Analytical treatment of the nonlinear electron cloud effect and the combined effects with beam-beam and space charge nonlinear forces in storage rings*

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Abstract In this paper we treat first some nonlinear beam dynamics problems in storage rings, such as beam dynamic apertures due to magnetic multipoles, wiggles, beam-beam effects, nonlinear space charge effect, and then nonlinear electron cloud effect combined with beam-beam and space charge effects, analytically. This analytical treatment is applied to BEPCⅡ. The corresponding analytical expressions developed in this paper are useful both in understanding the physics behind these problems and also in making practical quick hand estimations.

Key words electron cloud effect, beam-beam effect, space charge effect, storage ring, BEPCⅡ

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1 Introduction

In storage rings many physical phenomena connected with particle motion are caused by nonlinear forces, either static or dynamic, acting on the moving particles. Among them one finds dynamic apertures limited by static magnetic multipoles, wigglers, beam-beam effects due to dynamic nonlinear beam-beam interaction forces, nonlinear space charge and electron cloud effects, which are separately treated in the following sections. It is aimed to demonstrate the validity of the analytical method in treating multi-nonlinear sources and there combined effects. Finally, the combined effects of electron cloud, beam-beam and space charge nonlinear forces are discussed and the analytical treatment is applied to BEPCII.

2 Dynamic apertures of multipoles

We start with the simplest case, which is the physical and mathematical basis for the analytical treating of other different subjects in the other sections, i.e., the dynamic aperture limited by a single nonlinear multipole located somewhere inside a storage ring. The Hamiltonian of this problem is expressed as follows

$$H=\frac{p^2}{2}+\frac{K(s)}{2}x^2+\frac{1}{m!B_0\rho}\frac{\partial^{m-1}B_z}{\partial x^{m-1}}x^mL\sum_{k=-\infty}^{\infty}\delta(s-kL), \quad (1)$$

with

$$B_z=B_0x^{m-1}b_{m-1}, \quad (2)$$

where ρ is the bending radius corresponding to B_0, and L is the circumference of the ring. The general formula for the dynamic aperture limited by this multipole reads[1]

$$A_{\rm dyna,2m,x} = \sqrt{2\beta_x(s)}\left(\frac{1}{m\beta_x^m(s(2m))}\right)^{\frac{1}{2(m-2)}}\times \left(\frac{\rho}{|b_{m-1}|L}\right)^{1/(m-2)}, \quad (3)$$

where $s(2m)$ is the location of this multipole. The dynamic aperture in vertical plane can be estimated as

$$A_{\rm dyna,2m,y}=\sqrt{\frac{\beta_x(s(2m))}{\beta_y(s(2m))}(A^2_{\rm dyna,2m,x}-x^2)}, \quad (4)$$

where $\beta_y(s(2m))$ is the vertical beta function where the multipole is located. If the are many independent

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multipoles, one can estimate their combined effects through following equation

$$A_{\rm dyna,total}=\frac{1}{\sqrt{\sum\limits_{i,m}\frac{1}{A^2_{{\rm dyna},2m,i}}}}. \quad (5)$$

The validity of Eqs. (3), (4), and (5) has been checked by numerical simulations[1].

3 Dynamic aperture limited by wigglers

Considering a wiggler of sinusoidal magnetic field variation, one can express the wiggler's magnetic field, which satisfies Maxwell equations, as follows

$$B_x=\frac{k_x}{k_y}B_0\sinh(k_xx)\sinh(k_yy)\cos(ks), \quad (6)$$

$$B_y=B_0\cosh(k_xx)\cosh(k_yy)\cos(ks), \quad (7)$$

$$B_z=-\frac{k}{k_y}B_0\cosh(k_xx)\sinh(k_yy)\sin(ks), \quad (8)$$

with

$$k_x^2+k_y^2=k^2=\left(\frac{2\pi}{\lambda_w}\right)^2, \quad (9)$$

where B_0 is the peak sinusoidal wiggler magnetic field, λ_w is the period length of the wiggler, and x, y, s represent horizontal, vertical, and beam moving directions, respectively.

The Hamiltonian describing the motion can be written as

$$H_w=\frac{1}{2}\left(p_z^2+(p_x-A_x\sin(ks))^2+(p_y-A_y\sin(ks))^2\right), \quad (10)$$

where

$$A_x=\frac{1}{\rho_wk}\cosh(k_xx)\cosh(k_yy), \quad (11)$$

$$A_x=-\frac{k_x}{k_y}\frac{\sinh(k_xx)\sinh(k_yy)}{\rho_wk}, \quad (12)$$

and $\rho_w=E_0/ecB_0$ is the radius of curvature of the wiggler peak magnetic field B_0 with E_0 being the electron energy. After making a canonical transformation to betatron variables, averaging the Hamiltonian over one period of wiggler, and expanding the hyperbolic functions to fourth order in x and y, one gets

$$\begin{aligned}\mathcal{H}_w &= \frac{1}{2}(p_x^2+p_y^2)+\frac{1}{4k^2\rho_w^2}(k_x^2x^2+k_y^2y^2)+\\ &\frac{1}{12k^2\rho_w^2}(k_x^4x^4+k_y^4y^4+3k^2k_x^2x^2y^2)-\\ &\frac{\sin(ks)}{2k\rho_w}\left(p_x(k_x^2x^2+k_y^2y^2)-2k_x^2p_yxy\right). \end{aligned}\quad (13)$$

Now we insert a "wiggler" of only one period (or one cell) into a storage ring located at s_w. The total Hamiltonian of the ring in the vertical plane can be expressed as follows

$$H=H_0+\frac{1}{4\rho^2}y^2+\frac{k_y^2}{12\rho^2}y^4\lambda_w\sum_{i=-\infty}^{\infty}\delta(s-{\rm i}L), \quad (14)$$

where H_0 is the Hamiltonian without the inserted wiggler, L is the circumference of the ring, and $k_y=k$. It is obvious that the perturbation is a delta function octupole. Comparing Eq. (1) with Eq. (14), one finds easily that

$$\frac{b_3}{\rho}L=\frac{k_y^2\lambda_w}{3\rho_w^2}, \quad (15)$$

and the dynamic aperture limited by this one period "wiggler" as

$$A_{1,y}(s)=\frac{\sqrt{\beta_y(s)}}{\beta_y(s_w)}\left(\frac{3\rho_w^2}{k_y^2\lambda_w}\right)^{1/2}, \quad (16)$$

where $\beta_y(s)$ is the unperturbed beta function. In fact, a wiggler is an insertion device which is composed of a large number of cells, say, N_w, and the wiggler length $L_w=N_w\lambda_w$. Now, the first question which follows is what the combined effect of these N_w cells will be. According to Ref. [1], one has

$$\frac{1}{A^2_{N_w,y}(s)}=\sum_{i=1}^{N_w}\frac{1}{A^2_{i,y}}=\sum_{i=1}^{N_w}\left(\frac{k_y^2}{3\rho_w^2\beta_y(s)}\right)\beta_y^2(s_{i,w})\frac{L_w}{N_w}, \quad (17)$$

