Lie algebraic analysis and simulation of high-current pulsed beam transport in a solenoidal lens*

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Abstract Recent calculations of the transport of a high-current beam in a solenoidal lens have made progress to the second order with the Lie algebraic method. A review of the theory and our simulation to realize it will be described. Then we will present the results of simulation. A brief discussion on the space charge effect's contribution to the transportation will also be made.

Key words solenoidal lens, Lie algebraic, space charge effect, nonlinearity

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1 Introduction

Nonlinearity in particle orbit dynamics plays an important role in affecting the computing precision of particle trajectories and thus the accelerator design. A variety of methods are used for the purpose to calculate it reasonably and effectively. The Lie algebraic method, whose validity has been testified by a comprehensive study and application, because of its concision in dealing with nonlinear effects, has been a method for particle tracing and accelerator design for decades.

Solenoidal lenses are focusing elements frequently used in induction accelerators, linear electron accelerators, linear proton accelerators, beam transport system at low energy and a variety of cathode-ray tubes. Sometimes the beam current is too high for the space charge effect to be neglected. In this paper, we use the Lie algebraic method to calculate the nonlinear transport of a beam considering the space charge effect. Mapping expressions to the second order are derived. For plainness and clarity, we also give the simulation results by a program developed on our own, my Beam Orbit Code (my BOC), a program designed for simulation of high-current pulsed beam transport in transporting elements including solenoidal lenses.

2 Lie map and factorization

The main idea of the Lie algebraic method is that the Lie transformation associated with an analytic function produces an analytic symplectic map and that conversely, under certain general conditions, an analytic symplectic map can be written as a product of Lie transformations [1]. When a particle is transported through a beam transporting system, its final coordinates are determined by its initial coordinates and the Hamiltonian of the system within the course. The total of trajectories of particles with all possible initial conditions together is called the Hamiltonian flow. A map following the Hamiltonian flow is a symplectic map. Let us treat the problem from the point of view of the Lie algebraic. Write the final coordinates as a map acting on the initial ones $\xi = M(\xi^0)$. The map can be expressed as a product of a series of Lie transformations. Our job is to find these Lie transformations and act them on the initial coordinates. Then we will be able to derive the final coordinates. We can see that with the aid of the Lie algebraic method and tools, the problem can be solved concisely.

For a Hamiltonian system, the mapping can be expressed by $M = \exp(:-\int_{z_0}^{z} H\mathrm{d}z:)$. The double colon

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along with the function between them denotes a Lie operator. When acting on any function g it will take the Poisson brackets operation

$$:f:g=[f,g] = \sum_i(\partial f/\partial q_i)(\partial g/\partial p_i)-(\partial f/\partial p_i)(\partial g/\partial q_i).$$

Assume that H has been expanded as a power series. Insert the expansion of H into the expression above and suppose that the result is written in factored product form; we can write the map as

$$M = \exp\left(-\int_0^z(:H_2:+:H_3:+\cdots)\mathrm{d}z\right) = \cdots\exp(:f_3:)\exp(:f_2:),$$

where f_2 and f_3 are calculated by [2]

$$f_2=-\int_{z_0}^z H_2\mathrm{d}z,\quad f_3=-\int_{z_0}^z H_3^{\mathrm{int}}\mathrm{d}z,$$

with

$$:H_3^{\mathrm{int}}:=M_2:H_3:M_2^{-1}=:M_2H_3:.$$

According to the property of the Lie operator, we have

$$H_3^{\mathrm{int}}(\xi)=H_3(M_2(\xi))=H_3(\xi_1).$$

The linear part ξ_1 and the second order ξ_2 of the particle's final coordinates can be obtained by carrying out the calculation of

$$\xi_1=\exp(:f_2:)\xi \text{ and } \xi_2=:f_3:\xi_1.$$

3 Hamiltonian and expansion

In Cartesian coordinates, for the motion of a charged particle in the solenoidal lens, the relativistic Hamiltonian with the axis z as an independent variable has the following form [3],

$$K=-qA_z-\frac{1}{c}\times\sqrt{-m_0^2c^4-c^2((p_x-qA_x)^2+(p_y-qA_y)^2)+(p_t+q\psi_{\mathrm{s}})^2}\,. \tag{1}$$

