)} &\equiv \partial^i_{\tilde{y}}\,{}^\star\mathcal{A}^j_0(V)+\partial^j_{\tilde{y}}\,{}^\star\mathcal{A}^i_0(V)\\ [1mm] &= -\frac{1}{4}(G^{-1}F_{(s)}G_E^{-1}+G_E^{-1}F_{(s)}G^{-1})^{ij}.\end{aligned}\tag{134}$$

Because according to Eq. (81), ${}^\star\mathcal{A}^i_1(V)=0$, all corresponding field strengths (both symmetric and antisymmetric parts) vanish, ${}^\star_y\mathcal{F}^{ij}_1=0={}^\star_{\tilde{y}}\mathcal{F}^{ij}_1$.

Comparing Eq. (132) with Eq. (106) we find

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(a)} &= {}^\star_y\mathcal{F}^{ab}_{0(a)}+{}^\star_{\tilde{y}}\mathcal{F}^{ab}_{1(a)}\\ [1mm] &= \partial^a_y\,{}^\star\mathcal{A}^b_0(V)-\partial^b_y\,{}^\star\mathcal{A}^a_0(V)+\partial^a_{\tilde{y}}\,{}^\star\mathcal{A}^b_1(V)-\partial^b_{\tilde{y}}\,{}^\star\mathcal{A}^a_1(V).\end{aligned}\tag{135}$$

Similarly, comparing Eq. (133) with Eq. (107) we have

$$\begin{aligned}{}^\star\mathcal{F}^{ab}_{(s)} &= 2\big({}^\star_{\tilde{y}}\mathcal{F}^{ab}_{0(s)}+{}^\star_y\mathcal{F}^{ab}_{1(s)}\big)\\ [1mm] &= 2\big(\partial^a_{\tilde{y}}\,{}^\star\mathcal{A}^b_0(V)+\partial^b_{\tilde{y}}\,{}^\star\mathcal{A}^a_0(V)+\partial^a_y\,{}^\star\mathcal{A}^b_1(V)+\partial^b_y\,{}^\star\mathcal{A}^a_1(V)\big).\end{aligned}\tag{136}$$

For the Dirichlet sector, comparing Eq. (116) with the first relation in Eq. (134), we obtain

$${}^\star\mathcal{F}^{ij}_{(a)}={}^\star_y\mathcal{F}^{ij}_{0(a)}=\partial^i_y\,{}^\star\mathcal{A}^j_0(V)-\partial^j_y\,{}^\star\mathcal{A}^i_0(V),\tag{137}$$

while comparing Eq. (117) with the second relation in Eq. (134) we have

$${}^\star\mathcal{F}^{ij}_{(s)}=2\,{}^\star_{\tilde{y}}\mathcal{F}^{ij}_{0(s)}=2\Big(\partial^i_{\tilde{y}}\,{}^\star\mathcal{A}^j_0(V)+\partial^j_{\tilde{y}}\,{}^\star\mathcal{A}^i_0(V)\Big).\tag{138}$$

Taking into account that ${}^\star\mathcal{A}^i_1(V)=0$, we can conclude that the same relations are valid for both Neumann and Dirichlet sectors. Consequently, such a form is valid for complete field strengths, and with μ,ν indices we have

$$\begin{aligned}{}^\star\mathcal{F}^{\mu\nu}_{(a)} &=\partial^\mu_y\,{}^\star\mathcal{A}^\nu_0(V)-\partial^\nu_y\,{}^\star\mathcal{A}^\mu_0(V)+\partial^\mu_{\tilde{y}}\,{}^\star\mathcal{A}^\nu_1(V)-\partial^\nu_{\tilde{y}}\,{}^\star\mathcal{A}^\mu_1(V),\\ {}^\star\mathcal{F}^{\mu\nu}_{(s)} &=2\big[\partial^\mu_{\tilde{y}}\,{}^\star\mathcal{A}^\nu_0(V)+\partial^\nu_{\tilde{y}}\,{}^\star\mathcal{A}^\mu_0(V)+\partial^\mu_y\,{}^\star\mathcal{A}^\nu_1(V)+\partial^\nu_y\,{}^\star\mathcal{A}^\mu_1(V)\big].\end{aligned}\tag{139}$$

If we define $y^\alpha_\mu=\{y^0_\mu=y_\mu, y^1_\mu=-\tilde{y}_\mu\}$ and $\partial^\mu_\alpha\equiv\frac{\partial}{\partial y^\alpha_\mu}=\{\frac{\partial}{\partial y_\mu},\frac{\partial}{\partial\tilde{y}_\mu}\}$, we can rewrite the above equations in a compact form,

$$\begin{aligned}{}^\star\mathcal{F}^{\mu\nu}_{(a)} &= \eta^{\alpha\beta}\Big[\partial^\mu_\alpha\,{}^\star\mathcal{A}^\nu_\beta(V)-\partial^\nu_\alpha\,{}^\star\mathcal{A}^\mu_\beta(V)\Big],\\ {}^\star\mathcal{F}^{\mu\nu}_{(s)} &= -2\varepsilon^{\alpha\beta}\Big[\partial^\mu_\alpha\,{}^\star\mathcal{A}^\nu_\beta(V)+\partial^\nu_\alpha\,{}^\star\mathcal{A}^\mu_\beta(V)\Big].\end{aligned}\tag{140}$$

Finally, we have

$$\begin{aligned}{}^{\star}\mathcal{F}^{\mu\nu} &= {}^{\star}\mathcal{F}^{\mu\nu}_{(a)}+\frac{1}{2}{}^{\star}\mathcal{F}^{\mu\nu}_{(s)}\\ &= \eta^{\alpha\beta}\Big[\partial^{\mu}_{\alpha}{}^{\star}\mathcal{A}^{\nu}_{\beta}(V)-\partial^{\nu}_{\alpha}{}^{\star}\mathcal{A}^{\mu}_{\beta}(V)\Big]\\ &\quad -\varepsilon^{\alpha\beta}\Big[\partial^{\mu}_{\alpha}{}^{\star}\mathcal{A}^{\nu}_{\beta}(V)+\partial^{\nu}_{\alpha}{}^{\star}\mathcal{A}^{\mu}_{\beta}(V)\Big]\\ &= (\eta^{\alpha\beta}-\varepsilon^{\alpha\beta})\partial^{\mu}_{\alpha}{}^{\star}\mathcal{A}^{\nu}_{\beta}(V)-(\eta^{\alpha\beta}+\varepsilon^{\alpha\beta})\partial^{\nu}_{\alpha}{}^{\star}\mathcal{A}^{\mu}_{\beta}(V).\end{aligned} \tag{141}$$

