A new model for calculating the binding energy of the lithium nucleus under the generalized Yukawa potential and Hellmann potential

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Abstract: In this paper, the Schrödinger equation for a 6-body system is studied. We solve this equation for the lithium nucleus by using a supersymmetry method with several specific potentials. These potentials are the Yukawa potential, the generalized Yukawa potential and the Hellmann potential. The results of our model for all calculations show that the ground state binding energy of the lithium nucleus with these potentials is very close to that obtained experimentally.

Key words: lithium nucleus; binding energy, Schrödinger equation, supersymmetry, Yukawa potential, generalized Yukawa potential, Hellmann potential

PACS: 21.10.-k, 21.10.Dr, 24.10.Cn   **DOI:** 10.1088/1674-1137/39/6/063106

1   Introduction

A small number of exact solutions to the Schrödinger equation were obtained historically in the genesis of quantum mechanics. One of the important tasks of quantum mechanics is to solve the Schrödinger equation with physical potentials. Over the past several decades, much effort has been made to study the stationary Schrödinger equation with central potentials. Until now, the Schrödinger equation for 2 and 3 particles has been solved for different potentials using several methods such as the Nikiforov-Uvarov (NU) method, Supersymmetry method (SUSY), and Ansatz method [1–9]. Similar work has also been done for other equations (for example, the Dirac equation and Klein-Gordon equation) [10]. We now want to do this for 6-body particles (such as the lithium nucleus). In this work the Supersymmetry method is used to study the 6-body Schrödinger equation with central potential.

In Sections 2 and 3 we briefly review the Yukawa potential, generalized Yukawa potential and Hellmann potential. In Section 4, the Supersymmetry method is reviewed. We then study an analytical solution to the Schrödinger equation for a 6-body system and report the numerical results. Section (6) gives a summary of the paper and our conclusions.

2   The Yukawa potential

The Yukawa potential is of interest in many areas of physics [11]. In high energy physics, it is used to model strong interactions due to meson exchange [11–13]. To display the cloud of electronic charges around the nucleus leads to a screened Coulomb potential, in atomic and molecular physics, it is known, or to account for the shielding by outer charges of the Coulomb field experienced by an atomic electron in hydrogen plasma. The basic form of this potential is given by [14]:

$$V = -\frac{V_0 e^{-ar}}{r}, \tag{1}$$

where a is the screening parameter and V_0 is the strength of the Yukawa potential. This potential is often used to compute bound-state normalizations and energy levels of neutral atoms. To get better physical results, we now add a term to this potential, giving the generalized Yukawa potential, which takes the form:

$$V = \frac{-V_0 e^{-ar}}{r} + \frac{k}{r^2}, \tag{2}$$

where $\frac{k}{r^2}$ removes the degeneracies and k is the strength of the generalized Yukawa potential.

3   The Hellmann potential

The Hellmann potential has the form:

$$V = \frac{V_0 e^{-ar}}{r} - \frac{k}{r}. \tag{3}$$

where k and V_0 are the strengths of the Coulomb and the Yukawa potential respectively, and a is the screening

parameter. This potential reduces to Yukawa, Coulomb and Cornell potentials. For $k=0$, the Hellmann potential changes into the Yukawa potential $\left(V=\dfrac{V_0 e^{-ar}}{r}\right)$ which is a nuclear potential. For $V_0=0$, the Hellmann potential turns into the Coulomb potential $\left(V=-\dfrac{k}{r}\right)$, which is a nuclear potential. For $a=0$, this potential reduces into the Cornell potential $\left(V=\dfrac{V_0-k}{r}\right)$ which is also a nuclear potential. The Hellmann potential has many applications in atomic physics and condensed-matter physics [15–25]. This potential, with V_0 positive, was suggested originally by Hellmann [16, 26] and henceforth called the Hellmann potential if V_0 is positive or negative. The Hellmann potential has been used as a model for alkali hydride molecules [18]. It has also been used to represent the electron-ion [19, 20] and electron core interactions [21, 22]. It has also been shown that the main properties of the effective two-particle interaction for charged particles in polar crystals may be described by this potential [23–25].

