Covariant open string field theory on multiple Dp-branes*

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Abstract: We study covariant open bosonic string field theories on multiple Dp-branes by using the deformed cubic string field theory, which is equivalent to string field theory in the proper-time gauge. Constructing the Fock space representations of the three-string vertex and the four-string vertex on multiple Dp-branes, we obtain the field theoretical effective action in the zero-slope limit. On multiple $D0$-branes, the effective action reduces to the Banks-Fishler-Shenker-Susskind (BFSS) matrix model. We also discuss the relation between open string field theory on multiple D-instantons in the zero-slope limit and the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model. The covariant open string field theory on multiple Dp-branes could be useful to study the non-perturbative properties of quantum field theories in (p+1)-dimensions in the framework of the string theory. The non-zero-slope corrections may be evaluated systematically by using covariant string field theory.

Keywords: open string, Dp-brane, covariant string field theory, Yang-Mills gauge theory, matrix model

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1 Introduction

String theories are defined only in critical dimensions: 10 dimensions for the super-string theories and 26 dimensions for the bosonic string theories. However, the quantum field theories, which describe open strings in the low energy region, can be defined in any dimension less than or equal to the critical dimension d_{critical} if we construct the string field theories on Dp-branes, $-1 \leq p \leq d_{\text{critical}}-1$. Thus, string field theory provides a unique framework to explore low dimensional quantum field theories in a unified manner. The purpose of this work is twofold. First, we shall construct covariant string field theories on Dp-branes of which zero-slope limits correspond to the quantum field theories in dimensions lower than the critical dimension. These covariant string field theories will be useful to understand various non-perturbative features of quantum field theories, which could not have been approached by the conventional perturbation theory. Second, we wish to understand the origins of actions for the matrix models [1, 2], which have served as important tools to study the non-perturbative effects of superstring theories and M-theory [3–5] within the framework of the covariant string field theory.

The core strategy we shall adopt in the present work is deformed cubic open string field theory [6, 7], which is equivalent to covariant string field theory in the proper-time gauge [8]. We have shown that deformed cubic open string field theory, if defined on the space-filling D-brane, yields the non-Abelian Yang-Mills theory in the zero-slope limit. The main reason we adopt the deformed cubic open string field theory is that we can obtain the exact results without using field redefinition [9] or level truncation [10–14]. The deformed cubic string field theory may also provide a systematic means to calculate the non-zero-slope corrections [15] and string scattering amplitudes [16–20]. In fact, deformation of the cubic interaction is not a new idea. Hua and Kaku [21] have discussed deformation of the midpoint overlapping interaction of Witten's cubic string field theory into the endpoint interaction in the context of closed string field theory. In recent works [6, 7] we developed the deformed cubic open string field theory by defining the theory on space-filling D-branes. On space-filling D-branes, the end points of the string satisfy only the Neumann boundary condition, so that the light-cone string field theory technique [22–28] is readily available. To deal with open strings on multiple Dp-branes, of which string coordinates along the directions orthogonal to the Dp-brane world volume satisfy the Dirichlet boundary condition, we need to extend the previous works appropriately.

The deformation procedure transforms the non-planar world sheet diagrams of Witten's cubic open string field theory [29, 30] into equivalent planar dia-

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grams of string field theory in the proper-time gauge. In the present work, we shall show that the deformation procedure is also applicable to open strings which satisfy the Dirichlet boundary condition. Then by mapping the planar diagrams of the deformed cubic string field theory on multiple Dp-branes onto the upper half plane, we will be able to evaluate the Neumann functions of the three-string vertex and the four-string vertex for the string on multiple Dp-branes. With the Neumann functions, we shall construct the Fock space representation of the string vertices and calculate the three-string and the four-string scattering amplitudes. In the zero-slope limit the external string states are $U(N)$ matrix valued non-Abelian gauge fields and $(d_{\text{critical}}-p-1)$ scalar fields in $(p+1)$ dimensions. From the three-string scattering amplitude and the four-string scattering amplitude in the zero-slope limit, we get the correct $U(N)$ gauge invariant matrix valued scalar field theory, which describes the dynamics of multiple Dp-branes in the low energy region. In particular, for multiple $D0$-branes the covariant open string field theory reduces to $U(N)$ matrix quantum mechanics, which has been the main subject of the Banks-Fishler-Shenker-Susskind (BFSS) matrix model [1]. Choosing multiple D-instantons may bring us an open string field theory of which action can be expressed solely in terms of matrices. In the zero-slope limit, the cubic string field theory on the multiple D-instantons is expected to reduce to the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model [2], of which action comprises only the contact quartic term of $U(N)$ matrix valued vector fields.

2 Open string fields on Dp-branes

On a Dp-brane, the string coordinates X^μ, $\mu=0,1,\ldots,p$ are tangential to the Dp-brane world-volume and the string coordinates X^i, $i=p+1,\ldots,d=d_{\text{critical}}-1$, are normal to the Dp-brane world-volume. The end points of X^μ, $\mu=0,1,\ldots,p$ satisfy the Neumann condition and the end points of X^i, $i=p+1,\ldots,d$ satisfy the Dirichlet condition,

$$\frac{\partial X^\mu}{\partial\sigma}\Big|_{\sigma=0,\pi} = 0, \quad \text{for} \quad \mu=0,1,\ldots,p, \tag{1a}$$

$$X^i\Big|_{\sigma=0,\pi} = 0, \quad \text{for} \quad i=p+1,\ldots,d. \tag{1b}$$

In accordance with the boundary conditions, the string coordinates X^I, $I=0,1,\ldots,d$ may be expanded in terms of the normal modes as

$$X^\mu(\sigma) = x^\mu+\sqrt{2}\sum_{n=1} x^\mu_n\cos(n\sigma), \quad \mu=0,1,\ldots,p, \tag{2a}$$

$$X^i(\sigma) = \sqrt{2}\sum_{n=1} x^i_n\sin(n\sigma), \quad i=p+1,\ldots,d. \tag{2b}$$

Note that the string coordinates X^i, $i=p+1,\ldots,d$ do not contain zero modes.

The string propagator is obtained by evaluating the path integral on a strip with the Polyakov string action,

$$G[X_1;X_2] = \int D[h]D[X]\exp(iS), \tag{3a}$$

$$S = -\frac{1}{4\pi\alpha'}\int_M \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-h}h^{\alpha\beta}\frac{\partial X^I}{\partial\sigma^\alpha}\frac{\partial X^J}{\partial\sigma^\beta}\eta_{IJ}, \quad I,J=0,\ldots,d \tag{3b}$$

where $\sigma^1=\tau$, $\sigma^2=\sigma$ and α' is the Regge slope parameter. We may fix the reparametrization invariance by choosing the proper-time gauge where the proper-time on the string world sheet is defined properly [8],

$$\partial_\tau N_{10}=0, \quad N_{1n}=0, \quad N_{2n}=0, \quad n\neq 0, \tag{4}$$

where $N_{\alpha n}$ is the normal modes of the lapse and shift functions $N_\alpha=\sum_n N_{\alpha n}\mathrm{e}^{in\sigma}$, with $\alpha=1,2$, of the two-dimensional metric on the world sheet,

$$\sqrt{-h}h^{\alpha\beta}=\frac{1}{N_1}\begin{pmatrix} -1 & N_2 \\ N_2 & (N_1)^2-(N_2)^2 \end{pmatrix}. \tag{5}$$

Evaluating the Polyakov path integral leads us to the open string field propagator on the Dp-branes

$$G[X_1;X_2) = \int_0^\infty \mathrm{d}s\langle X_1|\exp[-\mathrm{i}s(L_0-\mathrm{i}\epsilon)]|X_2\rangle = \langle X_1|\frac{1}{L_0-\mathrm{i}\epsilon}|X_2\rangle, \tag{6a}$$

$$L_0 = \frac{p^\mu p_\mu}{2}+\sum_{n=1}\frac{1}{2}\left(p^I_n p^J_n+n^2 x^I_n x^J_n\right)\eta_{IJ}-1, \tag{6b}$$

where p^I_n, $I=0,1,\ldots,d$ are normal modes of the momentum operators P^I,

$$P^\mu(\sigma) = \frac{1}{\pi}\left(p^\mu+\sqrt{2}\sum_{n=1}p^\mu_n\cos(n\sigma)\right), \ \mu=0,1,\ldots,p, \tag{7a}$$

$$P^i(\sigma) = \frac{\sqrt{2}}{\pi}\sum_{n=1}p^i_n\sin(n\sigma), \quad i=p+1,\ldots,d. \tag{7b}$$

(Throughout this paper, we suppress the ghost sector for the sake of simplicity.)