We can check this expression in another way. From Eqs. (110) and (120) we have,

$$\begin{aligned}{}^{\star}S_A[y] &= {}^{\star}S^D_A[y]+{}^{\star}S^N_A[y]\\ &= 2\kappa\eta^{\alpha\beta}\int \mathrm{d}\tau\Big({}^{\star}\mathcal{A}^{\mu}_{\alpha}[V]\partial_{\beta}y_{\mu}/_{\sigma=\pi}-{}^{\star}\mathcal{A}^{\mu}_{\alpha}[V]\partial_{\beta}y_{\mu}/_{\sigma=0}\Big).\end{aligned} \tag{142}$$

After transition from integration over τ to integration over $d^2\xi=d\tau d\sigma$ and partial integration over τ, we obtain

$${}^{\star}S_A[y]=\kappa\int \mathrm{d}^2\xi\partial_{+}y_{\mu}{}^{\star}\mathcal{F}^{\mu\nu}\partial_{-}y_{\nu}, \tag{143}$$

where ${}^{\star}\mathcal{F}^{\mu\nu}$ is just Eq. (141), obtained previously in the other way.

Let us stress that the field strength of the initial theory is a particular case of Eq. (141). In fact, in that case background fields depend only on x^{μ} and not on $\tilde{x}^{\mu}$. So, if in Eq.(141) we omit terms which contain derivatives with respect to the tilde variable $\tilde{y}_{\mu}=-y^1_{\mu}$, we obtain a relation of the same form as that in Eq. (131).

Equation (141) we can consider as a general definition of the field strength for both geometric and non-geometric theories. Note that besides the antisymmetric part ${}^{\star}\mathcal{F}^{\mu\nu}_{(a)}$ it also contains a symmetric part ${}^{\star}\mathcal{F}^{\mu\nu}_{(s)}$. In the definition of both parts, the derivatives with respect to both T-dual coordinate y_{μ} and to its double $\tilde{y}_{\mu}$ contribute.

The unusual form of ${}^{\star}\mathcal{F}^{\mu\nu}$ is a consequence of two facts. First, the T-dual vector field ${}^{\star}A^a_D(V)$ is not multiplied by $\dot{y}_a$ but by T-dual σ-momentum ${}^{\star}G^{-1}_{ab}{}^{\star}\gamma^b_{(0)}$; and second, the T-dual vector fields depend on V^{μ} (see Eq. (84)) which is a function of both y_{μ} and $\tilde{y}_{\mu}$.

6 Genuinely non-geometric theories

Until now we have used a generalized Buscher procedure to establish a new structure for non-geometric theories defined in terms of effective vector fields $\mathcal{A}^{\mu}_0(V)$,$\mathcal{A}^{\mu}_1(V)$ and effective field strength $\mathcal{F}^{\mu\nu}$. It is important to stress that effective vector fields are not independent. The initial vector fields are connected with Eqs. (65) and (66), and the T-dual ones with Eq. (82).

Now we are able to separate from the Buscher approach and establish new kinds of non-geometric theories. We can preserve the obtained structure Eqs. (141)-(143), and omit the relation between effective vector fields. Consequently, in all obtained theories we will have a nontrivial field $\mathcal{A}_{1a}(V)$. Moreover, we can define new background field dependence on the arguments. As well as T-dual background fields dependent on

$$V^{\mu}(y)=-\kappa\theta^{\mu\nu}y_{\nu}+G^{-1\mu\nu}_{E}\tilde{y}_{\nu}=-2^{\star}B^{\mu\nu}y_{\nu}+{}^{\star}G^{\mu\nu}\tilde{y}_{\nu}, \tag{144}$$

(see Eq. (84)), which is a solution for x^{μ} of the finite part (for $\mathcal{A}_{\pm\mu}=0$) of T-dual transformation laws (86), we will take that the initial background fields are dependent on $V_{\mu}(x)$, which is a solution for y_{μ} of the finite part of the inverse T-dual transformation laws (87). So it takes the form

$$V_{\mu}(x)=-2B_{\mu\nu}x^{\nu}+G_{\mu\nu}\tilde{x}^{\nu}, \tag{145}$$

and depends on $\tilde{x}^{\mu}=\int(d\tau x'^{\mu}+d\sigma\dot{x}^{\mu})$, which makes the theory non-geometric. Therefore all theories, including the initial one, will be non-geometric. Our vector fields of genuinely non-geometric theories are $\mathcal{A}_{\pm\mu}[V_{\mu}(x)]$ and ${}^{\star}\mathcal{A}^{\mu}_{\pm}[V^{\mu}(y)]$, where the arguments are defined in the above expressions.