where the index i labels the different cell. When N_w is a large number, Eq. (17) can be simplified as:

$$\frac{1}{A^2_{N_w,y}(s)}=\frac{k_y^2}{3\rho_w^2\beta_y(s)}\int_{s_{w_0}-L_w/2}^{s_{w_0}+L_w/2}\beta_y^2(s){\rm d}s, \quad (18)$$

where s_{w_0} correspond to the center of the wiggler. For practical purposes, one can replace $\beta_y^2(s)$ inside the integral by $\beta_{y,m}^2$ which is the beta function value in the middle of the wiggler, and one gets

$$A_{N_w,y}(s)=\sqrt{\frac{3\beta(s)}{\beta_{y,m}^2}}\frac{\rho_w}{k_y\sqrt{L_w}}, \quad (19)$$

$$A_{N_w,x}(s)=\sqrt{\frac{\beta_y(s)}{\beta_x(s)}(A_{N_w,y}(s)^2-y^2)}. \quad (20)$$

If there are more than one wiggler in a storage ring, the total dynamic aperture limited by these wigglers can be estimated by applying Eq. (5).

Eq. (19) has been checked by numerical simulations[2].

4 Beam-beam effects and limitations

For two head-on colliding bunches, the incoherent kick felt by each particle can be calculated as

$$\delta y'+\mathrm{i}\delta x'=-\frac{N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*}f(x,y,\sigma_x,\sigma_y), \tag{21}$$

where x' and y' are the horizontal and vertical slopes, N_{e} is the particle population in the bunch, r_{e} is the electron radius (2.818×10^{-15} m), σ_x and σ_y are the standard deviations of the transverse charge density distribution of the counter-rotating bunch at IP, γ_* is the normalized particle's energy, and $*$ denotes the test particle and the bunch to which the test particle belongs. When the bunch is Gaussian $f(x,y,\sigma_x,\sigma_y)$ can be expressed by Basseti-Erskine formula

$$f(x,y,\sigma_x,\sigma_y)=\sqrt{\frac{2\pi}{\sigma_x^2-\sigma_y^2}}\times w\left(\frac{x+\mathrm{i}y}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)-$$

$$\sqrt{\frac{2\pi}{\sigma_x^2-\sigma_y^2}}\times\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)w\left(\frac{\frac{\sigma_y}{\sigma_x}x+\mathrm{i}\frac{\sigma_x}{\sigma_y}y}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right), \tag{22}$$

where w is the complex error function expressed as

$$w(z)=\exp(-z^2)(1-\mathrm{erf}(-\mathrm{i}z)). \tag{23}$$

For the round beam (RB) and the flat beam (FB) cases one has the incoherent beam-beam kicks expressed as[3]

$$\delta r'[\mathrm{RB}]=-\frac{2N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*r}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right), \tag{24}$$

$$\delta x'[\mathrm{FB}]=-\frac{2\sqrt{2}N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*\sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\int_0^{\frac{x}{\sqrt{2}\sigma_x}}\exp(u^2)\mathrm{d}u, \tag{25}$$

$$\delta y'[\mathrm{FB}]=-\frac{\sqrt{2\pi}N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*\sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right), \tag{26}$$

where $r=\sqrt{x^2+y^2}$. Since the probability to find the transverse displacement of the test particle is not constant (in fact, the probability function is the same as the charge distribution of the bunch to which the test particle belongs), one is interested in the average kick felt by the test particle. In the following we assume that the transverse sizes for the two colliding bunches at IP are exactly the same. For the round beam case after averaging one gets

$$\delta\bar{r}'[\mathrm{RB}]=-\frac{2N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*\bar{r}}\left(1-\exp\left(-\frac{\bar{r}^2}{4\sigma^2}\right)\right). \tag{27}$$

Although this expression is the same as that of the coherent beam-beam kick for round beams, one should keep in mind that we are not finding coherent beam-beam kick. The difference will be obvious when we treat the vertical motion in the case of flat beams. For the flat beam case, we will treat the horizontal and vertical planes separately. As far as the horizontal kick is concerned, it depends only on one displacement variable similar to the round beam case. We will use its coherent form given by the follow expression

$$\delta x'[\mathrm{FB}]=-\frac{2N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*\sigma_x}\exp\left(-\frac{x^2}{4\sigma_x^2}\right)\int_0^{\frac{x}{2\sigma_x}}\exp(u^2)\mathrm{d}u. \tag{28}$$

As for the vertical kick, one has to make an average over Eq. (26) with the horizontal probability distribution function of the test particle. This leads to

$$\delta y'[\mathrm{FB}]=-\frac{\sqrt{2\pi}N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*\sigma_x}\left\langle\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\right\rangle_x\mathrm{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right), \tag{29}$$

where $\langle\rangle_x$ means the average over the horizontal probability distribution function of the test particle, and for two identical colliding Gaussian beams $\langle\rangle_x=1/\sqrt{2}$. It is obvious that Eq. (29) is not the expression for the coherent beam-beam kick. The average over Eqs. (24) and (26) is only a technical operation to simplify (or to make equivalent) a two dimensional problem to a one dimensional one. To study both round and flat beam cases, we expand $\delta\bar{r}'$ at $x=0$ (for round beam we study only vertical plane since the formalism in the horizontal plane is the same), $\delta x'$ and $\delta y'$ of Eqs. (27), (28) and (29), into Taylor series

$$\begin{aligned}\delta y'[\mathrm{RB}] &= \frac{N_{\mathrm{e}}r_{\mathrm{e}}}{\gamma_*}\left(\frac{1}{2\sigma^2}y-\frac{1}{16\sigma^4}y^3+\frac{1}{192\sigma^6}y^5-\right.\\ &\quad \frac{1}{3072\sigma^8}y^7+\frac{1}{61440\sigma^{10}}y^9-\\ &\quad \left.\frac{1}{1474560\sigma^{12}}y^{11}+\cdots\right),\end{aligned} \tag{30}$$

$$\begin{aligned}\delta_x'[\mathrm{FB}] &= -\frac{N_{\mathrm{e}}r_{\mathrm{e}}}{2\gamma_*}\left(\frac{2}{\sigma_x^2}x-\frac{1}{3\sigma_x^4}x^3+\frac{1}{30\sigma_x^6}x^5-\right.\\ &\quad \frac{1}{420\sigma_x^8}x^7+\frac{1}{7560\sigma_x^{10}}y^9-\\ &\quad \left.\frac{1}{166320\sigma_x^{12}}x^{11}+\cdots\right),\end{aligned} \tag{31}$$

$$\delta'_y[\text{FB}] = -\frac{N_\text{e}r_\text{e}}{\sqrt{2}\gamma_*}\left(\frac{2}{\sigma_x\sigma_y}y-\frac{1}{3\sigma_x\sigma_y^3}y^3+\frac{1}{20\sigma_x\sigma_y^5}y^5-\frac{1}{168\sigma_x\sigma_y^7}y^7+\frac{1}{1728\sigma_x\sigma_y^9}y^9-\frac{1}{21120\sigma_x\sigma_y^{11}}y^{11}+\cdots\right). \quad (32)$$