Because the end point of the open string is attached to one of N Dp-branes, the open string has N^2 different quantum states and consequently, the string field Ψ carries the group indices of $U(N)$

$$\Psi[X]=\frac{1}{\sqrt{2}}\Psi^0[X]+\Psi^a[X]T^a, \quad a=1,\ldots,N^2-1, \tag{8}$$

where T^a $a=1,\ldots,N^2-1$ are generators of the $SU(N)$ group. Now the string propagator on the multiple Dp-

branes, carrying the group indices, may be written as

$$\begin{aligned} G^{ab}[X_1,X_2] &= \mathrm{i}\langle T\Psi^a[X_1]\Psi^b[X_2]\rangle \\ &= \mathrm{i}\int D[X]\Psi^a[X_1]\Psi^b[X_2] \\ &\quad\times\exp\left\{-\mathrm{i}\int D[X]\,\mathrm{tr}\,\Psi(L_0+i\epsilon)\Psi\right\}. \end{aligned} \tag{9}$$

From this expression for the string propagator, the action of the string field theory follows,

$$\mathcal{S}_0=\int D[X]\,\mathrm{tr}\,\Psi(L_0+\mathrm{i}\epsilon)\Psi. \tag{10}$$

If we introduce the BRST ghosts, we may cast the free string field action into a BRST invariant form,

$$\mathcal{S}_0 = \int \mathrm{tr}\,\Psi{*}Q\Psi, \tag{11}$$

where Q is the BRST operator.

3 Deformation of cubic open string field theory on multiple Dp-branes

It is not difficult to extend Witten's cubic open string field theory [29] defined on a space-filling D-brane to the cubic open string field theory on multiple Dp-branes. It only takes replacing normal mode expansions of the string coordinates X^I and the momentum operators P^I, $I=0,1,\ldots,d$ by those given as Eqs. (2a, 2b) and Eqs. (7a, 7b):

$$\mathcal{S} = \int \mathrm{tr}\left(\Psi{*}Q\Psi+\frac{2g}{3}\Psi{*}\Psi{*}\Psi\right), \tag{12}$$

where the star product between the string field operators is defined as:

$$\begin{aligned} (\Psi_1{*}\Psi_2)[X(\sigma)] &= \int \prod_{\frac{\pi}{2}\leqslant\sigma\leqslant\pi} DX^{(1)}(\sigma)\prod_{0\leqslant\sigma\leqslant\frac{\pi}{2}} DX^{(2)}(\sigma) \\ &\quad\times\prod_{\frac{\pi}{2}\leqslant\sigma\leqslant\pi}\delta\left[X^{(1)}(\sigma)-X^{(2)}(\pi-\sigma)\right] \\ &\quad\times\Psi_1[X^{(1)}(\sigma)]\Psi_2[X^{(2)}(\sigma)], \end{aligned} \tag{13a}$$

$$X(\sigma) = \begin{cases} X^{(1)}(\sigma) & \text{for } 0\leqslant\sigma\leqslant\dfrac{\pi}{2}, \\ X^{(2)}(\sigma) & \text{for } \dfrac{\pi}{2}\leqslant\sigma\leqslant\pi. \end{cases} \tag{13b}$$

The star product is associative and the string field action is invariant under the BRST gauge transformation,

$$\delta\Psi=Q{*}\epsilon+\Psi{*}\epsilon-\epsilon{*}\Psi. \tag{14}$$

Now we shall deform the cubic open string field theory on multiple Dp-branes in a fashion similar to the deformation of the cubic open string field theory on multiple space-filling D-branes [6, 7]. Firstly, we extend the range of the world sheet spatial coordinate σ as

$$0\leqslant\sigma\leqslant\pi \implies 0\leqslant\sigma\leqslant 2\pi \tag{15}$$

and redefine the star product as

$$\begin{aligned} (\Psi_1{*}\Psi_2)[X(\sigma)] &= \int \prod_{\pi\leqslant\sigma\leqslant 2\pi} DX^{(1)}(\sigma)\prod_{0\leqslant\sigma\leqslant\pi} DX^{(2)}(\sigma) \\ &\quad\times\prod_{\pi\leqslant\sigma\leqslant 2\pi}\delta\left[X^{(1)}(\sigma)-X^{(2)}(2\pi-\sigma)\right] \\ &\quad\times\Psi_1[X^{(1)}(\sigma)]\Psi_2[X^{(2)}(\sigma)], \end{aligned} \tag{16a}$$

$$X(\sigma) = \begin{cases} X^{(1)}(\sigma) & \text{for } 0\leqslant\sigma\leqslant\pi, \\ X^{(2)}(\sigma) & \text{for } \pi\leqslant\sigma\leqslant 2\pi. \end{cases} \tag{16b}$$

To be consistent, the normal mode expansions of the string coordinates X^I, $I=0,1,\ldots,d$ are also redefined as

$$X^\mu(\sigma) = x^\mu+\sqrt{2}\sum_{n=1} x^\mu_n\cos\left(\frac{n}{2}\sigma\right),\ \mu=0,1,\ldots,p, \tag{17a}$$

$$X^i(\sigma) = \sqrt{2}\sum_{n=1} x^i_n\sin\left(\frac{n}{2}\sigma\right),\quad i=p+1,\ldots,d. \tag{17b}$$

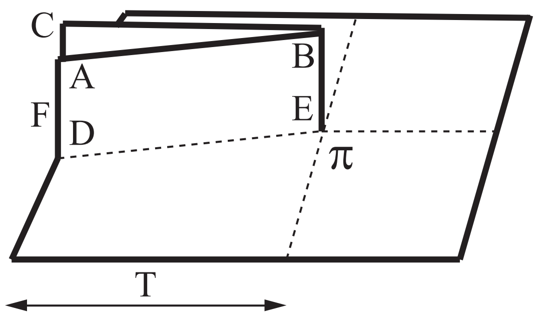

Fig. 1. The world sheet diagram of the three-string scattering.

Figure 1 depicts the world sheet diagram of three-string scattering. The world sheet of three-string interaction described by the cubic string field theory is not planar but a conic surface with an excess angle π. It is this non-planarity that hinders us from applying the fully developed techniques of the light-cone string field theory to obtain the Fock space representations of multi-string vertices. In recent works [6, 7], we discussed the deformation of the cubic open string field theory on multiple space-filling D-branes and application of the light-cone string field theory technique to the covariant string field theory. Our discussion on the cubic open string field theory on multiple Dp-branes will be parallel to the previous one. As we may see in Fig. 1, in the process of three-string scattering physical information, encoded on half of the first string $\overline{AD}$ and half of the second string $\overline{CF}$, is not carried over to the third string. Thus, the roles of these halves of two strings are auxiliary, and it may be appropriate to encode physical information only on the other halves of the two strings. The strings satisfy the Neumann condition or the Dirichlet condition on the boundary $\overline{ABC}$, depending on whether the string coordinate X^I is parallel or perpendicular to the world volume of the Dp-branes. It is convenient to separate the auxiliary patch (Fig. 2) M_A from the rest of the world sheet of the three-string scattering. On the patch we may redefine the local coordinates by interchanging the temporal coordinate τ and the spatial coordinate σ, $\tau\leftrightarrow\sigma$. In accordance with the local coordinates we redefine the string coordinates X^I, $I=0,1,\ldots,d$ as follows:

$$\begin{aligned} X^I(\sigma) &= x^I+\sqrt{2}\sum_{n=1} x^I_n \cos\left(\frac{n\pi\sigma}{2T}\right) \\ &= x^I+\sum_{n=1}\frac{i}{\sqrt{n}}\left(a^I_n-a^{I\dagger}_n\right)\cos\left(\frac{n\pi\sigma}{2T}\right), \quad I=0,1,\ldots,d, \end{aligned} \tag{18}$$

and express the string state on $\overline{ABC}$ as the following boundary state:

$$|N,D\rangle=c\exp\left(-\frac{1}{2}\sum_{n=1}a^{\mu\dagger}_n a^{\nu\dagger}_n\eta_{\mu\nu}+\frac{1}{2}\sum_{n=1}a^{i\dagger}_n a^{j\dagger}_n\eta_{ij}\right)|0\rangle, \tag{19}$$

satisfying the boundary condition

$$\partial_\tau X^\mu|N,D\rangle=0, \quad \partial_\sigma X^i|N,D\rangle=0. \tag{20}$$

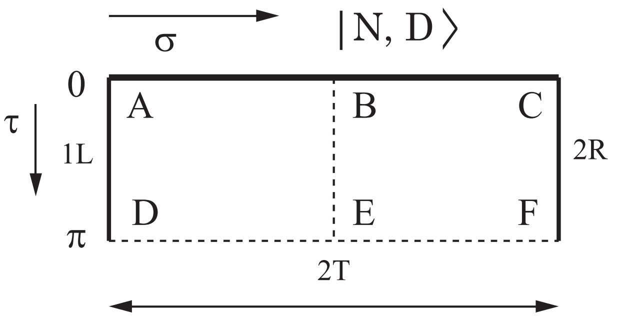

Fig. 2. Auxiliary patch to be effectively removed by deformation.