Now new duality transformations take a simple form,

$$\mathcal{A}_{\pm\mu}[V_{\mu}(x)]\to{}^{\star}\mathcal{A}^{\mu}_{\pm}[V^{\mu}(y)]=\kappa\theta^{\mu\nu}_{\pm}\mathcal{A}_{\pm\nu}[V^{\mu}(y)]. \tag{146}$$

Then, for example, the inverse T-dual transformation produces non-trivial relations

$$\begin{aligned}&2B_{\mu\nu}{}^{\star}\mathcal{A}^{\nu}_0[V^{\mu}(y)]-G_{\mu\nu}{}^{\star}\mathcal{A}^{\nu}_1[V^{\mu}(y)]\to\mathcal{A}_{0\mu}[V_{\mu}(x)],\\ &2B_{\mu\nu}{}^{\star}\mathcal{A}^{\nu}_1[V^{\mu}(y)]-G_{\mu\nu}{}^{\star}\mathcal{A}^{\nu}_0[V^{\mu}(y)]\to\mathcal{A}_{1\mu}[V_{\mu}(x)].\end{aligned} \tag{147}$$

The constraints (82) on the T-dual effective fields force the Neumann part to zero, $\mathcal{A}_{1a}=0$, but without these constraints $\mathcal{A}_{1a}$ is non-trivial. Also, the fields $\mathcal{A}_{\alpha\mu}$ depend on $V_{\mu}(x)$ and we have truly non-geometric theories. In Section 6.2. we will introduce a matter non-geometric field. The part $\psi_1(V)$, corresponding to gauge field $\mathcal{A}^{\mu}_1(V)$, prevents regression to the geometric theory after T-dualization.

There exist a few different approaches to genuinely non-geometric theories. For more details see Ref. [27] and references therein.

6.1 Local gauge symmetries of non-geometric theories

We are ready to find gauge transformations of the vector fields in non-geometric theories. To be definite, we will use T-dual fields ${}^{\star}\mathcal{A}^{\mu}_{\alpha}[V^{\mu}(y)]$, but similar expressions are valid for initial fields $\mathcal{A}_{\alpha\mu}[V_{\mu}(x)]$. By our definition the action (123) is proportional to field strength, and consequently it is gauge invariant. So, it is enough to find a transformation which leaves the action (123)

unchanged. It is easy to see that transformation,

$$ {}^\star\mathcal{A}^\mu_\alpha[V]\to{}^\star\mathcal{A}^\mu_\alpha[V]+\partial^\mu_\alpha\lambda_\alpha(y^\alpha),\quad (\lambda_\alpha(y^\alpha)\equiv\{\lambda_0(y),\lambda_1(\tilde{y})\}) \tag{148}$$

or equivalently in components,

$$ {}^\star\mathcal{A}^\mu_0[V]\to{}^\star\mathcal{A}^\mu_0[V]+\partial^\mu_y\lambda_0(y),\quad {}^\star\mathcal{A}^\mu_1[V]\to{}^\star\mathcal{A}^\mu_1[V]+\partial^\mu_{\tilde{y}}\lambda_1(\tilde{y}), \tag{149}$$

satisfy this condition because

$$\begin{aligned}&\eta^{\alpha\beta}\int \mathrm{d}\tau\,\partial^\mu_\alpha\lambda_\alpha(y^\alpha)\,\partial_\beta y_\mu/_{\partial\Sigma}\\ =&\int \mathrm{d}\tau\left(\partial^\mu_y\lambda_0(y)\dot{y}_\mu-\partial^\mu_{\tilde{y}}\lambda_1(\tilde{y})\dot{\tilde{y}}_\mu\right)/_{\partial\Sigma}\\ =&\int \mathrm{d}\tau\left(\dot{\lambda}_0-\dot{\lambda}_1\right)/_{\partial\Sigma}=0.\end{aligned} \tag{150}$$

Consequently, the expression for field strength in Eq. (141) should be invariant under gauge transformations,

$$\delta\,{}^\star\mathcal{A}^\mu_\alpha[V]=\partial^\mu_\alpha\lambda_\alpha(y^\alpha), \tag{151}$$

or in components,

$$\delta\,{}^\star\mathcal{A}^\mu_0[V]=\partial^\mu_y\lambda_0(y),\qquad \delta\,{}^\star\mathcal{A}^\mu_1[V]=\partial^\mu_{\tilde{y}}\lambda_1(\tilde{y}). \tag{152}$$

It easy to check that this is true. In fact variation of the antisymmetric part (the coefficient in front of $\eta^{\alpha\beta}$) vanishes in the same way as in geometric theory (partial derivatives commute). Variation of the symmetric part (the coefficient in front of $\varepsilon^{\alpha\beta}$) vanishes because we have derivatives with respect to both y_μ and $\tilde{y}_\mu$ of the parameter λ which depend on only one of these variables.

The transformation (149) we can take as the definition of gauge transformations for non-geometric theories.

6.2 Non-geometric matter fields

In the description of T-dual non-geometric fields we introduced a pair of T-dual coordinates $y^0_\mu=y_\mu$ and $y^1_\mu=\tilde{y}_\mu$, as well as a pair of effective vector fields $\mathcal{A}^\mu_0(V)$ and $\mathcal{A}^\mu_1(V)$. Each vector field transforms with its gauge parameter $\lambda_0(y)$ and $\lambda_1(\tilde{y})$. So, it is natural to introduce a pair of spinor matter fields $\psi_0(V)$ and $\psi_1(V)$ with Lagrangian

$$\mathcal{L}=\bar{\psi}_0(V)i\gamma_\mu\partial^\mu_y\psi_0(V)+\bar{\psi}_1(V)i\gamma_\mu\partial^\mu_{\tilde{y}}\psi_1(V). \tag{153}$$

As well as in the standard electrodynamics, it is invariant under global symmetries,

$$\psi^{(\lambda_0)}_0(V)=\mathrm{e}^{-\mathrm{i}\lambda_0}\psi_0(V),\quad \psi^{(\lambda_1)}_1(V)=\mathrm{e}^{-\mathrm{i}\lambda_1}\psi_1(V).\quad (\lambda_0,\lambda_1=\mathrm{const}) \tag{154}$$

Now, we can gauge these symmetries requiring that Lagrangian (153) is invariant under corresponding local symmetries with parameters $\lambda_0(y)$ and $\lambda_1(\tilde{y})$. This can be achieved by introducing covariant derivatives,

$$\partial^\mu_y\to D^\mu_y=\partial^\mu_y+i\,{}^\star\mathcal{A}^\mu_0(V),\ \ \partial^\mu_{\tilde{y}}\to\tilde{D}^\mu_{\tilde{y}}=\partial^\mu_{\tilde{y}}+i\,{}^\star\mathcal{A}^\mu_1(V). \tag{155}$$