The differential equations for the motion of the test particle in the transverse planes are given by

$$\frac{\text{d}^2y}{\text{d}s^2}+K_y(s)y=-\frac{N_\text{e}r_\text{e}}{\gamma_*}\left(\frac{1}{2\sigma^2}y-\frac{1}{16\sigma^4}y^3+\frac{1}{192\sigma^6}y^5-\frac{1}{3072\sigma^8}y^7+\frac{1}{61440\sigma^{10}}y^9-\frac{1}{1474560\sigma^{12}}y^{11}+\frac{1}{41287680\sigma^{14}}y^{13}-\cdots\right)\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{RB}), \quad (33)$$

$$\frac{\text{d}^2x}{\text{d}s^2}+K_x(s)x=-\frac{N_\text{e}r_\text{e}}{2\gamma_*}\left(\frac{2}{\sigma_x^2}x-\frac{1}{3\sigma_x^4}x^3+\frac{1}{30\sigma_x^6}x^5-\frac{1}{420\sigma_x8}x^7+\frac{1}{7560\sigma_x^{10}}x^9-\frac{1}{166320\sigma_x^{12}}x^{11}+\frac{1}{4324320\sigma_x^{14}}x^{13}-\cdots\right)\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{FB}), \quad (34)$$

$$\frac{\text{d}^2y}{\text{d}s^2}+K_y(s)y=-\frac{N_\text{e}r_\text{e}}{\sqrt{2}\gamma_*}\left(\frac{2}{\sigma_x\sigma_y}y-\frac{1}{3\sigma_x\sigma_y^3}y^3+\frac{1}{20\sigma_x\sigma_y^5}y^5-\frac{1}{168\sigma_x\sigma_y^7}y^7+\frac{1}{1728\sigma_x\sigma_y^9}y^9-\frac{1}{21120\sigma_x\sigma_y^{11}}y^{11}+\frac{1}{299520\sigma_x\sigma_y^{13}}y^{13}-\cdots\right)\times\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{FB}), \quad (35)$$

where $K_x(s)$ and $K_y(s)$ describe the linear focusing of the lattice in the horizontal and vertical planes. The corresponding Hamiltonians are given by

$$H = \frac{p_y^2}{2}+\frac{K_y(s)}{2}y^2+\frac{N_\text{e}r_\text{e}}{\gamma_*}\left(\frac{1}{4\sigma^2}y^2-\frac{1}{64\sigma^4}y^4+\frac{1}{1152\sigma^6}y^6-\frac{1}{24576\sigma^8}y^8+\cdots\right)\times\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{RB}), \quad (36)$$

$$H_x = \frac{p_x^2}{2}+\frac{K_x(s)}{2}x^2+\frac{N_\text{e}r_\text{e}}{2\gamma_*}\left(\frac{1}{\sigma_x^2}x^2-\frac{1}{12\sigma_x^4}x^4+\frac{1}{180\sigma_x^6}x^6-\frac{1}{3360\sigma_x^8}x^8+\cdots\right)\times\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{FB}), \quad (37)$$

$$H_y = \frac{p_y^2}{2}+\frac{K_y(s)}{2}y^2+\frac{N_\text{e}r_\text{e}}{\sqrt{2}\gamma_*}\left(\frac{1}{\sigma_x\sigma_y}y^2-\frac{1}{12\sigma_x\sigma_y^3}y^4+\frac{1}{120\sigma_x\sigma_y^5}y^6-\frac{1}{1344\sigma_x\sigma_y^7}y^8+\cdots\right)\times\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{FB}), \quad (38)$$

with $p_x=\text{d}x/\text{d}s$ and $p_y=\text{d}y/\text{d}s$.

Using the general information from section 2 and comparing Eq. (1) with the Hamiltonians for beam-beam interactions, we derive the beam-beam effects and the limitations on the beam lifetimes for a rigid flat beam[3]

$$\tau_{bb,y,\text{flat}}=\frac{\tau_y}{2}\left(\frac{3}{\sqrt{2}\pi\xi_y}\right)^{-1}\exp\left(\frac{3}{\sqrt{2}\pi\xi_y}\right), \quad (39)$$

$$\tau_{bb,x,\text{flat}}=\frac{\tau_x}{2}\left(\frac{3}{\pi\xi_x}\right)^{-1}\exp\left(\frac{3}{\pi\xi_x}\right), \quad (40)$$

and a rigid round beam

$$\tau_{bb,y,\text{round}}=\frac{\tau_y}{2}\left(\frac{4}{\pi\xi_x}\right)^{-1}\exp\left(\frac{4}{\pi\xi_x}\right). \quad (41)$$

From Eqs. (39) and (40) one finds that for the same $\tau_{y,bb,\text{flat}}/\tau_y$, $\tau_{x,bb,\text{flat}}/\tau_x$, and $\tau_{y,bb,\text{round}}/\tau_y$, one has $\xi_{x,\text{flat}}=\sqrt{2}\xi_{y,\text{flat}}$, and $\xi_{y,\text{round}}=\frac{4\sqrt{2}}{3}\xi_{y,\text{flat}}=1.89\xi_{y,\text{flat}}$.

In reality the colliding bunch is not rigid. The transverse emittance will increase due to additional heating. In the following we will show how emittance blow-up is included into the beam-beam lifetime expressions.

In e^+e^- storage ring colliders, due to strong quantum excitation and synchrotron damping effects, the particles are confined inside a bunch. The state of the particles can be regarded as a gas, where the positions of the particles follow statistic laws. When two bunches undergo collision at an interaction point (IP, denoted by "$*$") the particles in each bunch will receive some additional heating. Taking the vertical plane for example, one has beam-beam induced kicks in y and $y'=\text{d}y/\text{d}s$ (see Ref. [4])

$$\delta y=-\frac{\sigma_s}{f_y}y\ , \quad (42)$$

$$\delta y' = -\frac{y}{f_y} y \ , \tag{43}$$

$$\frac{1}{f_y} = \frac{2N_{\rm e}r_{\rm e}}{\gamma\sigma_{y,*,+}(\sigma_{x,*,+}+\sigma_{y,*,+})} \ , \tag{44}$$

where σ_s is the bunch length, $N_{\rm e}$ is the particle number inside the bunch, $r_{\rm e}$ is the electron radius, $\sigma_{x,*,+}$ and $\sigma_{y,*,+}$ are the transverse dimensions just before the two colliding bunches overlap each other, and $\sigma_{x,*}$ and $\sigma_{y,*}$ are defined as the transverse dimensions when the two bunches fully overlap at IP. The invariant of vertical betatron motion can be expressed as[5]

$$a_y^2 = \frac{1}{\beta_y^*}\left(y_*^2 + \left(\beta_{y,*}y_*' - \frac{1}{2}\beta_{y,*}'y_*\right)^2\right). \tag{45}$$