Here A is the magnetic potential vector with $A_x\cong(-B/2)\cdot y$ the linear approximation of the horizontal projection, $A_y\cong(B/2)\cdot x$ that of the vertical projection, and $A_z=0$ that of the longitudinal projection. In the function above, $B=B^{(0)}(z)$ is the magnetic induction intensity excited by the solenoidal lens along the z axis. Ψ_{s} is the self-electric potential that the particle suffered from the beam; for a beam observing the uniform distribution, Ψ_{s} is described by $\Psi_{\mathrm{s}}=-U(\mu_xx^2+\mu_yy^2+\mu_zz_r^2)$, where U is a parameter defined as $U=3IT_{\mathrm{rf}}/(8\pi\varepsilon_0\gamma_0D_xD_yD_z)$. In the function above, I is the average beam current, T_{rf} is the period of the beam pulses, D_x, D_y and D_z are the beam dimensions, z_r is the relative longitudinal position of the particle to the reference particle and is expressed as $z_r=z-v_0t$, with v_0 representing the velocity of reference particle. γ_0 is the beam relativistic energy and $\gamma_0=1/\sqrt{1-\beta_0^2}$ with $\beta_0=v_0/c$, the beam velocity normalized by the speed of light. μ_x, μ_y, μ_z are factors dependent on the beam dimensions,

$$\mu_x=\frac{D_xD_yD_z\gamma_0}{2}\times\int_0^\infty\frac{1}{(D_x^2+\xi)\sqrt{(D_x^2+\xi)(D_y^2+\xi)(D_z^2\gamma_0^2+\xi)}}\mathrm{d}\xi,$$

$$\mu_y=\frac{D_xD_yD_z\gamma_0}{2}\times\int_0^\infty\frac{1}{(D_y^2+\xi)\sqrt{(D_x^2+\xi)(D_y^2+\xi)(D_z^2\gamma_0^2+\xi)}}\mathrm{d}\xi,$$

$$\mu_z=\frac{D_xD_yD_z\gamma_0}{2}\times\int_0^\infty\frac{1}{(D_z^2\gamma_0^2+\xi)\sqrt{(D_x^2+\xi)(D_y^2+\xi)(D_z^2\gamma_0^2+\xi)}}\mathrm{d}\xi.$$

With the original variables, following the Hamiltonian flow generated by K along the design orbit does not lead to an analytical map. For convenience, one can transform the variables by the canonical transformation arising from the function

$$F_2=xp_x+yp_y+(t-z/v_0)(p_\tau+p_t^0)$$

and define the "new" variables (τ, x, y, p_τ, p_x, p_y) with $\tau=t-z/v_0$ and $p_\tau=p_t-p_t^0$. In terms of these new variables, the design orbit is expressed as $\tau=x=y=p_\tau=p_x=p_y=0$, which suggests that following the Hamiltonian flow generated by Hamiltonian in terms of the new variables along the design orbit does lead to an analytical map. The variables τ, x, y, and their canonical momenta are measured as deviation from the design orbit. Let H denote the Hamiltonian for the new variables. According to the relation $H=K+\partial F_2/\partial z$, we have the result,

$$H=-\frac{p_\tau+p_t^0}{c\beta_0}-\frac{1}{c}\sqrt{-m_0^2c^4-c^2\left(\left(p_x+\frac{qBy}{2}\right)^2+\left(p_y-\frac{qBx}{2}\right)^2\right)+(p_\tau+p_t^0+q\psi_{\mathrm{s}})^2}\,. \tag{2}$$

We expand the Hamiltonian H into Taylor series. Here we enumerate the first few polynomials,

$$\begin{aligned}
H_0 &= p_0\left(-1+\frac{1}{\beta_0^2}\right),\\
H_1 &= 0,\\
H_2 &= \frac{p_x^2}{2p_0}+\frac{p_y^2}{2p_0}-k_x^2x^2\frac{p_0}{2}-k_y^2y^2\frac{p_0}{2}-kxp_y+kp_xy+\\
&\quad \frac{p_\tau^2}{2p_0\gamma_0^2\beta_0^2c^2}-k_\tau^2\tau^2\frac{p_0\gamma_0^2\beta_0^2c^2}{2},\\
H_3 &= \frac{p_x^2p_\tau}{2p_0^2\beta_0c}+\frac{p_y^2p_\tau}{2p_0^2\beta_0c}+x^2p_\tau\frac{\beta_0}{2c}\left(k^2-\frac{k_x^2}{\gamma_0^2\beta_0^2}\right)+\\
&\quad y^2p_\tau\frac{\beta_0}{2c}\left(k^2-\frac{k_y^2}{\gamma_0^2\beta_0^2}\right)+\frac{p_\tau^3}{2p_0^2\gamma_0^2\beta_0^3c^3}-k_\tau^2\tau^2p_\tau\frac{\beta_0c}{2}
\end{aligned} \tag{3}$$