With the help of Eqs. (149) and (154) we can easily check that the covariant derivatives really transform as

$$\begin{aligned}\left[D^\mu_y\psi_0(V)\right]^{(\lambda_0)}&=\mathrm{e}^{-\mathrm{i}\lambda_0(y)}D^\mu_y\psi_0(V),\\ \left[\tilde{D}^\mu_{\tilde{y}}\psi_1(V)\right]^{(\lambda_1)}&=\mathrm{e}^{-\mathrm{i}\lambda_1(\tilde{y})}\tilde{D}^\mu_{\tilde{y}}\psi_1(V).\end{aligned} \tag{156}$$

Consequently, the interaction Lagrangian obtains the form

$$\mathcal{L}_{\mathrm{int}}=-\bar{\psi}_0(V)\gamma_\mu\psi_0(V)\,{}^\star\mathcal{A}^\mu_0(V)-\bar{\psi}_1(V)\gamma_\mu\psi_1(V)\,{}^\star\mathcal{A}^\mu_1(V). \tag{157}$$

It is possible to form the Lagrangian for gauge fields in non-geometric theories by constructing the scalar from the gauge invariant field strength (141). In analogy with electrodynamics we can write ${}^\star\mathcal{L}\sim{}^\star\mathcal{F}^{\mu\nu}\,{}^\star\mathcal{F}_{\mu\nu}$, while in analogy with Born-Infeld theory we have ${}^\star\mathcal{L}\sim\sqrt{-\det(\eta_{\mu\nu}+2\pi\alpha'\,{}^\star\mathcal{F}_{\mu\nu})}$. The equations of motion and other features of non-geometric theories, which follow from these Lagrangians, will be discussed elsewhere.

We offer a possible interpretation of such a non-geometric theory. The fields with index 0, $\psi_0(V)$ and ${}^\star\mathcal{A}^\mu_0(V)$, are standard ones and we can suppose that they represent known spinor and gauge fields. The fields with index 1, $\psi_1(V)$ and ${}^\star\mathcal{A}^\mu_1(V)$, are new ones and we can suppose that they describe some so far unknown physics. It might be interesting to consider its possible relation with dark matter and dark energy.

7 Example: three-torus with D_1-brane

In this section we will take the example of a three-torus with D_1-brane. We will perform T-dualization along all coordinates and obtain the T-dual three-torus with D_0-brane.

7.1 Initial theory

We will start with definition of the background fields of the initial theory and introduce effective vector background fields and effective field strengths for a three-torus with D_1-brane.

7.1.1 Background fields of initial theory

The coordinates of the $D=3$ dimensional torus are denoted by x^0,x^1,x^2. In our particular example, nontrivial components of the background are

$$G_{\mu\nu}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad B_{\mu\nu}=\begin{pmatrix}0&B&0\\-B&0&0\\0&0&0\end{pmatrix}, \tag{158}$$

which produce

$$\Pi_{\pm\mu\nu}\equiv B_{\mu\nu}\pm\frac{1}{2}G_{\mu\nu}=\frac{1}{2}\begin{pmatrix}\pm1&2B&0\\-2B&\mp1&0\\0&0&\mp1\end{pmatrix}. \tag{159}$$

We will examine a D_1-brane defined with the Dirichlet boundary conditions $x^2(\tau,\sigma)/_{\sigma=0} = x^2(\tau,\sigma)/_{\sigma=\pi} = const$. This means that according to our convention we will have $p=1$, $a,b=0,1$ and $i,j=2$. So, we will work with Neumann background fields A^0_N and A^1_N and Dirichlet background field A^2_D.

Such a configuration produces $\gamma^{(0)}_2=\kappa x'^2$ and the action (45) takes the form

$$\begin{aligned} S_{\rm open}[x] \;=\; & \kappa\int_\Sigma {\rm d}^2\xi\partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu \\ & +2\kappa\int{\rm d}\tau\Big[\Big(A^N_0[x]\dot{x}^0+A^N_1[x]\dot{x}^1+A^D_2[x]x'^2\Big)/_{\sigma=\pi} \\ & -\Big(A^N_0[x]\dot{x}^0+A^N_1[x]\dot{x}^1+A^D_2[x]x'^2\Big)/_{\sigma=0}\Big]. \quad (160) \end{aligned}$$

Note an unusual coupling of A^D_2 with x'^2.

It is easy to find effective metric and noncommutativity parameters

$$G^E_{\mu\nu}=\begin{pmatrix} G_E & 0 & 0 \\ 0 & -G_E & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \theta_{\mu\nu}=\begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (161)$$

where

$$G_E\equiv 1-4B^2, \qquad \theta\equiv\frac{2B}{\kappa G_E}. \quad (162)$$

We will also need an expression for the combination of background fields,

$$\theta^{\mu\nu}_\pm=\theta^{\mu\nu}\mp\frac{1}{\kappa}G^{-1\mu\nu}_E=\begin{pmatrix} \mp\dfrac{1}{\kappa G_E} & \theta & 0 \\ -\theta & \pm\dfrac{1}{\kappa G_E} & 0 \\ 0 & 0 & \pm\dfrac{1}{\kappa} \end{pmatrix}. \quad (163)$$

According to Eq. (46) the nontrivial vector background fields are:

$$\begin{aligned} A^N_0(x) \;&=\; A^0_0-\frac{1}{2}F^{(a)}x^1, \\ [1mm]A^N_1(x) \;&=\; A^0_1+\frac{1}{2}F^{(a)}x^0, \\ A^D_2(x) \;&=\; A^0_2-\frac{1}{4}F^{(s)}x^2, \end{aligned} \quad (164)$$

where $F^{(a)}\equiv F^{(a)}_{01}=\partial_0A^N_1-\partial_1A^N_0$ and $F^{(s)}\equiv F^{(s)}_{22}=-4\partial_2A^D_2$. Consequently, the field strength of the initial theory is

$$F_{\mu\nu}=F^{(a)}_{\mu\nu}+\frac{1}{2}F^{(s)}_{\mu\nu}=\begin{pmatrix} 0 & F^{(a)} & 0 \\ -F^{(a)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{2}F^{(s)} \end{pmatrix}. \quad (165)$$

Note the unusual appearance of symmetric field strength $F^{(s)}$.