If we choose the Neumann condition as the boundary conditions for the end points of the string on the patch, we may think of the patch as a world sheet of an open string propagating freely from the initial state on $\overline{ABC}$ to the final state on $\overline{DEF}$. Then we find that the string state on $\overline{DEF}$ turns out to be the state $|N,D\rangle$ from Eq. (19) again,

$$\exp(-\mathrm{i}\pi L_0)|N,D\rangle=|N,D\rangle, \tag{21}$$

and the Polyakov string path integral over the patch M_A does not contribute to the string scattering amplitude because

$$\int_{M_A}\exp(iS)=\langle N,D|\mathrm{e}^{-\mathrm{i}\pi L_0}|N,D\rangle=1. \tag{22}$$

Therefore, we may effectively remove this auxiliary patch M_A from the non-planar world sheet to render the diagram planar.

It follows from consideration of this deformation that the initial states of the first string and the second string should be given as

$$|N_1\rangle\otimes|\Psi_1\rangle, \quad |\Psi_2\rangle\otimes|N_2\rangle, \tag{23a}$$

where

$$|N_1\rangle=\mathrm{e}^{-\frac{1}{2}\sum_{n=1}a^{(1)\dagger}_n a^{(1)\dagger}_n}|0\rangle, \quad |N_2\rangle=\mathrm{e}^{-\frac{1}{2}\sum_{n=1}a^{(2)\dagger}_n a^{(2)\dagger}_n}|0\rangle. \tag{23b}$$

Here the oscillator operators $a^{(1)\dagger}_n$ and $a^{(2)\dagger}_n$ act only on the left half of the first string and the right half of the second string respectively. As discussed in Refs. [31–35], we may treat a single string as two halves in string field theory. We choose a particular string state to encode the physical information only on the one of the halves for the first and second strings. It would be more convenient to express the external string state $|N_1\rangle\otimes|\Psi_1\rangle$, Eq. (23a) in momentum space. Let us denote the string momentum operator on the original (undeformed) string as $\tilde{P}(\sigma)$?

$$\tilde{P}(\sigma)=\frac{1}{2\pi}\left\{\tilde{p}+\sqrt{2}\sum_{n=1}\tilde{p}_n\cos\left(\frac{n\sigma}{2}\right)\right\}, \quad 0\leqslant\sigma\leqslant 2\pi. \tag{24}$$

It may also be written in terms of the string momentum operator defined on the half of the string $P(\sigma)$ as

$$\tilde{P}(\sigma)=\begin{cases} 0 & \text{for } \pi<\sigma\leqslant 2\pi, \\ \frac{1}{\pi}\left(p+\sqrt{2}\sum_{n=1}p_n\cos(n\sigma)\right) & \text{for } 0\leqslant\sigma\leqslant\pi. \end{cases} \tag{25}$$

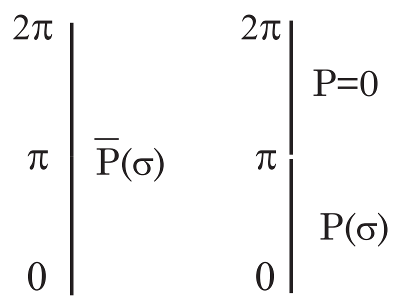

Fig. 3. Comparison of two string momentum bases.

It is important to note that we deform the cubic open string field theory only by choosing the external string states given as Eq. (23a), whereas the cubic string action is kept intact. Thus, the deformed cubic open string field theory is still invariant under the BRST gauge transformation, Eq. (14). A simple algebra yields the relation between the two momentum operators in terms of normal modes as:

$$\begin{aligned} \tilde{p} &= p, \\ \tilde{p}_{2k+1} &= \frac{p}{\pi}\frac{\sqrt{2}(-1)^k}{(k+\frac{1}{2})}+\sum_{n=1}\frac{p_n}{\pi}\frac{2k(-1)^{k-n}}{k^2-n^2}, \quad k\geqslant 0, \\ \tilde{p}_{2k} &= p_k, \quad k\geqslant 1. \end{aligned} \tag{26}$$

This relation between two momentum operators implies that the momentum space representations of the physical string states $\langle\{n_n^r\}|\Psi_r\rangle$, $r=1,2$ are not invariant under the deformation. The momentum space representations of the physical states transform under the deformation as the momentum space representation of the number eigen-states $\langle P_r|\{n_n^r\}\rangle$ change:

$$\langle P_r|\Psi_r\rangle=\int \mathrm{d}p^{(r)}\sum_{\{n_n^r\}}\langle P_r|\{n_n^r\}\rangle\langle\{n_n^r\}|\Psi_r\rangle,\ r=1,2. \quad (27)$$

As we shall show in this paper, if we choose the deformed string states as the external string states, we can get the gauge covariant Yang-Mills action directly. We may recall that in conventional works, which make use of the undeformed string state, one has to apply the method of field redefinition [36] to the effective string field action to obtain the usual covariant Yang-Mills action. The relation between two momentum operators, Eq. (25), may indicate that deformation of the external string states, adopted in the present work, may be equivalent to the procedure of field redefinition in conventional works.

4 Three-string vertex for open string on multiple *Dp*-branes

Effectively removing the auxiliary patch from the world sheet diagram of the three-string scattering by choosing the external string states appropriately, we find that the deformed world sheet diagram is the same as the planar diagram of the string field theory in the proper-time gauge [8]. It corresponds to the planar world sheet diagram of covariantized light-cone string field theory [37] with length parameters which are fixed as

$$\alpha_1=1,\quad \alpha_2=1,\quad \alpha_3=-2. \quad (28)$$

On the planar world sheet a global coordinate ρ may be introduced such that its real part is the proper-time $\mathrm{Re}\rho=\tau$ and the planar world sheet may be mapped onto the upper half plane by the Schwarz-Christoffel transformation,

$$\rho=\sum_r \alpha_r\ln(z-Z_r)=\ln(z-1)+\ln z, \quad (29)$$

where $Z_1=1$, $Z_2=0$, $Z_3=\infty$. The three temporal boundaries labelled a, b and c in Fig. 4 are mapped to form the real line on the upper half plane. The local coordinates on the individual string world sheet patches, ζ_r, $r=1,2,3$ are related to z as follows:

$$\mathrm{e}^{-\zeta_1} = \mathrm{e}^{\tau_0}\frac{1}{z(z-1)}, \quad (30a)$$

$$\mathrm{e}^{-\zeta_2} = -\mathrm{e}^{\tau_0}\frac{1}{z(z-1)}, \quad (30b)$$

$$\mathrm{e}^{-\zeta_3} = -\mathrm{e}^{-\frac{\tau_0}{2}}\sqrt{z(z-1)}, \quad (30c)$$

where $\tau_0=-2\ln2$. To obtain the Fock space representation of the three-string vertex, we need to solve Green's equation on the world sheet of the three-string scattering. However, it is not a simple task to solve Green's equation directly on the world sheet. Green's functions on the world sheet may be obtained by using a conformal transformation (inverse Schwarz-Christoffel transformation) of the well-known Green's functions on the upper half plane, which are given by

$$G_N(z,z') = \ln|z-z'|+\ln|z-z'^*|, \quad \text{for the Neumann boundary condition,} \quad (31a)$$

$$G_D(z,z') = \ln|z-z'|-\ln|z-z'^*|, \quad \text{for the Dirichlet boundary condition,} \quad (31b)$$

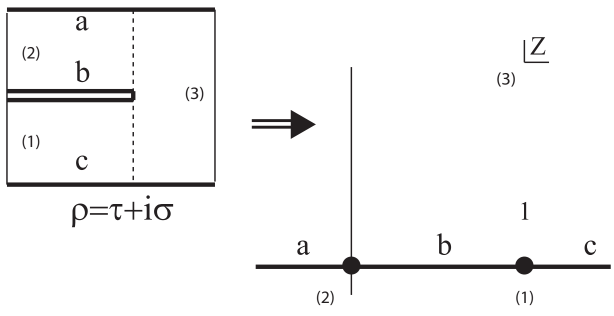

Fig. 4. Three-string scattering diagram of string field theory in the proper-time gauge.