From Eqs. (42) and (43) one finds that

$$\delta a_y^2 = \frac{1}{\beta_{y,*}}\left(\frac{\sigma_s}{f_y}\right)^2 y_*^2\left(1+\left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right), \tag{46}$$

where y_* is the vertical displacement of the test particle with respect to the center of the colliding bunch. Due to the gaseous nature of the particles, one has to take an average over all possible values of y_* according to its statistical distribution function. From Eq. (46) one obtains

$$\langle\delta a^2\rangle = \frac{1}{\beta_{y,*}}\left(\frac{\sigma_s\sigma_{y,*}}{f_y}\right)^2\left(1+\left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right). \tag{47}$$

The resulting vertical dimension combining the synchrotron radiation and beam-beam effects can be expressed as follows

$$\sigma_{y,*}^2 = \frac{1}{4}\tau_y\beta_{y,*}Q_y + \frac{1}{4}\tau_y\beta_{y,*}\left(\frac{1}{T_0\beta_{y,*}}\left(\frac{\sigma_s\sigma_{y,*}}{f_y}\right)^2\left(1+\left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right)\right), \tag{48}$$

where T_0 is the revolution time, τ_y is the radiation damping time, and Q_y is defined according to Ref. [5] as $\sigma_{y,*,0}^2 = \frac{1}{4}\tau_y\beta_{y,*}Q_y$ with $\sigma_{y,*,0}$ being the natural vertical dimension at IP. Solving Eq. (48) for $\sigma_{y,*}$ gives:

$$\sigma_{y,*}^2 = \frac{\sigma_{y,*,0}^2}{\left(1-\frac{\tau_y}{4T_0}\left(\frac{e^2N_{\rm e}K_{bb,y}}{E_0}\right)^2\right)} \ , \tag{49}$$

where E_0 is the energy, and

$$K_{bb,y} = \frac{\sigma_s}{2\pi\epsilon_0\sigma_{y,*,+}(\sigma_{x,*,+}+\sigma_{y,*,+})}\times \left(1+\left(\frac{\beta_{y,*,+}}{\sigma_s}\right)^2\right)^{1/2}. \tag{50}$$

Since $\sigma_y(s)=\sqrt{\epsilon_y\beta_y(s)}$ one gets from from Eq. (49)

$$\epsilon_y = \frac{\epsilon_{y,0}}{\left(1-\frac{\tau_y}{4T_0}\left(\frac{e^2N_{\rm e}K_{bb,y}}{E_0}\right)^2\right)}, \tag{51}$$

where $\epsilon_{y,0}$ is the natrual transverse emittance. For a flat bunch ($\sigma_{y,*,+}\ll\sigma_{x,*,+}$), we obtain from Eq. (51) the following relation:

$$\sigma_{x,*,+}\sigma_{y,*,+} > \left(\frac{3RN_{IP}(e^2fN_{\rm e}\beta_{y,*})^2}{8\pi^2\epsilon_0m_0c^2\gamma^5}\right)^{1/2}. \tag{52}$$

We define

$$H = \frac{\sigma_{x,*,+}\sigma_{y,*,+}}{\sigma_{x,*}\sigma_{y,*}}, \tag{53}$$

which is a measure of the plasma pinch effect and assume that it can be expressed as

$$H = \frac{H_0}{\sqrt{\gamma}}. \tag{54}$$

Recalling the beam-beam parameter definition

$$\xi_y = \frac{N_{\rm e}r_{\rm e}\beta_{y,*}}{2\pi\gamma\sigma_{y,*}(\sigma_{x,*}+\sigma_{y,*})}, \tag{55}$$

where β_y^* is the beta function value at the interaction point, σ_x^* and σ_y^* are the bunch transverse dimensions after the plasma pinch effect, a combination of Eqs. (52), (54) and (55) leads finally to the following results:

$$\xi_y \leqslant \xi_{y,\max,em,\text{flat}} = \frac{H_0}{2\pi F}\sqrt{\frac{T_0}{\tau_y\gamma N_{IP}}} \tag{56}$$

for the general case and

$$\xi_y \leqslant \xi_{y,\max,em,\text{flat}} = \frac{H_0\gamma}{F}\sqrt{\frac{r_{\rm e}}{6\pi RN_{IP}}} \tag{57}$$

for isomagnetic case. Here $H_0\approx 2845$, R is the local dipole bending radius and F is expressed as follows

$$F = \frac{\sigma_s}{\sqrt{2}\beta_{y,*}}\left(1+\left(\frac{\beta_{y,*}}{\sigma_s}\right)^2\right)^{1/2}. \tag{58}$$

The subscript *em* in Eqs. (56) and (57) denotes the beam-beam emittance parameter, limited by blow-up. For $\sigma_s=\beta_{y,*}$ we get $F=1$.

Now taking into account the emittance blow-up effect due to beam-beam interactions, in a heuristic way, one gets

$$\tau_{bb,y,\text{flat}} = \frac{\tau_y}{2}\left(\frac{3\xi_{y,\max,em,\text{flat}}}{\sqrt{2}\pi\xi_{y,\max,0}\xi_yN_{IP}}\right)^{-1}\times \exp\left(\frac{3\xi_{y,\max,em,\text{flat}}}{\sqrt{2}\pi\xi_{y,\max,0}\xi_yN_{IP}}\right) \tag{59}$$

and

$$\tau_{bb,y,\text{round}} = \frac{\tau_y}{2}\left(\frac{3\xi_{y,\max,em,\text{round}}}{\sqrt{2}\pi\xi_{y,\max,0}\xi_y N_{IP}}\right)^{-1}\times \exp\left(\frac{3\xi_{y,\max,em,\text{round}}}{\sqrt{2}\pi\xi_{y,\max,0}\xi_y N_{IP}}\right), \qquad (60)$$

with

$$\xi_{y,\max,em,\text{round}} = 1.89\xi_{y,\max,em,\text{flat}}\ , \qquad (61)$$

where $\xi_{y,\max,0}$ refers to rigid beam case limiting value. Taking $\xi_{y,\max,0}=0.0447$ means that we quantify the term "beam-beam limit" for the beam-beam limited beam lifetime being one hour at $\tau_y = 30$ ms with $N_{IP}=1$.

Eqs. (56) and (59) have been checked with some machine operation results[4].