7.1.2 Effective vector background fields and effective field strength

We introduce effective vector background fields, which in our example of a three-torus with D_1-brane take the forms of Eqs. (62) and (67),

$$\begin{aligned} \mathcal{A}_{\pm0}(x) \;&=\; A^N_0(x), \\ \mathcal{A}_{\pm1}(x) \;&=\; A^N_1(x), \\ \mathcal{A}_{\pm2}(x) \;&=\; \mp A^D_2(x), \end{aligned} \quad (166)$$

or in components,

$$\begin{aligned} &\mathcal{A}_{00}(x)=A^N_0(x), \;\; \mathcal{A}_{01}(x)=A^N_1(x), \;\; \mathcal{A}_{10}(x)=0=\mathcal{A}_{11}(x), \\ &\mathcal{A}_{02}(x)=0, \qquad \mathcal{A}_{12}(x)=-A^D_2(x). \end{aligned} \quad (167)$$

Note that the constraints on the effective fields, Eqs. (65) and (66), are satisfied.

The effective field strength is equivalent to the initial field strength in Eqs. (94), (98) and (99):

$$\mathcal{F}_{\mu\nu}=F_{\mu\nu}, \quad \mathcal{F}^{(a)}_{\mu\nu}=\partial_\mu\mathcal{A}_{0\nu}(x)-\partial_\nu\mathcal{A}_{0\mu}(x), \quad (168)$$

$$\mathcal{F}^{(s)}_{00}=0, \quad \mathcal{F}^{(s)}_{11}=0, \quad \mathcal{F}^{(s)}_{22}=4\partial_2\mathcal{A}_{12}(x). \quad (169)$$

7.2 T-dual theory

Using the method described above, we will compute background fields and field strengths after T-dualization along all coordinates. We will obtain a T-dual three-torus with D_0-brane.

7.2.1 Background of T-dual theory

According to Eq. (84), in T-dual theory the vector background fields depend not only on the dual coordinate y_μ but on the expression

$$V^\mu=\begin{pmatrix} \dfrac{1}{G_E}(-2By_1+\tilde{y}_0) \\ \dfrac{1}{G_E}(2By_0-\tilde{y}_1) \\ -\tilde{y}_2 \end{pmatrix}, \quad (170)$$

where $\tilde{y}_\mu$ is defined in Eq. (71). The T-dual action (77) takes the form

$$\begin{aligned} {}^\star S[y] \;=\; & \frac{\kappa^2}{2}\int{\rm d}^2\xi\;\partial_+y_\mu\theta^{\mu\nu}_-\partial_-y_\nu \\ & +2\kappa\int{\rm d}\tau\Big[\Big(-A^D_2(V)\dot{y}_2-\frac{1}{G_E}A^N_0(V)(2B\dot{y}_1-y'_0) \\ & -\frac{1}{G_E}A^N_1(V)(-2B\dot{y}_0+y'_1)\Big)/_{\sigma=\pi} \\ & -\Big(-A^D_2(V)\dot{y}_2-\frac{1}{G_E}A^N_0(V)(2B\dot{y}_1-y'_0) \\ & -\frac{1}{G_E}A^N_1(V)(-2B\dot{y}_0+y'_1)\Big)/_{\sigma=0}\Big], \end{aligned} \quad (171)$$

where we used Eq. (26) for σ-momenta,

$${}^\star\gamma^0_{(0)}=\frac{\kappa}{G_E}(2B\dot{y}_1-y'_0), \quad {}^\star\gamma^1_{(0)}=\frac{\kappa}{G_E}(-2B\dot{y}_0+y'_1). \quad (172)$$

Consequently, according to Eq. (79) the T-dual background fields are

$$
\begin{aligned}
{}^{\star}G^{\mu\nu} &= G_E^{-1\mu\nu}=\begin{pmatrix} \frac{1}{G_E} & 0 & 0 \\ 0 & -\frac{1}{G_E} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
{}^{\star}B^{\mu\nu} &= \frac{\kappa}{2}\theta^{\mu\nu}=\frac{\kappa}{2}\begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{173}
$$

and

$$
\begin{aligned}
{}^{\star}A_D^0(V) &= \frac{1}{G_E}A_0^N(V), \\
{}^{\star}A_D^1(V) &= -\frac{1}{G_E}A_1^N(V), \\
{}^{\star}A_N^2(V) &= -A_2^D(V).
\end{aligned} \tag{174}
$$

Note that now we have one T-dual Neumann and two T-dual Dirichlet vector fields. It means that the T-dual three-torus has a D_0-brane defined with the Dirichlet boundary conditions $y_0(\tau,\sigma)/_{\sigma=0} = y_0(\tau,\sigma)/_{\sigma=\pi} = const$ and $y_1(\tau,\sigma)/_{\sigma=0} = y_1(\tau,\sigma)/_{\sigma=\pi} = const$, as well as $\tilde{y}_0(\tau,\sigma)/_{\sigma=0}=\tilde{y}_0(\tau,\sigma)/_{\sigma=\pi}=const$ and $\tilde{y}_1(\tau,\sigma)/_{\sigma=0}=\tilde{y}_1(\tau,\sigma)/_{\sigma=\pi}=const$.