Construction of the Fock space representations of multi-string vertices in the case of the Neumann Green's function G_N is well studied in the context of the light-cone string field theory. Here we will focus on the construction of the Fock space representations by using the Dirichlet Green's function G_D. We shall begin with the Dirichlet Green's function on an infinite strip (the world sheet of the free string propagator). The strip is mapped onto the upper half plane by a simple conformal transformation,

$$\rho=\alpha\zeta=\alpha\ln z, \quad (32)$$

where α is the length parameter and $\zeta=\xi+i\eta$. The Dirichlet Green's function on the strip is found to be

$$\begin{aligned} D_{\mathrm{strip}}(\zeta,\zeta') &= \ln|\mathrm{e}^{\zeta}-\mathrm{e}^{\zeta'}|-\ln|\mathrm{e}^{\zeta}-\mathrm{e}^{\zeta'^*}| \\ &= -\sum_{n=1}\frac{2}{n}\mathrm{e}^{-n|\xi-\xi'|}\sin n\eta\sin n\eta'. \end{aligned} \quad (33)$$

On the world sheet of multi-string scattering, we may define the Dirichlet functions $\bar{D}^{rs}_{nm}$, which are analogous to the Neumann functions, as follows:

$$D(\rho_r,\rho_s') = -\delta_{rs}\Big\{\sum_{n\geqslant 1}\frac{2}{n}\mathrm{e}^{-n|\xi_r-\xi_s'|}\sin(n\eta_r)\sin(n\eta_s')\Big\} +2\sum_{n,m\geqslant 0}\bar{D}^{rs}_{nm}\mathrm{e}^{n\xi_r+m\xi_s'}\sin(n\eta_r)\sin(m\eta_s'), \tag{34}$$

where ρ_r is the coordinate on the patch of the r-th string. Taking the limit, $z'\to Z_s$ or $z'\to Z_r$ of Eq. (34), we have

$$\bar{D}^{rs}_{n0}=0, \quad \text{for} \quad n\geqslant 0. \tag{35}$$

By differentiating Eq. (34) with respect to ζ_r, we find

$$\bar{D}^{rs}_{nm} = -\frac{1}{nm}\oint_{Z_r}\frac{\mathrm{d}z}{2\pi i}\oint_{Z_s}\frac{\mathrm{d}z'}{2\pi i}\frac{1}{(z-z')^2}\mathrm{e}^{-n\zeta_r(z)-m\zeta_s'(z')}, \quad n,m\geqslant 1. \tag{36}$$

It turns out that

$$\bar{D}^{rs}_{nm}=-\bar{N}^{rs}_{nm}. \tag{37}$$

These results, Eq. (35) and Eq. (37), are not limited to the case of the three-string vertex. It is interesting that we only need to calculate the Neumann functions to construct the Fock space representations of the multi-string vertices on Dp-branes.

To be explicit, we may write the Fock space representation of the three-string vertex in terms of the Neumann function as

$$E[1,2,3]|0\rangle = \exp\Big\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m\geqslant 1}\bar{N}^{rs}_{nm}\alpha^{(r)\dagger}_{n\mu}\alpha^{(s)\dagger}_{m\nu}\eta^{\mu\nu} +\sum_{r=1}^{3}\sum_{n\geqslant 1}\bar{N}^{r}_{n}\alpha^{(r)\dagger}_{n\mu}\boldsymbol{P}^{\mu} +\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\left(\frac{(p^{(r)}_{\mu}p^{(r)\mu}}{2}-1\right) -\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m\geqslant 1}\bar{N}^{rs}_{nm}\alpha^{(r)\dagger}_{ni}\alpha^{(s)\dagger}_{mj}\eta^{ij}\Big\}|0\rangle, \tag{38}$$

where $\boldsymbol{P}=p^{(2)}-p^{(1)}$. The three-string interaction may be written as

$$\mathcal{S}_{[3]}=\int\prod_{r=1}^{3}dp^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\frac{2g}{3}\langle\Psi_1,\Psi_2,\Psi_3|E[1,2,3]|0\rangle. \tag{39}$$

5 Zero-slope limit of the three-string interaction

In the zero-slope limit, the external string states corepond to massless gauge fields A^{μ} or massless scalar fields φ^i. By choosing the external string state as follows,

$$\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}|=\Big\langle 0\Big|\prod_{r=1}^{3}\Big(A_{\mu}(p^{(r)})a^{(r)}_{1\nu}\eta^{\mu\nu}+\varphi_i(p^{(r)})a^{(r)}_{1j}\eta^{ij}\Big), \tag{40}$$

we can evaluate the effective interaction between the gauge fields A^{μ} and the scalar fields φ^i which describes the three-string interaction, Eq. (38) and Eq. (39), in the zero-slope limit:

$$\begin{aligned}\mathcal{S}_{[3]} &= \int\prod_{r=1}^{3}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\frac{2g}{3}\mathrm{tr}\Big\langle 0\Big|\prod_{r=1}^{3}\Big\{A_{\mu}(p^{(r)})a^{(r)}_{1\nu}\eta^{\mu\nu}+\varphi_i(p^{(r)})a^{(r)}_{1j}\eta^{ij}\Big\}\exp[E[1,2,3]]\Big|0\Big\rangle \\ &= \frac{2g}{3}\mathrm{e}^{-\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}}\int\prod_{r=1}^{3}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\mathrm{tr}\Big\langle 0\Big|\prod_{r=1}^{3}\Big\{A_{\mu}(p^{(r)})a^{(r)}_{1\nu}\eta^{\mu\nu}+\varphi_i(p^{(r)})a^{(r)}_{1j}\eta^{ij}\Big\} \\ &\quad\times\left(\frac{1}{2}\sum_{r,s=1}^{3}\bar{N}^{rs}_{11}a^{(r)\dagger}_{1\mu}a^{(s)\dagger}_{1\nu}\eta^{\mu\nu}-\frac{1}{2}\sum_{r,s=1}^{3}\bar{N}^{rs}_{11}a^{(r)\dagger}_{1i}a^{(s)\dagger}_{1j}\eta^{ij}\right)\left(\sum_{r=1}^{3}\bar{N}^{r}_{1}a^{(r)\dagger}_{1}\cdot\boldsymbol{P}\right)\Big|0\Big\rangle.\end{aligned} \tag{41}$$

From Eq. (41) it is clear that we only get a three-gauge interaction term S_{AAA} and an interaction term of type $S_{A\varphi\varphi}$. In previous works [6, 7] we have evaluated the three-gauge interaction term S_{AAA},

$$\begin{aligned}S_{AAA} &= g_{YM}\int\prod_{i=1}\mathrm{d}p^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right)(p_1^{\mu}-p_2^{\mu})\mathrm{tr}\Big(A_{\nu}(p_1)A^{\nu}(p_2)A(p_3)^{\mu}\Big) \\ &= -g_{YM}\int\mathrm{d}^{p+1}xi\,\mathrm{tr}\,(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})[A^{\mu},A^{\nu}],\end{aligned} \tag{42}$$

where g_{YM} is the Yang-Mills coupling constant,

$$g_{YM}=(\alpha')^{\frac{p+1}{4}-1}g. \tag{43}$$

Here we only need to evaluate the term $S_{A\varphi\varphi}$:

$$\begin{aligned}\mathcal{S}_{A\varphi\varphi} &= -\frac{2g_{YM}}{3}\times 2^3\times 2!\int\prod_{r=1}^{3}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)(p_2^\mu-p_1^\mu)\\ &\quad\times\mathrm{tr}\left\{-\frac{1}{2^4}\varphi_i(p_1)\varphi_j(p_2)\eta^{ij}A_\mu(p_3)+\frac{1}{2^3}\varphi_i(p_2)\varphi_j(p_3)\eta^{ij}A_\mu(p_1)+\frac{1}{2^3}\varphi_i(p_3)\varphi_j(p_1)\eta^{ij}A_\mu(p_2)\right\}\\ &= 2g_{YM}\int\prod_{r=1}^{3}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)p_1^\mu\,\mathrm{tr}\Big(\varphi_i(p_1)\left[A_\mu(p_3),\varphi^i(p_2)\right]\Big)=-2g_{YM}\int\mathrm{d}^dxi\,\mathrm{tr}\,\partial_\mu\varphi_i\left[A_\mu,\varphi^i\right].\end{aligned} \tag{44}$$

Putting the two interaction terms, Eq. (42) and Eq. (44), together, we get the cubic interaction term in the zero-slope limit:

$$\begin{aligned}\mathcal{S}_{[3]} &= \mathcal{S}_{AAA}+\mathcal{S}_{A\varphi\varphi}\\ &= -\mathrm{i}g_{YM}\int\mathrm{d}^dx\,\mathrm{tr}\Big\{(\partial_\mu A_\nu-\partial_\nu A_\mu)[A^\mu,A^\nu]\\ &\quad+2\partial_\mu\varphi_i\left[A_\mu,\varphi^i\right]\Big\}.\end{aligned} \tag{45}$$

6 Zero-slope limit of the four-string interaction

The four-string scattering amplitude may be written at the tree level as

$$\mathcal{F}_{\mathrm{Tree}[4]} = \int D[\Psi]\,\mathrm{tr}\prod_{r=1}^{4}\Psi^{(r)}\frac{1}{2!}\left(\frac{2g}{3}\right)^2\left[\int\mathrm{tr}(\Psi*\Psi*\Psi)\right]^2 \times\mathrm{e}^{[-\mathrm{i}\int\mathrm{tr}\Psi L_0\Psi]}. \tag{46}$$

The Wick contraction brings us to nine identical Feynman diagrams. We may deform the cubic string field theory at two levels: 1) we may deform the theory only by choosing external string states where the physical information is encoded only on the halves of the external string; or 2) we may deform the theory at the level of string field action. In the first case, where we still keep Witten's cubic string field action, we would get nine Feynman diagrams which are all identical. We only need to take into account of the combinatorics factor as in Eq. (46). In this paper, only case 1) will be discussed. Of course, we may also deform the theory at the level of action. In the second case we get Feynman diagrams of different types and should worry about the Wick contraction of string field operators with different length parameters. These problems can be resolved by using the properties of the string propagator and the Neumann functions of the three-string vertex: the string propagator does not depend on the length parameters and the Neumann functions of the three-string vertex depend only on the ratios of the length parameters. We would also get nine Feynman diagrams of four-string scattering in this case, which can be made planar. Although these Feynman diagrams are not identical, their contributions to the low energy effective action are all identical. The reason is that the string scattering amplitudes which the string Feynman diagrams produce depend only on the Koba-Nielsen variables, not on the length parameters. This point has been elaborated in some detail in Ref. [6].