where $p_0=\gamma_0m_0\beta_0c$ is the magnitude of the design relativistic mechanical momentum. The parameters k, k_x, k_y, and k_τ are defined by $k=qB/(2p_0)$, and

$$k_x^2=\frac{2qU\mu_x}{p_0\beta_0c}-k^2,\quad k_y^2=\frac{2qU\mu_y}{p_0\beta_0c}-k^2,\quad k_\tau^2=\frac{2qU\mu_z}{p_0\gamma_0^2\beta_0c}.$$

4 Simulation

For the existence of the terms associated with the space charge effect, the expressions of the particle's final coordinates have the form of a power series which is too complicated for us to list them in this paper. However, simulation can help us to make a more explicit statement of the result. At present, we are in the process of developing a particle tracing code, named my BOC, making use of the results from Lie algebraic analysis of transport elements including solenoidal lenses. In this section, we will illustrate a few of the characteristics of the nonlinear transport of a high-current beam in the solenoidal lens using some of the outcomes of my BOC.

Let the particles be transported in the z direction. The transporting system is constructed with three drift spaces, of which two are 20 cm long and one 40 cm long, and two 15 cm long solenoidal lenses with a central magnetic field of 0.4 T and 0.6 T respectively, arranged alternately along the axis. The inner radius of the tube is 5 cm.

Consider a 50 mA, 5 MeV proton beam. Suppose that the beam has a uniform distribution. Then with the initial emittance $\varepsilon_x=\varepsilon_y=75.0$ mm·mrad, $\varepsilon_z=0.04°\cdot$MeV, the Twiss parameter $\alpha_x=\alpha_y=0.2$, $\alpha_z=1.0$, $\beta_x=\beta_y=3.0$ mm/mrad, $\beta_z=160000.0°$/MeV, the beam transport is calculated. Here the emittance and the Twiss parameters are important parameters which reveal a lot of information about the beam property. The emittance ε is associated with the area of the phase elliptic. β is the envelope function of the transverse movement. α tells the slope of the beam envelope [4].

The particle trajectory is calculated by the function of single particle tracing. We calculated respectively the trajectories of four typical particles whose initial trajectories are parallel to the axis with an offset from the axis in the x direction. To have an overall view of the particles' behavior, we plotted the trajectories in both the x and y directions in one figure (see Fig. 1). As is expected, the total effect of a set of drift spaces and solenoidal lenses arranged alternately is to minimize the beam profile. Note that the previous calculations are on the hypothesis that the particles are transported through the system one by one, without suffering from the space charge effect.

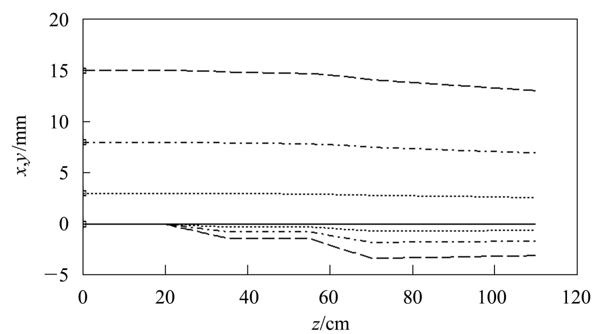

Fig. 1. Transverse trajectories. The initial coordinates of the particles are (0,0,0,0,0,0), (3,0,0,0,0,0), (8,0,0,0,0,0) and (15,0,0,0,0,0), as shown with the solid line, the dotted line, the dash-dotted line and the dashed line.