The T-dual effective vector background fields in term of initial ones are ${}^{\star}\mathcal{A}_{\pm}^{\mu}=\kappa\theta_{\pm}^{\mu\nu}\mathcal{A}_{\pm\nu}$ which, with the help of Eq. (166), is equivalent to Eq. (80). So, in the case of the present example we have:

$$
\begin{aligned}
{}^{\star}\mathcal{A}_{\pm}^0(V) &= \mp\frac{1}{G_E}\mathcal{A}_{\pm0}(V)+\kappa\theta\mathcal{A}_{\pm1}(V) \\
&= \mp\frac{1}{G_E}A_0^N(V)+\kappa\theta A_1^N(V),
\end{aligned} \tag{175}
$$

$$
\begin{aligned}
{}^{\star}\mathcal{A}_{\pm}^1(V) &= -\kappa\theta\mathcal{A}_{\pm0}(V)\pm\frac{1}{G_E}\mathcal{A}_{\pm1}(V) \\
&= -\kappa\theta A_0^N(V)\pm\frac{1}{G_E}A_1^N(V),
\end{aligned} \tag{176}
$$

$$
{}^{\star}\mathcal{A}_{\pm}^2(V) = \pm\mathcal{A}_{\pm2}(V)=-A_2^D(V). \tag{177}
$$

Rewriting this in components, or according to Eq. (81), we obtain:

$$
\begin{aligned}
{}^{\star}\mathcal{A}_0^0(V) &= \kappa\theta A_1^N(V), \\
{}^{\star}\mathcal{A}_0^1(V) &= -\kappa\theta A_0^N(V), \\
{}^{\star}\mathcal{A}_0^2(V) &= -A_2^D,
\end{aligned} \tag{178}
$$

$$
\begin{aligned}
[2mm]{}^{\star}\mathcal{A}_1^0(V) &= -\frac{1}{G_E}A_0^N(V), \\
{}^{\star}\mathcal{A}_1^1(V) &= \frac{1}{G_E}A_1^N(V), \\
{}^{\star}\mathcal{A}_1^2(V) &= 0,
\end{aligned} \tag{179}
$$

where G_E and θ are defined in Eq. (162). Note that the constraints of Eq. (82) are satisfied.

7.2.2 T-dual transformation laws

The T-dual transformation laws from a three-torus with D_1-brane to dual three-torus with D_0-brane, in accordance with Eq. (86), take the form

$$
\begin{aligned}
\partial_{\pm}x^0 &\cong \pm\tfrac{1}{G_E}(\partial_{\pm}y_0\mp4\mathcal{A}_{\pm0})-\kappa\theta(\partial_{\pm}y_1\mp4\mathcal{A}_{\pm1}), \\
\partial_{\pm}x^1 &\cong \kappa\theta(\partial_{\pm}y_0\mp4\mathcal{A}_{\pm0})\mp\tfrac{1}{G_E}(\partial_{\pm}y_1\mp4\mathcal{A}_{\pm1}), \\
\partial_{\pm}x^2 &\cong \mp(\partial_{\pm}y_2\mp4\mathcal{A}_{\pm2}),
\end{aligned} \tag{180}
$$

while the inverse, from Eq. (87), is

$$
\begin{aligned}
\partial_{\pm}y_0 &\cong \pm\partial_{\pm}x^0-2B\partial_{\pm}x^1\pm4\mathcal{A}_{\pm0}, \\
\partial_{\pm}y_1 &\cong 2B\partial_{\pm}x^0\mp\partial_{\pm}x^1\pm4\mathcal{A}_{\pm1}, \\
\partial_{\pm}y_2 &\cong \mp\partial_{\pm}x^2\pm4\mathcal{A}_{\pm2}.
\end{aligned} \tag{181}
$$

Note that the expression for V^{μ} (170) is a solution of the finite part of Eq. (180), for $\mathcal{A}_{\pm0}=\mathcal{A}_{\pm1}=0$.

7.2.3 T-dual field strength

In Eq. (165) we introduced the field strength of the vector background fields on the string end-points for initial theory of a three-torus with D_1-brane. Now, we are going to express the field strength of its T-dual three-torus with D_0-brane obtained after dualization along all coordinates.

In the T-dual Dirichlet sector, with the help of Eq. (163), we have

$$
\kappa\theta_{\pm}^{\mu\rho}\mathcal{F}_{\rho\sigma}^{(a)}\kappa\theta_{\pm}^{\sigma\nu}=\frac{\mathcal{F}^{(a)}}{G_E^2}\begin{pmatrix} \pm4B & -(1+4B^2) & 0 \\ 1+4B^2 & \mp4B & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{182}
$$

where $\mathcal{F}^{(a)} \equiv \mathcal{F}_{01}^{(a)} = F_{01}^{(a)} \equiv F^{(a)}$. So, according to Eqs. (108) and (109) we find

$$
{}^{\star}\mathcal{F}_{(a)}^{ab}=-\frac{\mathcal{F}^{(a)}}{G_E^2}\begin{pmatrix} 0 & -(1+4B^2) & 0 \\ 1+4B^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{183}
$$

and

$$
{}^{\star}\mathcal{F}_{(s)}^{ab}=2\frac{\mathcal{F}^{(a)}}{G_E^2}\begin{pmatrix} 4B & 0 & 0 \\ 0 & -4B & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{184}
$$

The only nontrivial term in the Neumann sector is the second term in Eq. (119), ${}^{\star}\mathcal{F}_{(s)}^{22}=-\mathcal{F}_{22}^{(s)}=-F_{22}^{(s)}\equiv-F^{(s)}$. Consequently, the complete field strength of the T-dual three-torus with D_0-brane is

$$ {}^\star\mathcal{F}^{\mu\nu} = {}^\star\mathcal{F}^{\mu\nu}_{(a)}+\frac{1}{2}{}^\star\mathcal{F}^{\mu\nu}_{(s)}=\begin{pmatrix} \frac{4\mathcal{F}^{(a)}B}{G_E^2} & \frac{\mathcal{F}^{(a)}(1+4B^2)}{G_E^2} & 0 \\ -\frac{\mathcal{F}^{(a)}(1+4B^2)}{G_E^2} & -\frac{4\mathcal{F}^{(a)}B}{G_E^2} & 0 \\ 0 & 0 & -\mathcal{F}^{(s)} \end{pmatrix}, \tag{185} $$

where $\mathcal{F}^{(a)}\equiv\mathcal{F}^{(a)}_{01}=\partial_0 A^N_1-\partial_1 A^N_0$ and $\mathcal{F}^{(s)}\equiv\mathcal{F}^{(s)}_{22}=-4\partial_2 A^D_2$. In terms of T-dual fields we have $\mathcal{F}^{(a)}=-G_E(\partial_0{}^\star A^1_D+\partial_1{}^\star A^0_D)$ and $\mathcal{F}^{(s)}=4\partial_2{}^\star A^2_N$.