4   Supersymmetry method in quantum physics

We start by noticing that we know the ground state function of a 1-dimension problem; we can find also the potential, up to a constant [9]. Taking the ground state energy to be zero, from the time-independent Schrödinger Equation (TISE) we have:

$$H_1\Psi_0(x)=-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}+V_1(x)\Psi_0=0, \qquad (4)$$

$$V_1=\frac{\hbar^2}{2m}\frac{\Psi_0''(x)}{\Psi_0(x)}. \qquad (5)$$

We can try to factorize the Hamiltonian with the ansatz

$$H_1=A^+A=H-E_0, \qquad (6)$$

where:

$$A=\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x}+W(x), \quad A^+=-\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}x}+W(x), \qquad (7)$$

and $W(x)$ is called the superpotential and can be written in terms of the ground state as

$$W(x)=-\frac{\hbar}{\sqrt{2m}}\frac{\Psi_0'}{\sqrt{\Psi_0}}. \qquad (8)$$

By writing V_1 in terms of $W(x)$, we obtain the Ricatti equation:

$$V_1(x)=W^2(x)-\frac{\hbar}{\sqrt{2m}}W'(x). \qquad (9)$$

We can now build up a SUSY theory searching for the SUSY partner Hamiltonian associated to H_1, namely

$H_2=AA^+$. This second Hamiltonian corresponds to a new potential:

$$H_2=-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_2(x), \ V_2(x)=W^2(x)+\frac{\hbar}{\sqrt{2m}}W'(x). \qquad (10)$$

5   Exact solution of the Schrödinger equation for the Yukawa potential, generalized Yukawa potential and Hellmann potential

The Schrödinger equation in D-dimensions is:

$$\frac{-\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{D-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+D-2)}{r^2}\right)R_{n,l}(r)$$

$$+V(r)R_{n,l}(r)=E_{n,l}R_{n,l}(r), \ D=3N-3, \qquad (11)$$

where n is the number of particles and l is the angular momentum [27].

Here, consider the Schrödinger equation for a 6-body system with a potential $V(r)$ that depends only on the distance r from the origin:

$$\frac{-\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{14}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+13)}{r^2}\right)R_{n,l}(r)$$

$$+V(r)R_{n,l}(r)=E_{n,l}R_{n,l}(r), \qquad (12)$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{14}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+13)}{r^2}\right)R_{n,l}(r)$$

$$+\frac{2\mu}{\hbar^2}\left(E_{n,l}-V(r)\right)R_{n,l}(r)=0. \qquad (13)$$

By applying $U_{n,l}=R_{n,l}r^{\frac{D-1}{2}}=R_{n,l}r^7$, we can write:

$$\frac{\mathrm{d}R_{n,l}}{\mathrm{d}r}=\frac{\mathrm{d}U_{n,l}}{\mathrm{d}r}r^{-7}-7r^{-8}U_{n,l},$$

$$\frac{\mathrm{d}^2R_{n,l}}{\mathrm{d}r^2}=\frac{\mathrm{d}^2U_{n,l}}{\mathrm{d}r^2}r^{-7}-14r^{-8}\frac{\mathrm{d}U_{n,l}}{\mathrm{d}r}+56r^{-9}U_{n,l}. \qquad (14)$$

By substituting Eq. (14) in Eq. (13) we find the following form for Eq. (13):

$$\frac{\mathrm{d}^2U_{n,l}}{\mathrm{d}r^2}r^{-7}-42r^{-9}U_{n,l}-l(l+13)r^{-9}U_{n,l}$$

$$+\frac{2\mu}{\hbar^2}\left(E_{n,l}-V(r)\right)U_{n,l}r^{-7}=0. \qquad (15)$$

By some summarizing, Eq. (15) becomes:

$$\frac{\mathrm{d}^2U_{n,l}}{\mathrm{d}r^2}+\frac{2\mu}{\hbar^2}\left(E_{n,l}-V(r)-\frac{\hbar^2(l+6)(l+7)}{2\mu r^2}\right)U_{n,l}=0, \qquad (16)$$

where μ is the reduced mass.