If we choose the external string states appropriately to encode physical information only on the halves of the external strings, as in the case of the three-string scattering, the non-planar diagram of the cubic string field theory may reduce to the planar diagram of string field theory in the proper-time gauge, as depicted in Fig. 5. Then, by applying the Cremmer-Gervais identity [26], we may cast the four-string scattering amplitude into a $SL(2,R)$ invariant form:

$$\begin{aligned}\mathcal{F}_{[4]} &= 2g^2\int\left|\frac{\prod_{r=1}^4\mathrm{d}Z_r}{\mathrm{d}V_{abc}}\right|\prod_{r<s}|Z_r-Z_s|^{p_r\cdot p_s}\exp\left[-\sum_{r=1}^{4}\bar{N}_{00}^{[4]rr}\right]\\ &\quad\times\mathrm{tr}\left\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)},\Psi^{(4)}\middle|\exp[E_{[4]}]\middle|0\right\rangle,\end{aligned} \tag{47a}$$

$$\begin{aligned}E_{[4]} &= \sum_{r,s=1}^{4}\left\{\frac{1}{2}\sum_{r,s=1}^{4}\sum_{m,n\geqslant 0}\bar{N}_{mn}^{[4]rs}\alpha_{m\mu}^{(r)\dagger}\alpha_{n\nu}^{(s)\dagger}\eta^{\mu\nu}\right.\\ &\quad\left.-\frac{1}{2}\sum_{r,s=1}^{4}\sum_{m,n=1}\bar{N}_{mn}^{[4]rs}\alpha_{mi}^{(r)\dagger}\alpha_{nj}^{(s)\dagger}\eta^{ij}\right\}.\end{aligned} \tag{47b}$$

The planar diagram, Fig. 5, corresponds to that of light-cone string field theory with length parameters fixed as

$$\alpha_1=1,\quad \alpha_2=1,\quad \alpha_3=-1,\quad \alpha_4=-1. \tag{48}$$

We may fix the $SL(2,R)$ invariance by choosing

$$Z_1=\infty,\quad Z_2=1,\quad Z_3=x,\quad Z_4=0,\quad 0\leqslant x\leqslant 1, \tag{49}$$

where x is the Koba-Nielsen variable of the four-string scattering. The Schwarz-Christoffel transformation which maps the four-scattering world sheet onto the upper half plane is given as

$$\rho = \sum_{r=1}^{4}\alpha_r\ln(z-Z_r)=\ln(z-1)-\ln z-\ln(z-x). \tag{50}$$

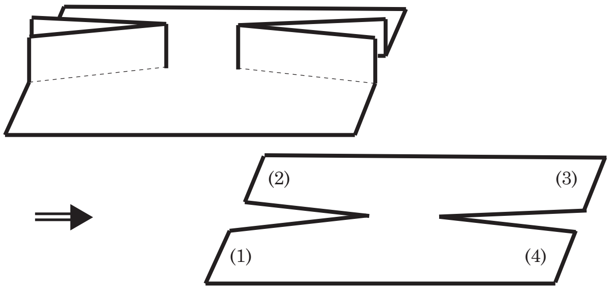

Fig. 5. Deformation of the four-string scattering diagram.

The local coordinates on individual string patches ζ_r, $r=1,2,3,4$ are related to the coordinate on the upper half plane z as follows [7]:

$$\mathrm{e}^{-\zeta_1} = \mathrm{e}^{\tau_1}\frac{z(z-x)}{1-z}, \quad \mathrm{e}^{-\zeta_2} = -\mathrm{e}^{\tau_1}\frac{z(z-x)}{1-z}, \tag{51a}$$

$$\mathrm{e}^{-\zeta_3} = \mathrm{e}^{-\tau_2}\frac{(1-z)}{(z-x)z}, \quad \mathrm{e}^{-\zeta_4} = -\mathrm{e}^{-\tau_2}\frac{(1-z)}{(z-x)z} \tag{51b}$$

where τ_1 and τ_2 are two interaction times on the world sheet,

$$\tau_0^{(1)} = \tau_0^{(2)} = \tau_1 = -2\ln\left(1+\sqrt{1-x}\right) < 0, \tag{52a}$$

$$\tau_0^{(3)} = \tau_0^{(4)} = \tau_2 = -2\ln\left(1-\sqrt{1-x}\right) > 0. \tag{52b}$$

To evaluate the effective action in the zero-slope limit, we choose the external string states as

$$\left\langle \Psi^{(1)},\Psi^{(2)},\Psi^{(3)},\Psi^{(4)}\right| = \left\langle 0\right|\prod_{r=1}^{4}\left\{A_\mu(p^{(r)})a_{1\nu}^{(r)}\eta^{\mu\nu}+\varphi_i(p^{(r)})a_{1j}^{(r)}\eta^{ij}\right\}. \tag{53}$$

It is expected from Eqs. (47a,—47b) and Eq. (53) that we would obtain interaction terms of the following three types:

$$AAAA, \quad AA\varphi\varphi, \quad \varphi\varphi\varphi\varphi.$$

In our previous works [6, 7], we calculated the effective four-gauge field action. The effective four-gauge field action, $S_{AAAA}^{\mathrm{effective}}$, obtained by evaluating the four-string scattering amplitude, contains both the contact quartic gauge field action S_{AAAA} and the effective four-gauge field interaction mediated by massless gauge field $S_{AAAA}^{\mathrm{massless}}$:

$$S_{AAAA}^{\mathrm{effective}} = S_{AAAA}+S_{AAAA}^{\mathrm{massless}}, \tag{54a}$$

$$S_{AAAA} = \frac{g_{YM}^2}{2}\int \mathrm{d}^{p+1}x\,\mathrm{tr}\,[A^\mu,A^\nu][A_\mu,A_\nu], \tag{54b}$$

$$S_{AAAA}^{\mathrm{massless}} = g_{YM}^2\int\prod_{i=1}^{4}\mathrm{d}p^{(i)}\delta\left(\sum_{i=1}^{4}p^{(i)}\right)\left(1+\frac{2u}{s}\right) \times\mathrm{tr}\left(A_\mu(p^{(1)})A^\mu(p^{(2)})A_\nu(p^{(3)})A^\nu(p^{(4)})\right). \tag{54c}$$

The effective four-scalar field action can be also calculated in a similar way. The four-scalar vertex is obtained by choosing the external string states as

$$\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)},\Psi^{(4)}| = \left\langle 0\right|\left\{\prod_{r=1}^{4}\varphi_i(p^{(r)})a_{1j}^{(r)}\eta^{ij}\right\}. \tag{55}$$

From Eq. (47a) and Eq. (55), we find:

$$S_{\varphi\varphi\varphi\varphi}^{\mathrm{effective}} = g_{YM}^2\int\prod_{r=1}^{4}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{4}p^{(r)}\right)\int\left|\frac{\prod_{r=1}^{4}\mathrm{d}Z_r}{\mathrm{d}V_{abc}}\right|\prod_{r<s}|Z_r-Z_s|^{p_r\cdot p_s}\exp\left[-\sum_{r=1}^{4}\bar{N}_{00}^{[4]rr}\right] \times\mathrm{tr}\left\langle 0\right|\left\{\prod_{r=1}^{4}\varphi_i(p^{(r)})a_{1j}^{(r)}\eta^{ij}\right\}\frac{1}{2!}\times\frac{1}{2^2}\left\{-\sum_{r,s=1}^{4}\bar{N}_{mn}^{[4]rs}a_{1i}^{(r)\dagger}a_{1j}^{(s)\dagger}\eta^{ij}\right\}^2\left|0\right\rangle. \tag{56}$$