5 Beam-beam effects with crossing angle

To get a higher luminosity one could run a circular collider in the multibunch operation mode with a definite collision crossing angle. Different from the head-on collision discussed above, the transverse kick received by a test particle due to the space charge field of the counter rotating bunch will depend on its longitudinal position with respect to the center of the bunch which the test particle belongs. In this section we consider first a flat beam colliding with another flat beam with a half crossing angle of ϕ in the horizontal plane. Due to the crossing angle the two curvilinear coordinates of the two colliding beams at the interaction point will no longer coincide. When the crossing angle is not too large one has

$$x^* = x + z\phi\ , \qquad (62)$$

where x^* is the horizontal displacement of the test particle to the center of the colliding bunch, z and x are the longitudinal and horizontal displacements of the test particle from the center of the bunch to which it belongs. Now we recall Eq. (37) which describes the Hamiltonian of the horizontal motion of a test particle in the head-on collision mode. By inserting Eq. (62) into Eq. (37) we get

$$H_x = \frac{p_x^2}{2}+\frac{K_x(s)}{2}x^2+\frac{N_e r_e}{2\gamma_*}\left(\frac{1}{\sigma_x^2}(x+z\phi)^2 - \frac{1}{12\sigma_x^4}(x+z\phi)^4+\frac{1}{180\sigma_x^6}(x+z\phi)^6 - \frac{1}{3360\sigma_x^8}(x+z\phi)^8+\cdots\right)\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (\text{FB}). \qquad (63)$$

Since the test particle can occupy a definite z within the bunch according to a certain probability distribution, say Gaussian, it is reasonable to replace z in Eq. (63) by σ_z. In this way we reduce the two dimensional Hamiltonian of Eq. (63) to a one dimensional. It should be noted that Eq. (63) takes only the longitudinal position of the test particle into account which is regarded as a small perturbation in the head-on collision case. This geometrical effect will included later. To simplify our analysis we consider only the lowest synchro-betatron nonlinear resonance, i.e., $3Q_x \pm Q_s = p$ (where Q_s is the synchrotron oscillation tune and p is an integer) which turns out to be the most dangerous one. Following the same procedure in section 4 one gets the dynamic aperture due to the lowest synchro-betatron nonlinear resonance as follows

$$A_{\text{syn-beta},x}(s) = \left(\frac{2\beta_x(s)}{3\beta_x(s_{IP})^3}\right)^{1/2}\frac{2\gamma_*\sigma_x^4}{N_e r_e \sigma_z \phi}, \qquad (64)$$

and

$$\mathcal{R}_{\text{syn-beta},x} = \frac{A_{\text{syn-beta},x}(s)^2}{\sigma_x(s)^2} = \frac{2}{3\pi^2}\left(\frac{1}{\xi_x^* \Phi}\right)^2, \qquad (65)$$

where $\Phi = \frac{\sigma_z}{\sigma_x}\phi$ is Piwinski angle. Now we are facing the problem of how to combine the two effects: the principal vertical beam-beam effect and the horizontal crossing angle induced perturbation. To solve this problem we assume that the total beam lifetime due to the vertical and the horizontal crossing angle beam-beam effects can be expressed as

$$\tau^*_{bb,\text{total}} = \frac{\tau_x^*+\tau_y^*}{4}\left(\frac{1}{\frac{1}{\mathcal{R}_{y,8,FB}}+\frac{1}{\mathcal{R}_{\text{syn-beta},x}}}\right)^{-1}\times \exp\left(\frac{1}{\frac{1}{\mathcal{R}_{y,8,FB}}+\frac{1}{\mathcal{R}_{\text{syn-beta},x}}}\right) \quad (\text{FB}), \qquad (66)$$

where τ_x^* and τ_y^* are damping times in horizontal and vertical plane, respectively, $\mathcal{R}_{y,8,FB}$ corresponds to Eq. (31) of Ref. [3], expressed explicitly as

$$\mathcal{R}_{y,8,FB} = \frac{A_{y,8,FB}(s)^2}{\sigma_y(s)^2} = \frac{3\sqrt{2}\gamma_*\sigma_x\sigma_y}{N_e r_e \beta_y(s_{IP})}. \qquad (67)$$

After the necessary preparations, we can try to answer two frequently asked questions. Firstly, for a machine working at the head-on collision beam-beam limit, how the beam lifetime depends on the crossing angle? Secondly, for a finite crossing angle, to keep the beam lifetime the same as that of the head-on collision at the beam-beam limit, how much one has to operate the machine below the designed head-on peak luminosity? To answer the first question we define a lifetime reduction factor:

$$R(\Phi)=\frac{\tau^*_{bb,\text{total}}}{\tau^*_{bb,y}} \qquad \text{(FB)}, \tag{68}$$

where $\tau^*_{bb,y}$ is given in Eq. (43) of Ref. [3], and $R(\Phi)$ will tell us to what extent one can increase Φ. Concerning the second question, one can imagine to reduce the luminosity at beam-beam limit by a factor of $f(\Phi)$ in order to keep the lifetime the same as that without the crossing angle. Physically, from Eq. (66) one requires:

$$\left(\frac{A_{\text{syn-beta},x}(s)^2}{\sigma_x(s)^2}\right)^{-1}+\left(\frac{A_{\text{dyna,crossing},8,y}(s)^2}{\sigma_y(s)^2}\right)^{-1}=\left(\frac{A_{\text{dyna,head-on},8,y}(s)^2}{\sigma_y(s)^2}\right)^{-1} \qquad \text{(FB)}. \tag{69}$$

Mathematically, one has to solve the following equation to find the peak luminosity reduction factor $f(\Phi)$:

$$\frac{3\pi^2\xi^2_{x,\text{design,FB}}f(\Phi)^2\Phi^2}{2}+\frac{\sqrt{2}\pi\xi_{y,\max,\text{FB}}f(\Phi)}{3}=\frac{\sqrt{2}\pi\xi_{y,\max,\text{FB}}}{3} \qquad \text{(FB)}, \tag{70}$$

$$f(\Phi)=\frac{-b_0+\sqrt{b_0^2+4a_0c_0}}{2a_0} \qquad \text{(FB)}, \tag{71}$$

where $a_0=3\pi^2\xi^2_{x,\text{design,FB}}\Phi^2/2$, $b_0=c_0=\sqrt{2}\pi\xi_{y,\max,\text{FB}}/3$, and $\xi_{x,\max,\text{FB}}$ is the designed maximum horizontal beam-beam parameter. In fact, $f(\Phi)$ corresponds to the luminosity reduction due to the synchrobetatron resonance, and to find out the total luminosity reduction factor, one has to include the geometrical effects[6, 7]. The total luminosity reduction factor can be expressed as follows:

$$F_1(\Phi)=f(\Phi)\left(1+\Phi^2\right)^{-1/2} \qquad \text{(FB)}, \tag{72}$$

where hourglass effect is not taken into account (i.e. $\beta_{y,IP}>\sigma_z$). Since Piwinski angle is proportional to the bunch length, the variation of luminosity with respect to the bunch length can be expressed as

$$F_2(\sigma_z)=\frac{L(\sigma_z)}{L(\sigma_{z,0})}=F(\Phi(\sigma_z))\frac{\sigma_{z,0}}{\sigma_z}, \tag{73}$$

where $\beta_{y,*}$ has been set to equal σ_z.