5.1   Solution with the Yukawa potential

Introducing the Yukawa potential and using the Taylor expansion, the potential takes the form:

$$V = -\frac{V_0 e^{-ar}}{r} = -V_0\frac{(1-ar)}{r} = -\frac{V_0}{r} + aV_0. \quad (17)$$

By putting Eq. (17) into Eq. (16) the Schrödinger equation changes to:

$$\frac{\mathrm{d}^2 U_{n,l}(r)}{\mathrm{d}r^2} + \frac{2\mu}{\hbar^2}\left(E_{n,l} - aV_0 + \frac{V_0}{r} - \frac{\Omega}{r^2}\right)U_{n,l}(r) = 0, \quad (18)$$

where Ω is defined as $\Omega = \dfrac{\hbar^2(l+6)(l+7)}{2\mu}$. By choosing $\varepsilon_{n,l} = \dfrac{2\mu}{\hbar^2}(E_{n,l} - aV_0)$, $\beta = \dfrac{4\mu}{\hbar^2}V_0$ and $\gamma = \dfrac{2\mu}{\hbar^2}\Omega$, Eq. (18) changes to:

$$\frac{\mathrm{d}^2 U_{n,l}(r)}{\mathrm{d}r^2} + \left(\varepsilon_{n,l} + \frac{\beta}{r} - \frac{\gamma}{r^2}\right)U_{n,l}(r) = 0. \quad (19)$$

In Supersymmetric quantum mechanics, the superpotential is defined as:

$$W_1 = -\frac{\hbar}{\sqrt{2\mu}}\left(A + \frac{B}{r}\right). \quad (20)$$

Substituting this superpotential into the Riccati equation has the form:

$$W_1^2(x) - \frac{\hbar}{\sqrt{2\mu}}W_1'(x) = \frac{2\mu}{\hbar^2}\left(V_1(x) - E_0^{(1)}\right). \quad (21)$$

We then obtain

$$\left(A^2 + \frac{B^2}{r^2} + \frac{2AB}{r} - \frac{B}{r^2}\right) = \left(-\varepsilon_{n,l} - \frac{\beta}{r} + \frac{\gamma}{r^2}\right). \quad (22)$$

By doing some calculations, we can get $A^2 = -\varepsilon_{n,l}$, $2AB = -\beta$, $B^2 - B = \gamma$ and the ground state binding energy for the lithium nucleus is given as:

$$E_{n,l} = \frac{-\hbar^2}{2\mu}\left(\frac{-\beta}{1 \pm \sqrt{1+4\gamma}}\right)^2 + V_0 a. \quad (23)$$

By using Eq. (24) from the SUSY method:

$$\psi_0^1(r) = N_0 \exp\left(-\frac{\sqrt{2\mu}}{\hbar}\int W(r')\mathrm{d}r'\right), \quad (24)$$

and the ground state normalized eigenfunctions are given as:

$$\psi_0^1(r) = N_0\exp[Ar + B\ln r]. \quad (25)$$

In Table 1, the fitted values of parameters of the ground state binding energy equations for the Yukawa potential are given.

From Table 1, it can be seen that for $a = 0.80\ \mathrm{fm}^{-1}$ and $V_0 = 40$ MeV, the calculated ground state binding energy (31.93 MeV) is in good agreement with the experimental data.

Table 1.   Fitted values of parameters of the ground state binding energy equations for the Yukawa potential. Column $B.E$ (our model) contains our calculation and column $B.E$ (experiment) contains the experimental data.

a/fm^{-1}	$V_0/(\mathrm{MeV \cdot fm})$	$B.E/\mathrm{MeV}$	$B.E$(experiment) [28]
0.59	50	29.39	31.995
0.70	45	31.42	31.995
0.95	33	31.31	31.995
0.70	40	27.93	31.995
0.80	40	31.93	31.995
0.80	30	23.96	31.995
0.60	50	29.90	31.995
0.50	50	24.90	31.995
0.90	35	31.45	31.995
0.64	50	31.90	31.995

As we know from SUSY, the potential is determined as:

$$V_\pm = W^2 \pm \frac{\hbar}{\sqrt{2\mu}}\frac{\mathrm{d}W}{\mathrm{d}r} = \frac{\hbar^2}{2\mu}\left[A^2 + \frac{B^2}{r^2} + \frac{2AB}{r^2} \pm \frac{B}{r^2}\right]. \quad (26)$$

We obtain $A = \dfrac{-\beta}{2(B^2-\gamma)}$ from Eq. (22). By substituting A in Eq. (26) we arrive at:

$$V_\pm = \frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2(B^2-\gamma)}\right)^2 + \frac{B^2}{r^2} + \frac{2\left(\dfrac{-\beta}{2(B^2-\gamma)}\right)B}{r^2} \pm \frac{B}{r^2}\right]. \quad (27)$$