The four-scalar field action may be calculated as:

$$\begin{aligned} S_{\varphi\varphi\varphi\varphi}^{\mathrm{effective}} &= g_{YM}^2\int\prod_{r=1}^{4}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{4}p^{(r)}\right)\int_0^1\mathrm{d}x\,\mathrm{tr}\left(x^{-\frac{s}{2}}(1-x)^{-\frac{t}{2}}\varphi^i(p_1)\varphi^j(p_2)\varphi_i(p_3)\varphi_j(p_4)\right. \\ &\quad \left.+2x^{-\frac{s}{2}-2}(1-x)^{-\frac{t}{2}}\varphi(p_1)^i\varphi(p_2)_i\varphi(p_3)^j\varphi(p_4)_j\right) \\ &= g_{YM}^2\int\prod_{r=1}^{4}\mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^{4}p^{(r)}\right)\mathrm{tr}\left(\varphi^i(p_1)\varphi^j(p_2)\varphi_i(p_3)\varphi_j(p_4)+\frac{2u}{s}\varphi^i(p_1)\varphi_i(p_2)\varphi^j(p_3)\varphi_j(p_4)\right). \end{aligned} \tag{57}$$

Here we define the Mandelstam variables as

$$s=-(p_1+p_2)^2, \quad t=-(p_1+p_4)^2, \quad u=-(p_1+p_3)^2 \tag{58}$$

and make use of

$$\int_0^1\mathrm{d}x x^{-\frac{s}{2}}(1-x)^{-\frac{t}{2}}=1, \quad \int_0^1\mathrm{d}x x^{-\frac{s}{2}-2}(1-x)^{-\frac{t}{2}}=\frac{u}{s} \tag{59}$$

in the zero-slope limit. This effective four-scalar field action $S_{\varphi\varphi\varphi\varphi}^{\mathrm{effective}}$ contains the contact quartic scalar action $S_{\varphi\varphi\varphi\varphi}$ as well as the effective four-scalar field interaction induced by intermediate massless gauge field $S_{\varphi\varphi\varphi\varphi}^{\mathrm{massless}}$, as depicted by Fig. 6

In the zero-slope limit, we have shown that there is an interaction term for scalar fields and the gauge fields

$S_{A\varphi\varphi}$ Eq. (44). This interaction term generates the effective four-scalar field interaction, which is mediated by the massless gauge field, perturbatively. By making use of the usual Feynman diagrams, we calculate the effective four-scalar field interaction term in the zero-slope limit as:

$$\begin{aligned} S^{\text{massless}}_{\varphi\varphi\varphi\varphi} &= -\frac{1}{2!}\times(2!)g^2_{YM}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\mathrm{tr}\Big(\varphi_i(p^{(1)})\varphi_j(p^{(2)})\eta^{ij}\left(p^{(1)}_\mu-p^{(2)}_\mu\right) \\ &\quad\times\frac{\eta^{\mu\nu}}{\left(p^{(1)}+p^{(2)}\right)^2}\left(p^{(3)}_\nu-p^{(4)}_\nu\right)\varphi_k(p^{(3)})\varphi_l(p^{(4)})\eta^{kl}\Big) \\ &= g^2_{YM}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{i=r}^4 p^{(r)}\right)\left(1+\frac{2u}{s}\right)\mathrm{tr}\Big(\varphi_i(p^{(1)})\varphi_j(p^{(2)})\eta^{ij}\varphi_k(p^{(3)})\varphi_l(p^{(4)})\eta^{kl}\Big). \end{aligned} \tag{60}$$

From Eq. (57) and Eq. (60) we may identify the contact quartic scalar field action $S_{\varphi\varphi\varphi\varphi}$:

$$S^{\text{effective}}_{\varphi\varphi\varphi\varphi} = S_{\varphi\varphi\varphi\varphi}+S^{\text{massless}}_{\varphi\varphi\varphi\varphi}, \tag{61a}$$

$$\begin{aligned} S_{\varphi\varphi\varphi\varphi} &= g^2_{YM}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\mathrm{tr}\left(\varphi^i(p^{(1)})\varphi^j(p^{(2)})\varphi_i(p^{(3)})\varphi_j(p^{(4)})-\varphi^i(p^{(1)})\varphi_i(p^{(2)})\varphi^j(p^{(3)})\varphi_j(p^{(4)})\right) \\ &= \frac{g^2_{YM}}{2}\int \mathrm{d}^{p+1}x\,\mathrm{tr}\left[\varphi^i,\varphi^j\right]\left[\varphi_i,\varphi_j\right] \end{aligned} \tag{61b}$$

Now we shall calculate the effective interaction term for the scalar field and the gauge field $S^{\text{effective}}_{AA\varphi\varphi}$ by choosing the external string state as

$$\begin{aligned} \langle AA\varphi\varphi| &= \langle 0|\Big\{\boldsymbol{A}(1)\boldsymbol{A}(2)\boldsymbol{\varphi}(3)\boldsymbol{\varphi}(4)+\boldsymbol{A}(1)\boldsymbol{\varphi}(2)\boldsymbol{A}(3)\boldsymbol{\varphi}(4)+\boldsymbol{A}(1)\boldsymbol{\varphi}(2)\boldsymbol{\varphi}(3)\boldsymbol{A}(4)+\boldsymbol{\varphi}(1)\boldsymbol{A}(2)\boldsymbol{A}(3)\boldsymbol{\varphi}(4) \\ &\quad+\boldsymbol{\varphi}(1)\boldsymbol{A}(2)\boldsymbol{\varphi}(3)\boldsymbol{A}(4)+\boldsymbol{\varphi}(1)\boldsymbol{\varphi}(2)\boldsymbol{A}(3)\boldsymbol{A}(4)\Big\}, \end{aligned} \tag{62a}$$

where

$$\boldsymbol{A}(r) = A_\mu(p^{(r)})a^{(r)\mu}_1, \quad \boldsymbol{\varphi}(r)=\varphi_i(p^{(r)})a^{(r)i}_1, \quad r=1,2,3,4. \tag{62b}$$

Making use of Eq. (47a) and Eq. (62a) we find

$$\begin{aligned} S^{\text{effective}}_{AA\varphi\varphi} &= -\frac{1}{2!}\times 2\times\frac{1}{2^2}g^2_{YM}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\int\left|\frac{\prod^4_{r=1}\mathrm{d}Z_r}{\mathrm{d}V_{abc}}\right|\prod_{r<s}|Z_r-Z_s|^{p_r\cdot p_s}\exp\left[-\sum_{r=1}^4\bar{N}^{[4]rr}_{00}\right] \\ &\quad\times\mathrm{tr}\Big\langle AA\varphi\varphi\Big|\left\{\sum_{r,s=1}^4\bar{N}^{[4]rs}_{11}a^{(r)\dagger}_{1\mu}a^{(s)\dagger}_{1\nu}\eta^{\mu\nu}\right\}\left\{\sum_{r,s=1}^4\bar{N}^{[4]rs}_{11}a^{(r)\dagger}_{1i}a^{(s)\dagger}_{1j}\eta^{ij}\right\}\Big|0\Big\rangle. \end{aligned} \tag{63}$$

In terms of the Koba-Nielson variable x, we may rewrite $S^{\text{effective}}_{AA\varphi\varphi}$ as

$$\begin{aligned} S^{\text{effective}}_{AA\varphi\varphi} &= -\frac{g^2_{YM}}{4}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\int_0^1 \mathrm{d}x\,x^{-\frac{s}{2}}(1-x)^{-\frac{t}{2}} \\ &\quad\times\mathrm{tr}\Big\{\frac{1}{x^2}A_\mu(p^{(1)})A^\mu(p^{(2)})\varphi_i(p^{(3)})\varphi^i(p^{(4)})+A_\mu(p^{(1)})\varphi_i(p^{(2)})A^\mu(p^{(3)})\varphi^i(p^{(4)}) \\ &\quad+\frac{1}{(1-x)^2}A_\mu(p^{(1)})\varphi_i(p^{(2)})\varphi^i(p^{(3)})A^\mu(p^{(4)})+\frac{1}{(1-x)^2}\varphi_i(p^{(1)})A_\mu(p^{(2)})A^\mu(p^{(3)})\varphi^i(p^{(4)}) \\ &\quad+\varphi_i(p^{(1)})A_\mu(p^{(2)})\varphi^i(p^{(3)})A^\mu(p^{(4)})+\frac{1}{x^2}\varphi_i(p^{(1)})\varphi^i(p^{(2)})A_\mu(p^{(3)})A^\mu(p^{(4)})\Big\} \\ &= -\frac{g^2_{YM}}{4}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\Big\{\frac{u}{s}A_\mu(p^{(1)})A^\mu(p^{(2)})\varphi_i(p^{(3)})\varphi^i(p^{(4)}) \\ &\quad+A_\mu(p^{(1)})\varphi_i(p^{(2)})A^\mu(p^{(3)})\varphi^i(p^{(4)})+\frac{u}{t}A_\mu(p^{(1)})\varphi_i(p^{(2)})\varphi^i(p^{(3)})A^\mu(p^{(4)}) \end{aligned}$$

$$+\frac{u}{t}\varphi_i(p^{(1)})A_\mu(p^{(2)})A^\mu(p^{(3)})\varphi^i(p^{(4)})+\varphi_i(p^{(1)})A_\mu(p^{(2)})\varphi^i(p^{(3)})A^\mu(p^{(4)})$$
$$+\frac{u}{s}\varphi_i(p^{(1)})\varphi^i(p^{(2)})A_\mu(p^{(3)})A^\mu(p^{(4)})\Bigg\}. \tag{64}$$