Taking BEPC-Ⅱ[8] as an example, one has $\sigma_x=380$ μm, $\sigma_z=1.5$ cm, $\phi=11$ mrad, $\Phi=0.434$, $\xi_{x,\text{design,FB}}=0.04$, and by putting $\Phi=0.434$ rad into Eq. (71) one finds $F(0.434)=85.7\%$. In Fig. 1 one finds $F_1(\Phi)$ as a function of Piwinski angle with bunch length keeping constant. In Fig. 2 we show how $F_2(\sigma_z)$ depends on bunch length. It is shown clearly that if due to bunch lengthening effect, BEPC-Ⅱ's bunch length reaches 2 cm, the luminosity will be 58% of the designed head-on collision luminosity, i.e., $L=5.8\times10^{32}$ ($\text{cm}^{-2}\cdot\text{s}^{-1}$).

Finally, when the crossing angle is in the vertical plane or the beam is round, one gets:

$$\mathcal{R}_{\text{syn-beta},y}=\frac{1}{3\pi^2}\left(\frac{r}{\xi^*_y\Phi}\right)^2 \qquad \text{(FB)}, \tag{74}$$

and

$$\mathcal{R}_{\text{syn-beta},y}=\frac{32}{27\pi^2}\left(\frac{1}{\xi^*_y\Phi}\right)^2 \qquad \text{(RB)}, \tag{75}$$

where $r=\sigma_y/\sigma_x$ and $\Phi=\frac{\sigma_z}{\sigma_x}\phi$ as defined before. Replacing $\mathcal{R}_{\text{syn-beta},x}$ in Eq. (66) by Eq. (74) or Eq. (75) and following the same procedure shown above one can easily make the corresponding discussion about the luminosity reduction effects. What should be remembered is that the geometrical luminosity reduction factors for the vertical crossing angle and the round beam cases are $(1+(\Phi/r)^2)^{-1/2}$ and $(1+\Phi^2)$, respectively.

Eq. (66) has been applied to KEK-B low energy ring to estimate the the luminosity reduction due to crossing angle effect[9].

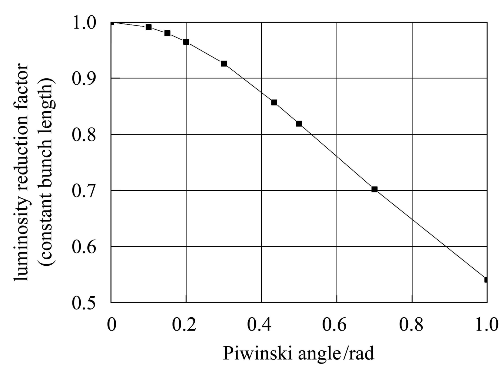

Fig. 1. The luminosity reduction factor $F_1(\Phi)$ of BEPC-Ⅱ vs Piwinski angle Φ with bunch length unchanged.

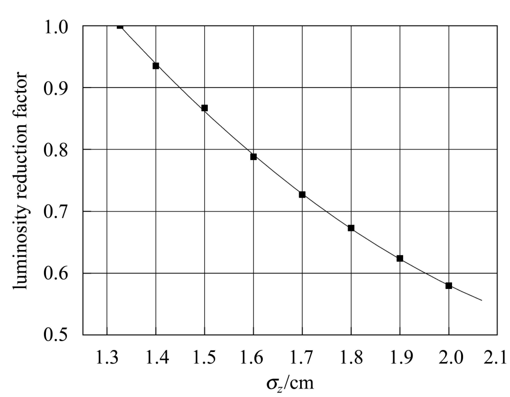

Fig. 2. The luminosity variation factor $F_2(\sigma_z)$ of BEPC-Ⅱ vs bunch length σ_z with $\beta_{y,*}=\sigma_z$.

6 Parasitic crossing effects

Parasitic crossings in e^+e^- storage ring colliders such as PEP-II working in by-2 mode will introduce additional beam lifetime limitation together with beam-beam effects at IP with or without crossing angle. The transverse separation of the two parasitic crossing bunches is $\Sigma_{PC} = \sqrt{d_x^2+d_y^2}$, where d_x and d_y are separations in horizontal and vertical plane, respectively. According to Ref. [10] the beam lifetime limited by one parasitic crossing

$$\tau_{PC,y,\mathrm{RB}} = \frac{\tau_y}{2}\left(\mathcal{R}_{y,PC,\mathrm{RB}}\right)^{-1}\exp\left(\mathcal{R}_{y,PC,\mathrm{RB}}\right) = \frac{\tau_y}{2}\left(\frac{4}{\pi\xi_{PC,y}}\right)^{-1}\exp\left(\frac{4}{\pi\xi_{PC,y}}\right), \quad (76)$$

with

$$\xi_{PC,y} = \frac{r_\mathrm{e}N_\mathrm{e}\beta_{PC,x}}{2\pi\gamma_*\Sigma_{PC}^2} = \frac{r_\mathrm{e}N_\mathrm{e}\beta_{PC,y}}{2\pi\gamma_* d_x^2}, \quad (77)$$

where $\beta_{PC,y}$ is the vertical beta function value at the parasitic crossing point, and d_y has been set to zero as a special case of a horizontal separation. The effects from the beam-beam interactions at IP and PC have to be combined to obtain the corresponding resultant beam lifetime as follows

$$\tau_{bb,\mathrm{total}} = \frac{\tau_y}{2}\left(\mathcal{R}_\mathrm{total}\right)^{-1}\exp\left(\mathcal{R}_\mathrm{total}\right), \quad (78)$$

where

$$\mathcal{R}_\mathrm{total} = \frac{1}{\dfrac{1}{\mathcal{R}_{y,IP,\mathrm{FB}}}+\dfrac{1}{\mathcal{R}_{y,PC,\mathrm{RB}}}}, \quad (79)$$

$$\mathcal{R}_{y,IP,\mathrm{FB}} = \frac{3}{\sqrt{2}\pi\xi_y}, \quad (80)$$

$$\mathcal{R}_{y,PC,\mathrm{RB}} = \frac{4}{\pi\xi_{PC,y}}. \quad (81)$$

If there are N_{PC} parasitic crossings per turn, Eq. (80) should be replaced by

$$\mathcal{R}_\mathrm{total} = \frac{1}{\dfrac{1}{\mathcal{R}_{y,IP,\mathrm{FB}}}+\displaystyle\sum_{i=1}^{N_{PC}}\dfrac{1}{\mathcal{R}_{y,PC,\mathrm{RB},i}}}, \quad (82)$$

where

$$\mathcal{R}_{y,PC,\mathrm{RB},i} = \frac{4}{\pi\xi_{PC,y,i}}, \quad (83)$$

$$\xi_{PC,y,i} = \frac{r_\mathrm{e}N_\mathrm{e}\beta_{PC,y,i}}{4\pi\gamma_*\Sigma_{PC,y,i}^2} = \frac{r_\mathrm{e}N_\mathrm{e}\beta_{PC,y,i}}{2\pi\gamma_* d_{x,i}^2}, \quad (84)$$

and d_y has been set to zero. To include emittance blow-up effects one should follow the same procedure as explained at the end of section 4.