The shape invariance concept that was introduced by Gendenshtein is [27]

$$V_+(a_0, r) = V_-(a_1, r) + R(a_1), \quad (28)$$

where a_1 is a function of a_0 and $R(a_1)$ is independent of r. Hence, the energy spectrum becomes:

$$E_0^{(k)} = \sum_{i=0}^{k} R(a_i), \quad E_{nl} = E_{nl}^- + E_0, \quad (29)$$

If we now consider a mapping of the form:

$$B \to B' = B - a, \quad (30)$$

then in Eq. (27), it is easily seen that apart from a constant, the partner potentials are the same – the chosen SUSY potential satisfies the shape invariance condition. On the other hand, we can obtain:

$$B_1 = B_0 - a, \quad B_n = B_0 - na, \quad (31)$$

$$R(a_1) = V_+(B,r) - V_-(B-a,r)$$

$$= \frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2(B^2-\gamma)}\right)^2 + \frac{B^2}{r^2} + \frac{2\left(\frac{-\beta}{2(B^2-\gamma)}\right)B}{r^2}\right.$$

$$+ \frac{B}{r^2} - \left(\frac{-\beta}{2((B-a)^2-\gamma)}\right)^2 - \frac{(B-a)^2}{r^2}$$

$$\left.- \frac{2\left(\frac{-\beta}{2((B-a)^2-\gamma)}\right)(B-a)}{r^2} + \frac{B-a}{r^2}\right], \quad (32)$$

$$\rightarrow R(a_i) = -\frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2((B-ia)^2-\gamma)}\right)^2\right.$$
$$\left.- \left(\frac{-\beta}{2((B-(i-1))^2-\gamma)}\right)^2\right], \quad (33)$$

$$R(a_i) = -\frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2((B-ia)^2-\gamma)}\right)^2\right.$$
$$\left.- \left(\frac{-\beta}{2((B-(i-1)a)^2-\gamma)}\right)^2\right]. \quad (34)$$

The remainder $R(a_i)$ is independent of r. Using Eqs. (23) and (29), the energy levels of the Yukawa potential are found as:

$$E_{n,l} = -\frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2((B-na)^2-\gamma)}\right)^2 - \left(\frac{-\beta}{2(B^2-\gamma)}\right)^2\right.$$
$$\left.+ \left(\frac{-\beta}{1+\sqrt{1+4\gamma}}\right)^2\right] + aV_0. \quad (35)$$

5.2 Solution with the generalized Yukawa potential

Now we change the potential in Eq. (16). To get better results, we use the generalized Yukawa potential which takes the form:

$$V = \frac{-V_0 e^{-ar}}{r} + \frac{k}{r^2}. \quad (36)$$

By substituting Eq. (36) in Eq. (16), we obtain:

$$\frac{d^2 U_{n,l}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left(E_{n,l} + \frac{V_0 e^{-ar}}{r} - \frac{k}{r^2}\right.$$
$$\left.- \frac{\hbar^2(l+6)(l+7)}{2\mu r^2}\right) U_{n,l}(r) = 0. \quad (37)$$

By using a Taylor expansion, Eq. (37) can be written as:

$$\frac{d^2 U_{n,l}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left(E_{n,l} - aV_0 + \frac{V_0}{r} - \frac{\Omega+k}{r^2}\right) U_{n,l}(r) = 0, \quad (38)$$

where Ω is defined as $\Omega = \dfrac{\hbar^2(l+6)(l+7)}{2\mu}$. If we get $\varepsilon_{n,l} = \dfrac{2\mu}{\hbar^2}(E - aV_0)$, $\beta = \dfrac{4\mu}{\hbar^2}V_0$ and $\gamma = \dfrac{2\mu}{\hbar^2}(\Omega+k)$, then we obtain:

$$\frac{d^2 U_{n,l}(r)}{dr^2} + \left(\varepsilon_{n,l} + \frac{\beta}{r} - \frac{\gamma}{r^2}\right) U_{n,l}(r) = 0. \quad (39)$$

As we know from SUSY the superpotential is defined as Eq. (20). When we put this into the Ricatti equation (Eq. (21)), as we did before, we get $A^2 = -\varepsilon_{n,l}$, $2AB = -\beta$ and $B^2 - B = \gamma$. The ground state binding energy is then obtained as:

$$E_{n,l} = \frac{-\hbar^2}{2\mu}\left(\frac{-\beta}{1\pm\sqrt{1+4\gamma}}\right)^2 + V_0 a. \quad (40)$$

By using Eq. (24) from the SUSY method, the ground state normalized eigenfunctions are given as Eq. (25). In Table 2, the fitted values of parameters of the ground state binding energy equations for the generalized Yukawa potential are given.