By rearranging terms in Eq. (64), we may express the effective action in the zero-slope limit as

$$S^{\text{effective}}_{AA\varphi\varphi} = -\frac{g_{YM}^2}{2}\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\left\{A_\mu(p^{(1)})\varphi_i(p^{(2)})A^\mu(p^{(3)})\varphi^i(p^{(4)})+\frac{2u}{s}A_\mu(p^{(1)})A^\mu(p^{(2)})\varphi_i(p^{(3)})\varphi^i(p^{(4)})\right\}. \tag{65}$$

From $S^{\text{effective}}_{AA\varphi\varphi}$ we should substract the effective gauge-scalar field interaction $S^{\text{massless}}_{AA\varphi\varphi}$, which is generated by the cubic intractions, S_{AAA} (Eq. (42)) and $S_{A\varphi\varphi}$ (Eq. (44)) to identify the contact gauge-scalar field interaction:

$$S^{\text{effective}}_{AA\varphi\varphi} = S_{AA\varphi\varphi}+S^{\text{massless}}_{AA\varphi\varphi}. \tag{66}$$

The Feynman diagrams corresponding to the effective gauge-scalar field interaction $S^{\text{massless}}_{AA\varphi\varphi}$, which is mediated by massless gauge fields, are depicted in Fig. 7. The effective gauge-scalar field interaction $S^{\text{massless}}_{AA\varphi\varphi}$ may be evaluated as

$$\begin{aligned} S^{\text{massless}}_{AA\varphi\varphi} &= g_{YM}^2\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\mathrm{tr}\left\{A_\mu(p^{(1)})A^\mu(p^{(2)})\left(p^{(1)}_\rho-p^{(2)}_\rho\right)\frac{\eta^{\rho\sigma}}{\left(p^{(1)}+p^{(2)}\right)^2}\left(p^{(3)}_\sigma-p^{(4)}_\sigma\right)\varphi_i(p_3)\varphi^i(p_4)\right\} \\ &= -g_{YM}^2\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\left(1+\frac{2u}{s}\right)\mathrm{tr}\left(A_\mu(p^{(1)})A^\mu(p^{(2)})\varphi_i(p_3)\varphi^i(p_4)\right). \end{aligned} \tag{67}$$

If Eq. (65) and Eq. (67) are used, the contact interaction term $S_{AA\varphi\varphi}$ is identified as

$$\begin{aligned} S_{AA\varphi\varphi} &= -g_{YM}^2\int\prod_{r=1}^4 \mathrm{d}p^{(r)}\delta\left(\sum_{r=1}^4 p^{(r)}\right)\mathrm{tr}\left(A_\mu(p^{(1)})\varphi_i(p^{(2)})A^\mu(p^{(3)})\varphi^i(p^{(4)})-A_\mu(p^{(1)})A^\mu(p^{(2)})\varphi_i(p_3)\varphi^i(p_4)\right) \\ &= -\frac{g_{YM}^2}{2}\int \mathrm{d}^{p+1}x\,\mathrm{tr}\left[A_\mu,\varphi_i\right]\left[A^\mu,\varphi^i\right]. \end{aligned} \tag{68}$$

It should be noted that the sign in front of the contact quartic interaction between gauge fields A_μ and scalar fields φ_i in Eq. (68) differs from those in front of the other two contact quartic interactions, S_{AAAA} in Eq. (54b) and $S_{\varphi\varphi\varphi\varphi}$ in Eq. (61b). It plays an important role, as we shall see in the next section. If we had applied a simple dimensional reduction to effective field theory which describes the zero-slope limit of the string field theory in the critical dimensions, we would have obtained a different result.

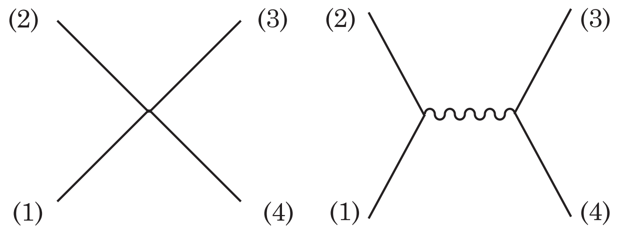

Fig. 6. Effective four-scalar field interactions.

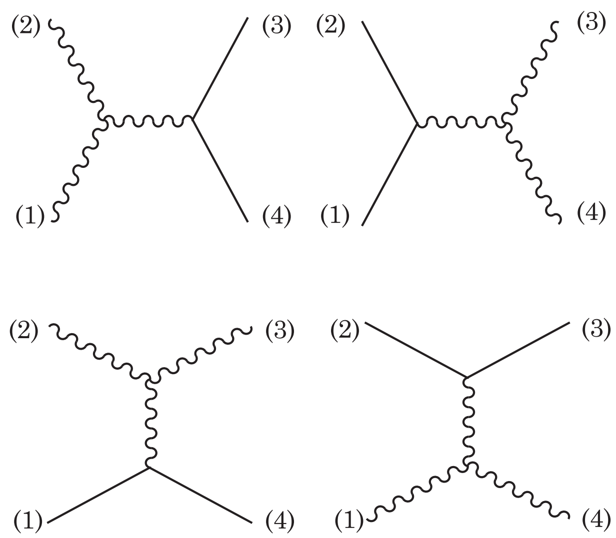

Fig. 7. Effective gauge-scalar field interactions.

7 Matrix models and cubic string field theory in the zero-slope limit

If we collect the effective actions which are represented by field theoretical actions for the $U(N)$ matrix valued gauge fields and scalar fields, we have

$$\begin{aligned} S &= S_0+S_{AAA}+S_{A\varphi\varphi}+S_{AAAA}+S_{AA\varphi\varphi}+S_{\varphi\varphi\varphi\varphi} \\ &= \int \mathrm{d}^{p+1}x\,\mathrm{tr}\left\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\left(D_\mu\varphi^i\right)^2 \right. \\ &\quad \left. +\frac{g_{YM}^2}{2}\left[\varphi^i,\varphi^j\right]\left[\varphi_i,\varphi_j\right]\right\}, \end{aligned} \tag{69a}$$

where

$$D_\mu\varphi^i=\frac{\partial\varphi^i}{\partial x^\mu}-\mathrm{i}g_{YM}\left[A_\mu,\varphi^i\right]. \tag{69b}$$

Here S_0 is the free field actions for the gauge fields and the scalar fields, which may be derived easily from the kinetic term of the string free field action $\mathrm{tr}\Psi*Q\Psi$. It is worth mentioning that we obtain the gauge invariant action without redefining fields. We only need to impose the Lorentz gauge fixing condition: $\partial_\mu A^\mu=0$. The resultant action, Eq. (69a), is precisely the effective field theoretical action on multiple Dp-branes which describes the dynamics of multiple Dp-branes in the low energy region.

If $p=0$, the covariant action for the gauge fields will be absent from the action and the field theoretical action reduces to a quantum mechanical action:

$$S=\int \mathrm{d}t\,\mathrm{tr}\left\{\frac{1}{2}\left(D_t\varphi^i\right)^2+\frac{g_{YM}^2}{2}\left[\varphi^i,\varphi^j\right]\left[\varphi_i,\varphi_j\right]\right\}, \tag{70a}$$

where

$$D_t\varphi^i=\frac{\mathrm{d}\varphi^i}{\mathrm{d}t}-ig_{YM}\left[A,\varphi^i\right]. \tag{70b}$$

This quantum mechanical action, Eq. (70a), is the bosonic part of the fundamental action of the BFSS matrix model where the $U(N)$ matrix valued scalar fields φ^i, $i=1,\cdots,d$, play the roles of $D0$-brane transverse coordinates. We may observe that in Eq. (70a) the gauge field A is auxiliary. However, in deriving the effective action for $D0$-branes from the string field theory, we should treat the gauge field A as dynamical. Otherwise, we cannot get the correct contact four-scalar interaction term.