Since in our particular case $\mathcal{F}^{(a)}=F^{(a)}$ and $\mathcal{F}^{(s)}=F^{(s)}$, we can write

$$ \begin{aligned} {}^\star\mathcal{F}^{\mu\nu} &= {}^\star\mathcal{F}^{\mu\nu}_{(a)}+\frac{1}{2}{}^\star\mathcal{F}^{\mu\nu}_{(s)} \\ &= \begin{pmatrix} \frac{4F^{(a)}B}{G_E^2} & \frac{F^{(a)}(1+4B^2)}{G_E^2} & 0 \\ -\frac{F^{(a)}(1+4B^2)}{G_E^2} & -\frac{4F^{(a)}B}{G_E^2} & 0 \\ 0 & 0 & -F^{(s)} \end{pmatrix}, \end{aligned} \tag{186} $$

where $F^{(a)}$ and $F^{(s)}$ have been introduced after Eq. (164). Again, besides antisymmetric field strength we have a non-trivial symmetric part of field strength $F^{(s)}$.

7.2.4 Non-geometric three-torus with D_0-brane

The basic relation (141) of the field strength of non-geometric theories can be expressed in the form

$$ {}^\star\mathcal{F}^{\mu\nu}=\hat\partial^\mu_+{}^\star\mathcal{A}^\nu_+-\hat\partial^\nu_-{}^\star\mathcal{A}^\mu_-, \tag{187} $$

where $\hat\partial^\mu_\pm=\partial^\mu_y\pm\partial^\mu_{\tilde y}$ and ${}^\star\mathcal{A}^\mu_\pm={}^\star\mathcal{A}^\mu_0\pm{}^\star\mathcal{A}^\mu_1$.

In the case of a three-torus it becomes the expression

$$ {}^\star\mathcal{F}^{\mu\nu}=\begin{pmatrix} \hat\partial^0_+{}^\star\mathcal{A}^0_+-\hat\partial^0_-{}^\star\mathcal{A}^0_- & \hat\partial^0_+{}^\star\mathcal{A}^1_+-\hat\partial^0_-{}^\star\mathcal{A}^1_- & \hat\partial^0_+{}^\star\mathcal{A}^2_+-\hat\partial^0_-{}^\star\mathcal{A}^2_- \\ \hat\partial^1_+{}^\star\mathcal{A}^0_+-\hat\partial^1_-{}^\star\mathcal{A}^0_- & \hat\partial^1_+{}^\star\mathcal{A}^1_+-\hat\partial^1_-{}^\star\mathcal{A}^1_- & \hat\partial^1_+{}^\star\mathcal{A}^2_+-\hat\partial^1_-{}^\star\mathcal{A}^2_- \\ \hat\partial^2_+{}^\star\mathcal{A}^0_+-\hat\partial^2_-{}^\star\mathcal{A}^0_- & \hat\partial^2_+{}^\star\mathcal{A}^1_+-\hat\partial^2_-{}^\star\mathcal{A}^1_- & \hat\partial^2_+{}^\star\mathcal{A}^2_+-\hat\partial^2_-{}^\star\mathcal{A}^2_- \end{pmatrix}. \tag{188} $$

The term with effective background fields of T-dual action (142) takes the form

$$ \begin{aligned} {}^\star S_A[y] = 2\kappa\int \mathrm{d}\tau\Big[&\Big({}^\star\mathcal{A}^\mu_0[V]\dot y_\mu-{}^\star\mathcal{A}^\mu_1[V]y'_\mu\Big)/_{\sigma=\pi} \\ &-\Big({}^\star\mathcal{A}^\mu_0[V]\dot y_\mu-{}^\star\mathcal{A}^\mu_1[V]y'_\mu\Big)/_{\sigma=0}\Big]. \end{aligned} \tag{189} $$

Note that ${}^\star\mathcal{A}^\mu_1[V]$ is multiplied by y'_μ, not by $\dot y_\mu$.

Let us stress that only the first term in Eq. (187) ${}^\star\mathcal{F}^{\mu\nu}_{standard}=\partial^\mu_y{}^\star\mathcal{A}^\nu_0-\partial^\nu_y{}^\star\mathcal{A}^\mu_0$ is the standard one. The other three terms are new and unusual. One is also antisymmetric, but with derivation with respect to $\tilde y_\mu$, while the other two are symmetric.

7.3 Genuinely non-geometric three-torus

If we preserve the relation between effective background fields we can go back to the initial geometric theory. As explained at the beginning of Section 6, if we want to introduce the new kind of theory we can use the obtained structure from Eqs.(187)-(189) and suppose that: firstly, the effective background fields and corresponding field strengths are independent; and secondly, that background fields depend on the solutions of T-duality transformations. In that case we will lose the possibility to go back to the initial theory with inverse T-dualization. So, all our theories will be genuinely non-geometric.

With the constraints for effective vector fields the Neumann part of $\mathcal{A}_{1\mu}$ is zero,

$$ \mathcal{A}_{1\mu}=\begin{pmatrix} 0 \\ 0 \\ -A^D_2 \end{pmatrix}, \tag{190} $$

but without constraints we have the non-trivial expression

$$ \mathcal{A}_{1\mu}=\begin{pmatrix} 2B{}^\star\mathcal{A}^1_1-{}^\star\mathcal{A}^0_0 \\ -2B{}^\star\mathcal{A}^0_1+{}^\star\mathcal{A}^1_0 \\ {}^\star\mathcal{A}^2_0 \end{pmatrix}. \tag{191} $$

The initial vector fields $\mathcal{A}_{\pm\mu}$ depend on the solution of the finite part of inverse T-dual transformation laws

(181),

$$V_\mu(x)=\begin{pmatrix} -2Bx^1+\tilde{x}^0 \\ 2Bx^0-\tilde{x}^1 \\ -\tilde{x}^2 \end{pmatrix}. \tag{192}$$

The T-dual ones ${}^\star\mathcal{A}^\mu_\pm$, depend on $V^\mu(y)$ defined in Eq. (170), which is the solution of the finite part of the T-dual transformation laws (180).