Table 2. Fitted values of parameters of the ground state binding energy equations for the generalized Yukawa potential. Column $B.E$ (our model) contains our calculation and column $B.E$ (experiment) contains the experimental data.

a/fm^{-1}	V_0/(MeV·fm)	$B.E$/MeV	$B.E$(experiment) [28]	k/(MeV·fm^2)
0.70	45	31.42	31.995	5
0.60	50	29.90	31.995	10
0.64	49.3	31.90	31.995	50
0.70	50	34.90	31.995	30
0.75	40	29.93	31.995	30
0.50	50	24.90	31.995	30
0.55	50	27.40	31.995	50
0.80	30	23.96	31.995	50
0.80	40	31.98	31.995	30
0.59	50	29.40	31.995	50

Table 3. Fitted values of parameters of the ground state binding energy equations for the Hellmann potential. Column $B.E$ (our model) contains our calculation and Column $B.E$ (experiment) contains the experimental data.

a/fm^{-1}	$V_0/(\text{MeV}\cdot\text{fm})$	$B.E/\text{MeV}$	$B.E$ (experiment) [28]	$k/(\text{MeV}\cdot\text{fm})$
0.60	-50	29.85	31.995	10
0.60	-50	29.80	31.995	20
0.65	-50	32.38	31.995	5
0.58	-50	28.85	31.995	10
0.65	-49	31.70	31.995	10
0.65	-49	31.65	31.995	20
0.70	-40	27.85	31.995	20
0.71	-45	31.83	31.995	10
0.50	-50	24.85	31.995	10
0.55	-50	27.35	31.995	10

From Table 2, it can be seen that for $a=0.80$ fm^{-1}, $k=30$ MeV and $V_0=40$ MeV, the calculated ground state binding energy (31.98 MeV) is in good agreement with the experimental data.

As before, the energy levels with this potential take the form:

$$E_{n,l} = -\frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2((B-na)^2-\gamma)}\right)^2 - \left(\frac{-\beta}{2(B^2-\gamma)}\right)^2\right.$$

$$\left. + \left(\frac{-\beta}{1+\sqrt{1+4\gamma}}\right)^2\right] + V_0 a. \tag{41}$$

5.3   Solution with the Hellman potential

Another potential that we use in Eq. (16) is the Hellmann potential. In this case, the Schrödinger equation takes the form:

$$\frac{\mathrm{d}^2U_{n,l}(r)}{\mathrm{d}r^2} + \frac{2\mu}{\hbar^2}\left(E_{n,l}+V_0 a+\frac{k-V_0}{r}-\frac{\Omega}{r^2}\right)U_{n,l}(r)=0, \tag{42}$$

where Ω is defined as $\Omega = \dfrac{\hbar^2(l+6)(l+7)}{2\mu}$. By choosing $\varepsilon_{n,l}=\dfrac{2\mu}{\hbar^2}(E_{n,l}+aV_0)$, $\beta=\dfrac{2\mu}{\hbar^2}(k-V_0)$ and $\gamma=\dfrac{2\mu}{\hbar^2}\Omega$, Eq. (41) takes the form:

$$\frac{\mathrm{d}^2U_{n,l}(r)}{\mathrm{d}r^2} + \left(\varepsilon_{n,l}+\frac{\beta}{r}-\frac{\gamma}{r^2}\right)U_{n,l}(r)=0. \tag{43}$$

We define the superpotential as Eq. (23) and, following the same steps as before, we can obtain the binding energy in the form

$$E_{n,l}=\frac{-\hbar^2}{2\mu}\left(\frac{-\beta}{1\pm\sqrt{1+4\gamma}}\right)^2 - V_0 a. \tag{44}$$

By using Eq. (24) from the SUSY method, the ground state normalized eigenfunctions are given as Eq. (25).

In Table 3, the fitted values of parameters of the ground state binding energy equations for the Hellmann potential are given.