Eq. (78) has been applied to the PEP-II low energy ring working in by-2 mode[10].

7 Nonlinear space charge effect

Considering an electron storage ring, particles inside a bunch will subject to collective space charge force from the bunch. As we will show later, in some special situations, the effect coming from this force cannot be neglected. We start with the linear incoherent space charge tune shift of the machine at the center of the bunch

$$\xi_{sc,y} = -\frac{r_\mathrm{e}N_\mathrm{e}\beta_{\mathrm{av},y}}{2\pi\gamma\sigma_y(\sigma_x+\sigma_y)}\left(\frac{L}{\sqrt{2\pi}\beta^2\gamma^2\sigma_z}\right), \quad (85)$$

where N_e is the particle population inside the bunch, σ_z is the bunch length, and $\beta_{\mathrm{av},y}$ is the average over the ring. In fact, as in the previous section, one can define the differential space charge tune shift from which the space charge tune shift of the ring can be obtained

$$\xi'_{sc,y}(s_0) = -\frac{r_\mathrm{e}N_\mathrm{e}\beta_y(s_0)}{2\pi\gamma\sigma_y(s_0)(\sigma_x(s_0)+\sigma_y(s_0))}\times\left(\frac{1}{\sqrt{2\pi}\beta^2\gamma^2\sigma_z}\right), \quad (86)$$

where $'$ denotes $\mathrm{d}/\mathrm{d}s$ and s_0 is an arbitrary position in the ring. Recalling the expression of the beam-beam tune shift of a storage ring collider, one has

$$\xi_{bb,y}(s_{IP}) = \frac{r_\mathrm{e}N_\mathrm{e}\beta_{y,IP}}{2\pi\gamma\sigma_y(s_{IP})(\sigma_x(s_{IP})+\sigma_y(s_{IP}))}, \quad (87)$$

where s_{IP} denotes the interaction point. By comparing Eq. (86) with Eq. (87), one finds the following relation between the transverse deflecting forces from the differential space charge and the beam-beam interactions.

$$f'_{sc}(s) = f_{bb}(s_{IP})G, \quad (88)$$

with

$$G = -\left(\frac{1}{\sqrt{2\pi}\beta^2\gamma^2\sigma_z}\right), \quad (89)$$

where f'_{sc} and f_{bb} are the total transverse forces including, of course, nonlinear parts. We conclude that the differential space charge effect can be made equivalent to the problem of beam-beam interaction in an storage ring collider.

By analogy one derives the dynamic aperture determined by the nonlinear (octupole is the lowest nonlinear multipole) differential space charge force

$$(A_{sc,y}(s)^2)' = \frac{\beta_y(s)}{\beta_y(s_0)^2}\left(\frac{3\sqrt{2}\gamma\sigma_x(s_0)\sigma_y^3(s_0)}{N_\mathrm{e}r_\mathrm{e}G}\right) \quad (\mathrm{FB}). \quad (90)$$

The total dynamic aperture limited by the space charge force can be calculated as

$$A_{\text{total},sc,y}(s)=\frac{1}{\sqrt{\sum_{s_0=0}^{L}\frac{1}{(A_{sc,y}(s)^2)'}}},\tag{91}$$

$$\frac{1}{A^2_{\text{total},sc,y}(s)}=$$

$$\int_{s_0=0}^{L}\frac{\beta_y(s_0)^2}{\beta_y(s)}\left(\frac{N_{\text{e}}r_{\text{e}}}{6\sqrt{\pi}\beta^2\gamma^3\sigma_x(s_0)\sigma_y(s_0)^3\sigma_z}\right)\text{d}s_0,\tag{92}$$

where the differential space charge forces are assumed to be independent. After some mathematical simplification and using Eq. (85), one gets

$$\mathcal{R}^2_{sc,y}=\left(\frac{A_{\text{total},sc,y}(s)}{\sigma_y(s)}\right)^2=\frac{3}{\sqrt{2}\pi\xi_{sc}}.\tag{93}$$

The lifetime of the particle limited by the nonlinear space charge forces can be estimated as:

$$\tau_{sc,y}(\xi_{sc,y})=\frac{\tau_y}{2}\left(\mathcal{R}^2_{sc,y}\right)^{-1}\exp\left(\mathcal{R}^2_{sc,y}\right)=\frac{\tau_y}{2}\left(\frac{3}{\sqrt{2}\pi\xi_{sc,y}}\right)^{-1}\exp\left(\frac{3}{\sqrt{2}\pi\xi_{sc,y}}\right).\tag{94}$$

With the result of Eq. (94) one can calculate the relative survival population of the particles, $R(\xi_{sc,y})$ at the moment of ejection ($t=\tau_{st}$).

$$R(\xi_{sc})=\exp\left(-\frac{\tau_{st}}{\tau_{sc,y}(\xi_{sc,y})}\right).\tag{95}$$

In the following we apply Eq. (95) to TESLA damping ring[11] with $\tau_y = 28$ ms, and storage time $\tau_{st} = 200$ ms, and calculate the relative survival population with respect to the linear space charge tune shift $\xi_{sc,y}$. From Eq. (95) one finds that to avoid particle loss due to nonlinear space charge forces, one has to choose $\xi_{sc,y}$ below 0.07 (less than 1% particles are lost). This coincides with the results from the numerical simulations in Ref. [11],where it was concluded that the condition $\xi_{sc,y} < 0.1$ should be fulfilled. Taking the TESLA parameters, $E_0 = 5$ GeV, $L = 17$ km, $N_{\text{e}} = 2\times10^{10}$, $\sigma_z = 6$ mm, and the normalized transverse emittances, $\epsilon_{x,n} = 9\times10^{-6}$ mrad and $\epsilon_{y,n} = 2\times10^{-8}$ mrad, one finds $\xi_{sc,y} = 0.248$ and $R(\xi_{sc,y}) = 7.7\%$, which are intolerable. In Ref. [11] a method was proposed to solve this problem without having to increase the energy of the damping ring. Instead it was proposed to increase the beam dimensions in the long straight sections of the "Dog-Bone" type damping ring by using skew quadrupoles. This reduces the space charge tune shift well below the threshold of $\xi_{sc,y} = 0.1$.