The local gauge symmetry is defined with Eq. (152), while the matter fields can be introduced with Eqs. (153) and (157).

8 Conclusions

In the present article, using T-duality of the vector fields, we are able to introduce a new definition for a geometrical feature (the field strength) in non-geometric (T-dual) theories.

We started with T-duality of the vector gauge fields. In string theory the gauge fields appear at the boundaries of the open string. Their role is to enable complete local gauge symmetries. In fact, there are two important symmetries of the closed string theory: local gauge symmetry of the Kalb-Ramond field and general coordinate transformations. In Section 3.2 we showed that the symmetry T-dual to local gauge symmetry includes transformations of the background fields but does not include transformations of the string coordinates. Both symmetries fail at the open string end-points. The function of gauge fields is to restore these symmetries at the end-points. So, they are defined only on the open string boundary and not on the whole world-sheet. The corresponding term in the action is a line integral over the world-sheet boundary.

To each of the above symmetries of the string theory there corresponds an appropriate vector gauge field. As a consequence of the boundary conditions only parts of these gauge fields survive. From a gauge field corresponding to local gauge symmetry of the Kalb-Ramond field, the components along coordinates with Neumann boundary conditions survive. From a gauge field corresponding to restricted general coordinate transformations, the components along coordinates with Dirichlet boundary conditions survive. So, we obtained one complete vector field $\{A^N_a,A^D_i\}, \mu=(a,i)$. The action which describes field A^N_a is a standard one (see for example Ref. [14]), while introduction of the action for field A^D_i is the contribution of the present article.

There are several important results in the present article. First, we added a new term $A^D_i[x]G^{-1ij}\gamma^{(0)}_j(x)/_{\partial\Sigma}$ in the action (45) which corresponds to the Dirichlet boundary conditions and which compensates not-implemented general coordinate transformations at string end-points. We considered the case when the vector gauge field is linear in coordinates, so that it satisfies the open string space-time equations of motion.

Second, we perform T-duality along all the coordinates. We used a new approach for T-dualization in the absence of global symmetry [6]. We showed that such T-dualization exchanges: 1) Neumann with Dirichlet boundary conditions; 2) initial Dirichlet vector fields $A^D_i(x)$ with T-dual Neumann vector fields ${}^\star A^i_N(V)$ (also initial Neumann vector fields $A^N_a(x)$ with T-dual Dirichlet vector fields ${}^\star A^a_D(V)$); and 3) local gauge transformations with general coordinate transformations. Note that in initial theory the gauge fields depend on x^μ while in T-dual non-geometric theory they depend on the non-local variable V^μ. This is the cause of many interesting consequences.

Third, we introduced field strength for T-dual theories. The final expression is in accordance with the result obtained in another way, with direct T-dualization of the action with field strength. Using the fact that T-duality transformation turns geometric to non-geometric theories we can express effective T-dual field strength ${}^\star\mathcal{F}^{\mu\nu}$ as a derivation of effective gauge fields ${}^\star\mathcal{A}^\mu_\alpha(V)$, see Eq. (141). Because the arguments of non-geometric theories depend on $V^\mu=-\kappa\theta^{\mu\nu}y_\nu+G^{-1\mu\nu}_E\tilde{y}_\nu$, the corresponding field strength contains derivatives with respect to both y_μ and $\tilde{y}_\mu$. We also find that field strength (141) is invariant under gauge transformations of non-geometric theories (149).

Fourth, when we omit the relation between effective background fields $\mathcal{A}^\mu_\alpha(V)$, proclaiming them independent, and introduce new arguments of background fields as a solution of T-dual transformation laws, we will not be able to go back to the initial geometric theory. So, all our theories in any duality frame become truly non-geometric.

In another paper [28] we reproduce the results of the present article in the double space introduced in Ref. [29]. Let us stress that there is an essential difference between our approach and that of double field theories [30, 31]. In double field theories there are two coordinates, the initial x^μ and its double, denoted as $\tilde{x}_\mu$. The variable $\tilde{x}_\mu$ corresponds to our y_μ but we have an additional dual coordinate $\tilde{y}_\mu$ defined in first relation of Eq. (71). It plays an essential role in the definition of field strength for non-geometric theories.

It will be interesting to establish a relation between our formulation and the recent work of other authors on double field theory. The fact that double field theory does not depend on $\tilde{y}_\mu$ suggests that in order to find a relation between these theories we should eliminate the variable $\tilde{y}_\mu$, expressing it in terms of y_μ. For example, it is possible to introduce a Lagrange multiplier λ^μ and add the term $\lambda^\mu(\dot{y}_\mu-\tilde{y}'_\mu)$ to the Lagrangian in order to

introduce the relation between these variables as a constraint. Then $\tilde{y}_\mu$ becomes an independent variable, but we pay the price of introducing a new variable λ^μ.

There may be a new, more general theory such that both double field theory and our theory formulated in double space are some particular cases of that general theory.

Consequently, in the present paper we extended features of non-geometric theories to the case of an open string. In the formulation with gauge fields, similar to the closed string case, the non-geometry can be noticed in arguments of T-dual gauge fields (non-local expressions of V^a). Non-standard couplings of Dirichlet field, not only with $\dot{x}^\mu$ but also with x'^μ, should also be mentioned. In the formulation with field strength, the T-dual field strength is derivative of vector gauge fields with respect to both the T-dual variable y_μ and its double $\tilde{y}_\mu$. It depends not only on the antisymmetric but also on the symmetric part. This can be seen from the T-dual expression for field strength, Eq. (141), which is the main contribution of this paper. All these features are completely new and in any case they require further investigations.

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