As we did before for obtaining $E_{n,l}$,

$$E_{n,l} = -\frac{\hbar^2}{2\mu}\left[\left(\frac{-\beta}{2((B-na)^2-\gamma)}\right)^2 - \left(\frac{-\beta}{2(B^2-\gamma)}\right)^2\right.$$

$$\left. + \left(\frac{-\beta}{1+\sqrt{1+4\gamma}}\right)^2\right] - V_0 a. \tag{45}$$

From Table 3, it can be seen that for $a=0.71$ fm^{-1}, $k=10$ MeV and $V_0=-45$ MeV, the calculated ground state binding energy (31.83 MeV) is in good agreement with the experimental data.

6   Conclusion

In this paper, we have obtained the exact solution of the Schrödinger equation for a 6-body system for the Yukawa potential, generalized Yukawa potential and Hellmann potential within the framework of SUSYQM. Also, we have calculated the binding energy of the lithium nucleus for the ground state with various potentials and found the wave function for this element. The results obtained from the SUSY method for these three potentials are in good agreement with the experimental data.

References

1 Salehi N, Rajabi A A, Ghalenovi Z. Chinese Journal of Physics, 2012, **50**: 1
2 Salehi N, Rajabi A A. Physica Scripta, 2012, **85**: 055101
3 Salehi N, Rajabi A A. Zahra Ghalenovi. Acta Physica Polonica B, 2011, **42**: o6
4 Salehi N, Rajabi A A. Mod. Phys. Lett. A, 2009, **24**: 2631
5 Panella O, Biondini S, Arda A. Math-ph, 2010, **2**: 1002.3723
6 Pekeris C L. Phys. Rev., 1934, **45**: 98
7 Flügge S. Practical Quantum Mechanics, 1994. **1**(Springer, Berlin)
8 Badalov V H, Ahmadov H I, Badalov S V. Int. J. Mod. Phys. E, 2010, **19**: 1463—1475
9 Bayrak O, Kocak G, Boztosun I. J. Phys. A: Math. Gen., 2006, **39**: 11521
10 Arda A, Sever R. Quant-Phy, 2009, **1**: 0902.2088
11 Mehta C H, Patil S H. Phys. Rev. A, 1978, **17**: 34
12 Dutt R, Varshni Y P. Z. Phys. A, 1983, **313**: 143
13 Alhaidari A D, Bahlouli H, Abdelmonem M S. J. Phys. A, Math. Theor., 2008, **41**: 032001
14 Imbo T, Pagnamenta A, Sukhatme U. Phys. Lett. A, 1984, **105**: 183
15 Hellmann H. Acta Physichim. URSS, 1935, **1**: 913; 1936, **4**: 225; 1936, **4**: 324; J. Chem. Phys., 1935, **3**: 61
16 Hellmann H, Kassatotchkin W. Acta Physicochim. URSS, 1936, **5**, 23; J. Chem. Phys., 1936, **4**: 324
17 Adamowski J. Phys. Rev. A, 1985, **31**: 43
18 Varshni Y P, Shukla R C. Rev Mod. Physi., 1963, **35**: 130
19 Gryaznov V K. Zh. Eksp. Teor. Fiz., 1980, **78**: 573; Sov. Phys. JETP, 1980, **51**: 288
20 Alekseev V A, Fortov V E, Yakubov I T. USP. Fiz Nauk, 1983, **139**: 193; Sov. Phys.-USP., 1983, **26**: 99
21 Gombas P. Die Statistische Theorie des Atoms und ihre Anwendungen. Springer: Berlin, 1949, 304
22 Callaway J. Phys. Rev., 1958, **112**: 322; Iafrate G J. J. Chem. Phys., 1966, **45**: 1072; Callaway J, Laghos P S. Phys. Rev., 1969, **187**: 192; McGinn G. J. Chem. Phys., 1970, **53**: 3635
23 Bebnarek S, Adamowski J, Saffczy'nski M. Solid State Commun., 1977, **21**: 1
24 Pollmann J, Buttner H. Phys. Rev. B, 1977, **16**: 4480; Buttner H, Pollmann J. Physica(Utrecht), B, 1983, **278**: 117/118
25 Roy A K, Jalbout A F, Proynov E I. Quant-Phys., 2013, **1**: 2983
26 Arda A, Aydogdu O, Sever R. Math-Ph., 2011, **1**: 3204
27 Hasanabadi H, Rajabi A A. Mod. Phys. Lett. A, 2009, **23**: 7
28 cdfe.sinp.msu.ru/index.en.html