8 Electron cloud effect combined with beam-beam and space charge effects

Electron clouds produced and trapped by the positron beam in the vacuum chamber can perturb the motion of positrons in return. In this section we focus ourselves to the special case where significant amount of electrons is traped near the positron beam axis with almost the same dimensions as those of trapping positron beam. We will in this section not be interested in electron-clouds far from the positron beam. We define the local electron-cloud and positron beam interaction force as $f'_{\text{ec}}(s_0)$, this differential force (where $'$ denotes $\text{d}/\text{d}s$), can be made equivalent to a virtual local beam-beam force $\mathcal{F}_{bb}(s_0)$. The relation between $f'_{\text{ec}}(s_0)$ and $\mathcal{F}_{bb}(s_0)$ can be expressed as

$$f'_{\text{ec}}(s_0)=\frac{1}{2L}\mathcal{F}_{bb}(s_0),\tag{96}$$

and the $f'_{\text{ec}}(s_0)$ induced differential positron linear tune shift is given by

$$\xi'_{\text{ec}}(s_0)=\frac{r_{\text{e}}N_{\text{ec}}\beta_{+,y}(s_0)}{2\pi\gamma_+\sigma_{+,y}(s_0)(\sigma_{+,x}(s_0)+\sigma_{+,y}(s_0))}\left(\frac{1}{2L}\right),\tag{97}$$

where $\sigma_{+,x}$ and $\sigma_{+,y}$ are the transverse rms dimensions of the electron-clouds and positron beam, L is the circumference of the storage ring, $\beta_{+,y}$ is the vertical beta function for positrons, γ_+ is the normalized positrons' energy, and finally N_{ec} is total electron-cloud charge numbers around the ring within a transverse cross section of $2\pi\sigma_{+,x}\sigma_{+,y}$. Making use of the analytical results for the beam-beam interactions in an e^+e^- storage ring collider developed in Ref. [3], one can estimate the vertical dynamic aperture limited by the differential electron-cloud nonlinear forces

$$\left(\frac{\sigma_{+,y}(s_0)}{A'_{\text{ec},y}(s_0)}\right)^2=\frac{N_{\text{ec}}r_{\text{e}}\beta_y(s_0)}{6\sqrt{2}\gamma_+\sigma_{+,x}(s_0)\sigma_{+,y}(s_0)L}.\tag{98}$$

The total contribution of the electron-cloud around the ring to the vertical dynamic aperture can be estimated according to Ref. [1] as

$$\left(\frac{\sigma_{+,y}}{A_{\text{ec},y}}\right)^2=\int_{s_0}^{s_0+L}\frac{N_{\text{ec}}r_{\text{e}}\beta_y(s_0)}{6\sqrt{2}\gamma_+\sigma_{+,x}(s_0)\sigma_{+,y}(s_0)L}\text{d}s_0.\tag{99}$$

One finds that

$$\mathcal{R}^2_{\text{ec},y}=\left(\frac{A_{\text{ec},y}}{\sigma_{+,y}}\right)^2\approx\frac{3\sqrt{2}\gamma_+}{\pi r_{\text{e}}\beta_{av,y}\rho_{\text{ec}}L},\tag{100}$$

where $\beta_{\text{av},y}$ is the average vertical beta function around the ring, and ρ_{ec} is the average electron-cloud density inside the vacuum chamber defined by:

$$\rho_{\text{ec}}=\frac{N_{\text{ec}}}{2\pi\sigma_{\text{av},+,x}\sigma_{\text{av},+,y}L},\tag{101}$$

where $\sigma_{\text{av},+,x}$ and $\sigma_{\text{av},+,y}$ are the average beam transverse dimensions around the ring. The total normalized vertical dynamic aperture limited by the beam-beam and the electron-cloud effects can be obtained from

$$\mathcal{R}^2_{\text{total},+,y} = \frac{1}{\frac{1}{\mathcal{R}^2_{bb,+,y}} + \frac{1}{\mathcal{R}^2_{\text{ec},y}} + \frac{1}{\mathcal{R}^2_{sc,y}}}, \qquad (102)$$

with $\mathcal{R}^2_{bb,+,y}$ expressed as

$$\mathcal{R}^2_{bb,+,y} = \left(\frac{A_{bb,y,IP}}{\sigma_{+,y,IP}}\right)^2 = \frac{3}{\sqrt{2}\pi\xi_{bb,+,y}}, \qquad (103)$$

where $\xi_{bb,+,y}$ is the linear beam-beam tune shift of the positron beam in the vertical plane, and the subscript IP denotes the interaction point. The positron's lifetime due to the combined beam-beam and electron-cloud effects can be estimated from:

$$\tau_{\text{total},+,y} = \frac{\tau_{+,y}}{2}\left(\mathcal{R}^2_{\text{total},+,y}\right)^{-1}\exp\left(\mathcal{R}^2_{\text{total},+,y}\right), \quad (104)$$

where $\tau_{+,y}$ is the damping time of positron in the vertical plane. Finally, also in case of an electron storage ring one can still use Eq. (104) to estimate the electron beam lifetime limited by the combined beam-beam and nonlinear electron-ion interactions.

9 Application to BEPCⅡ positron ring

Taking BEPCⅡ positron ring as an example[8], we show in Table 1 the ring parameters. The designed head-on vertical beam-beam parameter is 0.04. Fig. 3 shows the variation of the head-on vertical beam-beam parameter with respect to the electron cloud density, and Fig. 4 shows head-on collision beam lifetime ($\xi_{bb,y}=0.042$) variation with electron cloud density.

Table 1. The BEPCⅡ positron ring's parameters.

machine	γ	$\beta_{\text{av},+,y}$/m	L/km	$\tau_{+,y}$/ms
BEPCⅡ(e+)	3699	5.8	0.23753	25

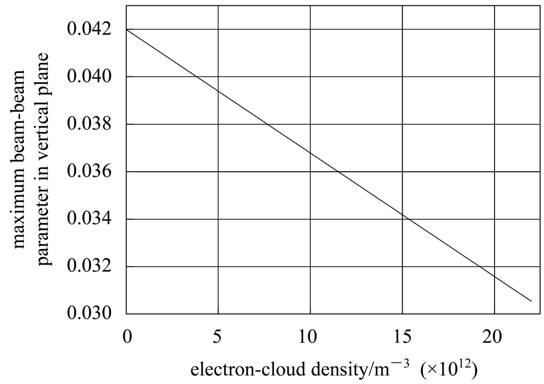

Fig. 3. The maximum attainable beam-beam tune shift in the vertical plane of BEPCⅡ positron ring as a function of the electron-cloud density, ρ_{ec}.

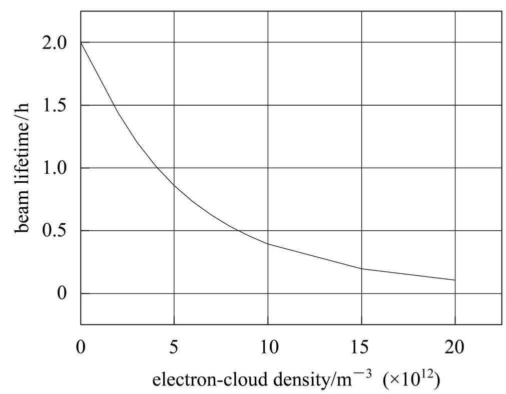

Fig. 4. BEPCⅡ positron ring, with $\xi_{+,y}=0.042$; the positron beam lifetime as a function of the electron-cloud density, ρ_{ec}.

10 Conclusion

Many complex phenomena in storage rings are connected with nonlinear beam dynamics, such as the subjects treated in this paper. Together with experiments and numerical simulations, analytical treatment plays an important role in understanding the relevant physical processes and is very helpful in designing and operating machines.

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