What is more interesting is the case where $p=-1$: if $p=-1$, all string coordinates X^I, $I=0,1,\ldots,d$, satisfy the Dirichlet condition so that there are no zero modes of string coordinates and their canonical conjugates. The string field action on the multiple D-instanton may be written as

$$\mathcal{S}=\mathrm{tr}\left\{\langle\Psi|(N-1)|\Psi\rangle+\frac{2g}{3}\langle\Psi|\Psi*\Psi\rangle\right\}, \tag{71}$$

where N is the total number operator,

$$N=\sum_{n=1} n a_{nI}^\dagger a_{nJ}\eta^{IJ}. \tag{72}$$

The free string field action, $\mathrm{tr}\langle\Psi|(N-1)|\Psi\rangle$, vanishes for the vector field states $|\varphi^I\rangle$, $I=0,1,\ldots,d$. The Fock space representation of the three-string vertex for the open strings on the D-instanton is given by

$$\begin{aligned} &E_D[1,2,3]|0\rangle \\ &= \exp\left\{-\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m\geqslant1}\bar{N}_{nm}^{rs}\alpha_{nI}^{(r)\dagger}\alpha_{mJ}^{(s)\dagger}\eta^{IJ}-\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\right\}|0\rangle, \end{aligned} \tag{73}$$

Choosing the external string state as

$$\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}|=\left\langle 0\right|\prod_{r=1}^{3}\left(\varphi_I a_{1J}^{(r)}\eta^{IJ}\right), \tag{74}$$

we find that there is no cubic term for the vector field φ^I:

$$S_{\varphi\varphi\varphi}=\frac{2g}{3}\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}|E[1,2,3]_D|0\rangle=0. \tag{75}$$

The four-string scattering amplitude for the open strings on multiple D-instantons may be expressed as

$$\begin{aligned} \mathcal{F}_{[4]D} &= 2g^2\int\left|\frac{\prod_{r=1}^4\mathrm{d}Z_r}{\mathrm{d}V_{abc}}\right|\mathrm{e}^{-\sum_{r=1}^4\bar{N}_{00}^{rr}} \\ &\quad\times\mathrm{tr}\left\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)},\Psi^{(4)}\right|\exp[E_{[4]D}]\left|0\right\rangle, \end{aligned} \tag{76a}$$

$$E_{[4]D} = \sum_{r,s=1}^{4}\left\{-\frac{1}{2}\sum_{r,s=1}^{4}\sum_{m,n=1}\bar{N}_{mn}^{[4]rs}\alpha_{mI}^{(r)\dagger}\alpha_{nJ}^{(s)\dagger}\eta^{IJ}\right\}. \tag{76b}$$

We may calculate the effective four-vector interaction by choosing the external string state as

$$\left\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)},\Psi^{(4)}\right|=\left\langle 0\right|\prod_{r=1}^{4}\left(\varphi_I a_{1J}^{(r)}\eta^{IJ}\right). \tag{77}$$

Using Eq. (76a), we find that the effective four-vector interaction is evaluated to be

$$\begin{aligned} S_{\varphi\varphi\varphi\varphi}^{\text{effective}} &= g_{YM}^2\int\left|\frac{\prod_{r=1}^4\mathrm{d}Z_r}{\mathrm{d}V_{abc}}\right|\mathrm{e}^{-\sum_{r=1}^4\bar{N}_{00}^{rr}} \\ &\quad\times\mathrm{tr}\left\langle 0\right|\left\{\prod_{r=1}^{4}\varphi_I a_{1J}^{(r)}\eta^{IJ}\right\} \\ &\quad\times\frac{1}{2!}\times\frac{1}{2^2}\left\{-\sum_{r,s=1}^{4}\bar{N}_{11}^{[4]rs}a_{1I}^{(r)\dagger}a_{1J}^{(s)\dagger}\eta^{IJ}\right\}^2\left|0\right\rangle. \end{aligned} \tag{78}$$

Making use of the Neumann functions for the four-string vertex in the proper-time gauge [6, 7] leads us to

$$\begin{aligned} S_{\varphi\varphi\varphi\varphi}^{\text{effective}} &= g_{YM}^2\int_0^1\mathrm{d}x\,\mathrm{tr}\left(\varphi^I\varphi^J\varphi_I\varphi_J+\frac{2}{x^2}\varphi^I\varphi_I\varphi^J\varphi_J\right) \\ &= g_{YM}^2\,\mathrm{tr}\left\{\frac{1}{2}[\varphi^I,\varphi^J][\varphi_I,\varphi_J]+\left(\frac{2}{\epsilon}-1\right)\left(\varphi^I\varphi_I\right)^2\right\}. \end{aligned} \tag{79}$$

By comparing the effective action with the bosonic part of the IKKT matrix model,

$$S_{IKKT} = \frac{g_{YM}^2}{2}\mathrm{tr}[\varphi^I,\varphi^J][\varphi_I,\varphi_J], \tag{80}$$

we find that the effective matrix action $S^{\text{effective}}_{\varphi\varphi\varphi\varphi}$ differs from the bosonic part of the IKKT matrix model action by $\left(\frac{2}{\epsilon}-1\right)\left(\varphi^I\varphi_I\right)^2$, which is divergent. It is hard to conceive that this term arises as an effective interaction between the vector φ^I and some other string states at low mass level. We may recall that the vector states φ^I are only string states at mass level 1 and there is no cubic interaction term for φ^I. This point may be clarified further in the supersymmetric, BRST invariant string field theory on multiple D-instantons.

8 Discussion and conclusions

In this present work, we have discussed cubic open string field theories on multiple Dp-branes. Interacting string field theories have been studied only for open strings on space-filling D-branes, which satisfy the Neumann boundary conditions. However, it is equally important to explore the open string field theory on multiple Dp-branes because we can define covariant field theories in dimensions lower than the critical dimensions within the framework of string theory. Even some non-renormalizable quantum field theories may be studied in a consistent manner as effective theories, describing the dynamics of open strings on Dp-branes in the low energy region. Open string field theories on multiple $D0$-branes and on multiple D-instantons are of particular importance, as they are intimately related to the matrix model of Banks-Fishler-Shenker-Susskind (BFSS) and the matrix model of Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) respectively.

We defined the open string field theory on multiple Dp-branes by extending Witten's cubic open string field for strings on a space-filling D-brane [38, 39]. Then, we showed that the deformation procedure, developed previously for Witten's cubic open string field theory, is also applicable to string field theory on multiple Dp-branes. By mapping the world sheet for three-string scattering onto the upper half plane and using the simple Green's functions on the upper half plane, we constructed the Fock space representation of the three-string vertex. The Fock space representation of the four-string vertex on multiple Dp-branes has also been constructed in a similar manner. The effective field theories, describing the open strings on multiple Dp-branes in the low energy region, are obtained by choosing the low mass level string states and evaluating the three-string and four-string scattering amplitudes. The resultant effective actions are those of the $U(N)$ matrix valued scalar fields, interacting with $U(N)$ non-Abelian gauge fields in $(p+1)$ dimensions. It must be emphasized that we have obtained the gauge covariant effective actions without using the field redefinition or the level truncations, in contrast to previous works in the literature. We only need to impose the usual Lorentz gauge fixing condition.

If we choose $p = 0$, the effective field theory action reduces to a quantum mechanical action, i.e., the bosonic part of the fundamental action of the BFSS matrix model. We confirmed this by an explicit evaluation. We also noted that one should treat the gauge field A as a dynamical field when one derives the effective quantum mechanical action from the string field theory on multiple Dp-branes, although the role of the gauge field becomes auxiliary in the final form of the matrix model action. If we further lower p to define the string field theory on multiple D-instantons, we expect that the effective theory is described by matrices only. However, the effective matrix action contains a divergent term and differs from the IKKT model action by this divergent term. We could not find a satisfactory resolution for this discrepancy within the framework of the open bosonic string theory. The resolution may be found in a complete string theory which is BRST invariant and super-symmetric.

We have shown that the deformed cubic open string field theory, if properly defined on multiple Dp-branes, correctly captures the dynamics of multiple Dp-branes as the theory reduces to the previously known effective gauge covariant field theory. The advantage of the string field theory approach over others is evident. Because the string field theory possesses the full degrees of freedom of the open string, it is possible to develop a systematic perturbation theory and to calculate the non-zero slope corrections which were beyond the scopes of previous approaches.

Recently, the cubic open string field theories on Dp-branes have been extended to closed string theory in the proper-time gauge. The three-closed-string scattering [40] and the four-closed-string scattering amplitudes [41] have been calculated. They are shown to reduce to the three-graviton scattering and the four-graviton scattering amplitudes of perturbative Einstein gravity in the zero-slope limit, respectively. Because the obtained scattering amplitudes of closed strings are valid for the full range of energy, they will be useful to study the ultraviolet completion of quantum gravity. Open string field theory in the proper-time gauge is also found to be useful to study the entanglement entropy of strings, which may differ from the entanglement entropy of quantum field theories: the entanglement entropies of string on Dp-branes [42] may be finite in contrast to their counterparts in quantum field theories. This may help us to understand black hole entropy as an entanglement entropy of strings.

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