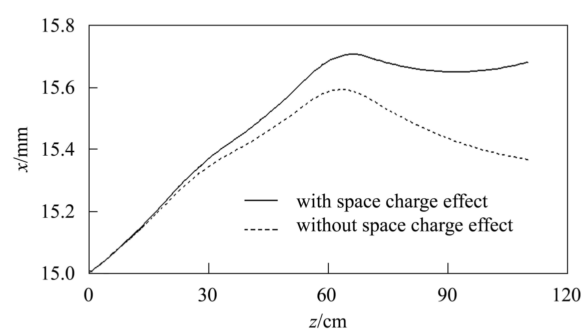

Fig. 2. Beam envelope.

Figure 2 shows space charge effect on the beam envelope. 10000 particles generated randomly within the distribution restriction are traced. As we can see from the figure, the space charge effect will accumulate as time goes on. Thus in long distance transport, the beam may deteriorate largely and the efficiency of the transporting system becomes lower. To conquer this problem, neutralization transport is one option.

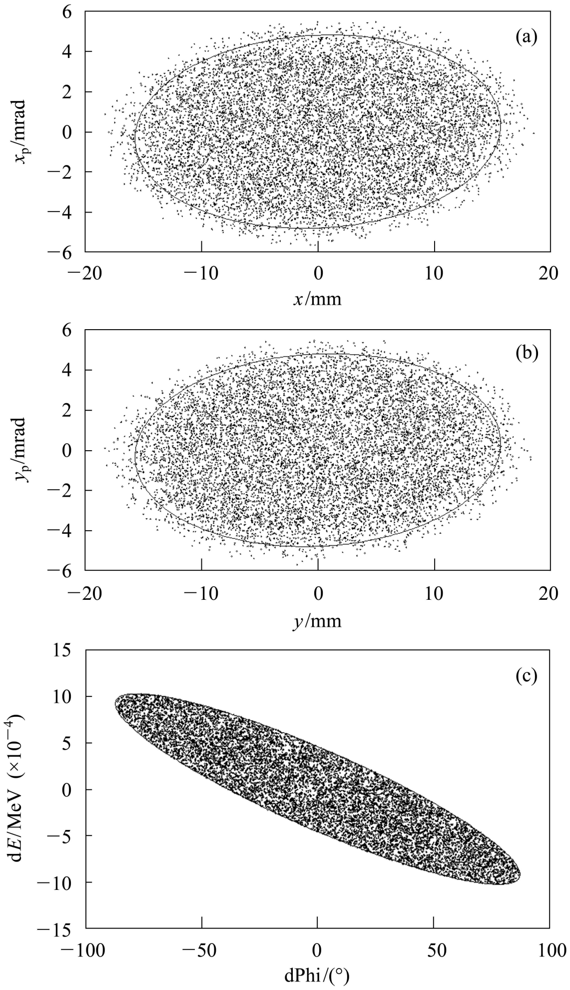

Fig. 3. Particle distribution in 6-dimension phase space. (a) ε_x =75 mm·mrad, α_x =0.07, β_x =3.3 mm/mrad; (b) ε_y =75 mm·mrad, α_y =0.07, β_y =3.3 mm/mrad; (c) ε_z =0.04° MeV, α_z =−2.0, β_z =190000°/MeV.

In the following, all the calculations are with space charge effect considered.

Nonlinear effects are also studied. Under a designated initial condition, we get a linear approximation of the beam envelope with the transfer matrix method and a nonlinear one with the multi-particle tracing method. The envelopes calculated with the two methods are almost the same. That is, the nonlinear effects in this transporting system are almost imperceptible.

Making use of the data calculated with the multi-particle tracing method (10000 particles included), we plotted the distribution of particles in the phase space at the exit of the transporting system, as shown in Fig. 3. The beam emittance ε, together with the Twiss parameters α and β, are also calculated and listed under the figure.

We can see from (a) and (b) that the phase ellipses in the two interceptive directions are the same within the calculation error, which is consistent with the uniform distribution assumption.

5 Conclusion

We have reviewed the main idea of the Lie algebraic applied to beam transport. Coordinate mapping formulae to the second order of high-current beam transport in a solenoidal lens have been derived. Programs are designed and a simulation has been done. We note that the space charge effect is one of the primary contributors to impairment of the quality of the beam. With the space charge effect considered, the simulation precision can be greatly improved.

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